Splines which are piecewise solutions of polyharmonic equation

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March 25, 2006

Abstract

This paper appeared in Proceedings of the Conference "Curves and Surfaces", Chamonix, 1993

1 Introduction

In recent years there has been a great deal of attempts to find a proper multivariate paradigma to occupy the position of what has to be called multivariate spline. The new progress in polyharmonic splines and radial basis functions shows that the search has not finished yet.

The present paper works with the notion of polyspline introduced and studied in [3,4,5,6]. The polysplines of order $2q$ are constructed by pieces of polyharmonic functions of order $2q$, which join up to some order. The very essence of polysplines is that the data are assumed to be given on hypersurfaces lying in a domain D in $\mathbb{R}^n$.

We suppose that $D = \bigcup_{j=1}^{N} D_j$, where the domains $D_j \subseteq D$, $D_j \bigcap D_k = \emptyset$ for $j \neq k$. We assume that $S = \bigcup_{j=1}^{N} \partial D_j$ is a union of smooth manifolds of dimension $n-1$, which do not intersect. The data $f(x)$ are supposed to be given on the set S.

There are two simplest nontrivial examples: The first is when D is a strip in $\mathbb{R}^2$ and D_j are strips parallel to it. The second is when D is a circle in $\mathbb{R}^2$ and D_j are annuli concentric with it.

The main purpose of the present paper is to study polysplines defined in the case when D_j are parallel strips in $\mathbb{R}^2$, and more generally, parallel layers in $\mathbb{R}^n$, and to show the analogy to many properties of the univariate splines.

In section 5 we prove existence of polysplines when D_j are parallel layers, thus making *the present paper completely independent of* [3,4,5]. We also prove that the approximating power of the polysplines (when the width of the layers goes to zero) is exactly $4q$, i.e. completely similar to that of the univariate splines. The proofs are based on the reduction of the polysplines to Tchebycheffian splines in one dimension (cf. [11, Ch. 9]).

The main technical device is Lemma 3 in Section 4, which provides uniform boundedness of a family of L-splines depending on a parameter.

Let us remark that the polysplines are genetically related to the theory of polyharmonic splines [9,10] and radial basis functions, (cf. [2,8]), since in view of the formula of Green [1, p. 10] they are "integrals over surfaces" of fundamental solutions to the polyharmonic equations up to order $2q$. Due to this fact the polysplines enjoy bigger smoothness compared to the radial basis functions.

2 Definition of Polysplines

Let $q \geq 1$ be an integer.

In [3,4,5] for bounded domains D, the *interpolation polyspline of order* $2q$, is defined as a solution to the following extremal problem:

$$\int_D \{\Delta^q u(x)\}^2 dx \to \inf, \tag{2.1}$$

where the infimum is taken over the family of functions $u(x)$ such that

$$u(x) = f(x), \quad x \in S, \tag{2.2}$$

$$\Delta^k u(x) = 0, \quad x \in \partial D, \quad k = 1, \ldots, q-1, \tag{2.3a}$$

$$(\partial/\partial n)\Delta^k u(x) = 0, \quad x \in \partial D, \quad k = 0, \ldots, q-1. \tag{2.3b}$$

In [3,4,5] the uniqueness and the existence is proved in the Sobolev spaces.

Let us put $T_{ij} = \partial D_i \bigcap \partial D_j$. We shall denote by $u_j(x)$ the restriction of u to the subdomain D_j. There it is proved that in a bounded domain D the solution u to problem (1.1-3) satisfies the following properties:

$$\Delta^{2q} u_j(x) = 0, \quad x \in D_j, \quad j = 1, 2, \ldots, N; \tag{2.4}$$

$$\Delta^k u(x) = 0, \quad x \in \partial D, \quad k = 1, \ldots, q-1; \tag{2.5a}$$

$$(\partial/\partial n)\Delta^k u(x) = 0, \quad x \in \partial D, \quad k = 0, \ldots, q-1; \tag{2.5b}$$

$$\Delta^p u_i(x) = \Delta^p u_j(x), \quad x \in T_{ij}, \tag{2.6}$$

for $i, j = 1, 2, \ldots, N$, $i \neq j$, and $p = 0, 1, \ldots, 2q-1$;

$$(\partial/\partial n_i)\Delta^p u_i(x) = (\partial/\partial n_j)\Delta^p u_j(x), \quad x \in T_{ij}, \tag{2.7}$$

for $i, j = 1, 2, \ldots, N$, $i \neq j$, and $p = 0, 1, \ldots, 2q-2$.

Equalities (1.4-7) should be considered like equalities between boundary values of functions in a Sobolev space (traces) (cf. [7]).

The space of all such polysplines will be denoted by $PS_{2q}(D)$ or, simply, by PS.

It is not difficult to modify the techniques of [5] based on apriori estimates for elliptic boundary value problems (cf. [7]) and to prove the equivalence of

(2.1-3) and (2.4-7)&(2.2) when the domain D is unbounded, and the data f decay at infinity, e.g. $f \in L_2$.

In the next section we consider the particular case when D_j are parallel layers, i.e. when S is a union of parallel hyperplanes and D itself is a layer between two parallel hyperplanes. By a Fourier transform we reduce problem (2.4-7)&(2.2) to Tchebycheffian splines [11].

3 Polysplines on Parallel Layers

We use the notation $x = (t, y)$, $t \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$. Consider the numbers $t_1 < \ldots < t_N$. Put $D_j = \{x \in \mathbb{R}^n : t_j < t < t_{j+1}\}$, and consider the domain $D = \bigcup_{j=1}^{N-1} D_j$. We assume that the data f are a function $f(t, y)$ given at least for $t = t_j, j = 1, \ldots, N$.

Proposition 1 *Let u be a solution to problem (2.4-7)&(2.2) and $u_j \in H^{4q}(D_j)$. Then it satisfies*

$$(\partial^k/\partial t^k)u_j(t_{j+1}, y) = (\partial^k/\partial t^k)u_{j+1}(t_{j+1}, y), y \in \mathbb{R}^{n-1}, \tag{3.1}$$

for $k = 0, 1, \ldots, 4q - 2$, $j = 1, \ldots, N - 2$;

$$(\partial^k/\partial t^k)u_1(t_1, y) = 0, \quad y \in \mathbb{R}^{n-1}, \tag{3.2a}$$

$$(\partial^k/\partial t^k)u_{N-1}(t_N, y) = 0, \quad y \in \mathbb{R}^{n-1}, \tag{3.2b}$$

both for $k = 1, ..., 2q - 1$, and

$$u_j(t_j, y) = f(t_j, y), \quad y \in \mathbb{R}^{n-1}, \quad j = 1, \ldots, N - 1; \tag{3.3a}$$

$$u_{N-1}(t_N, y) = f(t_N, y), y \in \mathbb{R}^{n-1}. \tag{3.3b}$$

Proof. We will show how (3.1) follows. For $p = 0$, from (2.6-7) we obtain

$$u_j(t_{j+1}, y) = u_{j+1}(t_{j+1}, y), y \in \mathbb{R}^{n-1},$$

and

$$(\partial/\partial t)u_j(t_{j+1}, y) = (\partial/\partial t)u_{j+1}(t_{j+1}, y), \quad y \in \mathbb{R}^{n-1}.$$

Consequently, for every multi-index α for which the derivatives exist, we have

$$D_y^\alpha u_j(t_{j+1}, y) = D_y^\alpha u_{j+1}(t_{j+1}, y), \quad y \in \mathbb{R}^{n-1}, \tag{3.4}$$

and

$$(\partial/\partial t)D_y^\alpha u_j(t_{j+1}, y) = (\partial/\partial t)D_y^\alpha u_{j+1}(t_{j+1}, y), \quad y \in \mathbb{R}^{n-1}.$$

For $p = 1$ (2.6) gives

$$\Delta_y u_j(t_{j+1}, y) + (\partial^2/\partial t^2)u_j(t_{j+1}, y) = \Delta_y u_{j+1}(t_{j+1}, y) + (\partial^2/\partial t^2)u_{j+1}(t_{j+1}, y)$$

for $y \in \mathbb{R}^{n-1}$.

Since (3.4) implies $\Delta_y u_j(t_{j+1}, y) = \Delta_y u_{j+1}(t_{j+1}, y)$, we obtain

$$(\partial^2/\partial t^2)u_j(t_{j+1}, y) = (\partial^2/\partial t^2)u_{j+1}(t_{j+1}, y), \quad y \in \mathbb{R}^{n-1}.$$

Thus we have established (2.1) for $k = 0, 1, 2$. Further we proceed by induction on p.

The rest of the equalities are proved in a similar way. ■

Remark 1. Proposition 1 implies that on the hyperplanes $t = t_j$ where the data are located the polyspline u has a smoothness bigger than that of the corresponding fundamental solution of Δ^{2q} , resp. the radial basis generated by it (cf. [2,8,9,10]) . Outside these hyperplanes it is a real analytic function [1]. In particular, for $q = 1, u_j$ satisfies $\Delta^2 u_j = 0$ in D_j, and $D_y^\alpha u_j(t_j, y)$ exists if $D_y^\alpha f(t, y)$ exists. Also, $(\partial^2/\partial t^2)u(t, y)$ is continuous throughout the whole D. For a C^2 smoothness only the mixed derivatives $(\partial^2/\partial t\partial y_k)u_j(t_j, y)$ are missing.

We make the Fourier transform $F_{y\to\xi}$ on the y-variable which reduces the above problem to a one-dimensional.

Proposition 2 *For every $\xi \in \mathbb{R}^{n-1}$ the function $Fu(t,\xi) = F_{y\to\xi}[u(t, y)]$ is a L-spline with knots t_j and data $Ff(t_j,\xi) = F_{y\to\xi}[f(t_j, y)]$, where the operator L is given by $L_\xi = (\partial^2/\partial t^2 - \xi^2)^{2q}$. Even more, $Fu(t,\xi)$ is a Tchebycheffian spline. (For the definition of Tchebycheffian and L-splines see [11, Ch. 9 and 10]).*

Proof. The proof easily follows from Proposition 1. To all equalities (3.1-3), we apply the Fourier transform $F_{y\to\xi}$ and simply replace $u_j(t, y)$ through $Fu_j(t,\xi)$. In fact, property (3.1) implies that $Fu_j(t,\xi) \in C_t^{4q-2}[t_1, t_N]$.

On the other hand, since the Fourier transform maps the Laplace operator Δ_x into $\partial^2/\partial t^2 - \xi^2$, it follows that Δ^{2q} is transformed into L_ξ. Thus the equation $\Delta^{2q}u_j = 0$ in D_j is transformed into the ordinary differential equation $L_\xi Fu_j(t,\xi) = 0$, $t_j < t < t_{j+1}$. The last, together with the C^{4q-2} -smoothness means exactly that $Fu(t,\xi)$ is a L_ξ-spline.

The equality

$$\begin{aligned} L_\xi g(t) &= (\partial/\partial t- \mid \xi \mid)^{2q}(\partial/\partial t+ \mid \xi \mid)^{2q} \\ &= e^{-2|\xi|t}(\partial^{2q}/\partial t^{2q})e^{2|\xi|t}(\partial^{2q}/\partial t^{2q})e^{-|\xi|t}g(t) \end{aligned}$$

follows from the easy-to-check one

$$(\partial/\partial t - \lambda) = e^{\lambda t}(\partial/\partial t)e^{-\lambda t}g(t), \quad \lambda \in \mathbb{R}.$$

Consequently, for the Tchebycheffian spline properties [11, §9.1], we have the functions w_j given by $w_1(t) = e^{|\xi|t}$, $w_{2q+1}(t) = e^{-2|\xi|t}$, and $w_j = 1$ for $j = 2, \ldots, 2q, 2q + 2, \ldots, 4q$. ■

4 Some Estimates of L-splines

A very important problem is to study how the L-splines behave when the coefficients of the operator L vary.

We prove the following fundamental:

Lemma 3 *Let $v(t,\xi)$ be a L_ξ-spline with data $v(t_j,\xi)=\gamma_j, j=1,...,N$, and boundary conditions (coming from (3.2)):*

$$(\partial^k/\partial t^k)v(t_1,\xi)=(\partial^k/\partial t^k)v(t_N,y)=0,\quad \xi\in\mathbb{R}^{n-1},$$

$k=1,\ldots,2q-1$. *Then*

$$\| v \|_C=\max_{t\in[t_1,t_N]} | v(t,\xi) |\leq C \| \gamma \|= C\max_{1\leq j\leq N} | \gamma_i |,\quad \xi\in\mathbb{R}^{n-1},$$

where the constant C depends on the knots t_i but not on ξ.

Proof. We will use the explicit formula for v.

1. Let v_k denote the restriction of v to the interval $[t_k,t_{k+1}]$. Then the theory of ordinary differential equations provides that

$$v_k(t,\xi)=P_k(t,\xi)e^{|\xi|t}+Q_k(t,\xi)e^{-|\xi|t},\quad k=1,\ldots,N-1,$$

where $P_k(t,\xi),Q_k(t,\xi)$, are polynomials in t of degree $\leq 2q-1$.

Further we will be interested in the behavior of P_k and Q_k for large values of ξ, i.e. for $|\xi|\geq\xi_0$ for some $\xi_0>0$.

2. Due to Proposition 2 we have the system, (Δ):

$$(\partial^j/\partial t^j)(P_ke^{|\xi|t}+Q_ke^{-|\xi|t})-(\partial^j/\partial t^j)(P_{k+1}e^{|\xi|t}+Q_{k+1}e^{-|\xi|t})=_{|t=t_{k+1}}0,$$

for $j=0,\ldots,4q-2$;

$$P_{k+1}e^{|\xi|t}+Q_{k+1}e^{-|\xi|t}=_{|t=t_{k+1}}\gamma_{k+1},\quad k=1,\ldots,N-2,$$

and the boundary conditions

$$\begin{aligned}(\partial^j/\partial t^j)(P_1e^{|\xi|t}+Q_1e^{-|\xi|t})&=_{|t=t_1}0,\quad j=1,\ldots,2q;\\ P_1e^{|\xi|t}+Q_1e^{-|\xi|t}&=_{|t=t_1}\gamma_1;\\ (\partial^j/\partial t^j)(P_{N-1}e^{|\xi|t}+Q_{N-1}e^{-|\xi|t})&=_{|t=t_N}0,\quad j=1,\ldots,2q;\\ P_{N-1}e^{|\xi|t}+Q_{N-1}e^{-|\xi|t}&=_{|t=t_N}\gamma_N.\end{aligned}$$

3. The basic observation is that after putting $P_k=\tilde{P}_ke^{-|\xi|t_{k+1}}$, $Q_{k+1}=\tilde{Q}_{k+1}e^{|\xi|t_{k+1}}$ (in fact, by dividing the columns by the corresponding exp) the above system has the following form, (Δ_{k+1}):

$$\begin{gathered}e^{-|\xi|t_{k+1}}(\partial^j/\partial t^j)(\tilde{P}_ke^{|\xi|t})+O_1(e^{-|\xi|T})\\ -e^{|\xi|t_{k+1}}(\partial^j/\partial t^j)(\tilde{Q}_{k+1}e^{-|\xi|t})=_{|t=t_{k+1}}0,\end{gathered}$$

for $j=0,\ldots,4q-2$;

$$O_2(e^{-|\xi|T})+\tilde{Q}_{k+1}=_{|t=t_{k+1}}\gamma_{k+1},\quad k=1,\ldots,N-2,$$

and the boundary conditions, (Δ_1):

$$O_3(e^{-|\xi|T}) + e^{|\xi|t_1}(\partial^j/\partial t^j)(\tilde{Q}_1 e^{-|\xi|t}) =_{|t=t_1} 0, \quad j = 1, \ldots, 2q;$$
$$O_4(e^{-|\xi|T}) + \tilde{Q}_1 =_{|t=t_1} \gamma_1;$$

and the system (Δ_N):

$$e^{-|\xi|t_N}(\partial^j/\partial t^j)(\tilde{P}_{N-1}e^{|\xi|t}) + O_5(e^{-|\xi|T}) =_{|t=t_N} 0, \quad j = 1, \ldots, 2q;$$
$$\tilde{P}_{N-1} + O_6(e^{-|\xi|T}) =_{|t=t_N} \gamma_N,$$

where $T = \min_{2\leq j\leq N}(t_j - t_{j-1})$, and the functions $O(.)$ satisfy $O(e^{-|\xi|T}) \leq Ce^{-|\xi|T}$ with a constant $C > 0$ independent of $|\xi|$.

4. Since $(\partial/\partial t - \lambda)^j g(t) = e^{\lambda t}(\partial/\partial t)^j e^{-\lambda t} g(t)$, $\lambda \in \mathbb{R}$, we see that the above system with respect to the new unknowns $\tilde{P}_k$, $\tilde{Q}_k$, has only polynomials of ξ and t on the left-hand side, thus up to $O(e^{-|\xi|T})$ it is splitted into blocks.

We have that the determinant of the system (Δ) is

$$\Delta(\xi) = \prod_{i=2}^{N} e^{|\xi|(t_i - t_{i-1})}\tilde{\Delta}(\xi) = e^{|\xi|(t_N - t_1)}\tilde{\Delta}(\xi)$$
$$= e^{|\xi|(t_N - t_1)}[\tilde{\Delta}_1(\xi) + O(e^{-|\xi|T})\tilde{\Delta}_2(\xi)],$$

where $\tilde{\Delta}_1(\xi)$ is the determinant obtained by cancelling the $O(.)$ terms, and $\tilde{\Delta}_2(\xi)$ is a bounded function of ξ.

We have $\tilde{\Delta}_1(\xi) = \prod_{k=1}^{N}\Delta_k(\xi)$, where Δ_k is the determinant of the system (Δ_k) with cancelled $O(.)$ terms.

Due to the above we may put zero where $O(.)$ terms occur in the systems (Δ_k). This will perturb the unknowns $\tilde{P}_k$, $\tilde{Q}_k$ only up to order $O(e^{-|\xi|T})$.

5. It is clear that the determinant Δ_k is a polynomial of ξ. We will prove that $\Delta_k \neq 0$. The last is equivalent to solubility of the corresponding block system (Δ_k) for every γ_k, or, equivalently, to prove that if $\gamma_k = 0$ then the system has only trivial solution.

6. Let $k = 2, \ldots, N-1$. We will use an idea of spline theory. Let us put $g(t) = P_k e^{|\xi|t}$ for $t \leq t_{k+1}$, and $g(t) = Q_{k+1}e^{-|\xi|t}$ for $t \geq t_{k+1}$. Then for every function $f \in H^{2q}(\mathbb{R})$ the following identity holds:

$$\int_{\mathbb{R}} (\partial/\partial t - |\xi|)^{2q} g(t)(\partial/\partial t - |\xi|)^{2q} f(t)dt$$
$$= \{\text{jump of } (\partial/\partial t - |\xi|)^{2q}(\partial/\partial t + |\xi|)^{2q-1}g(t) \text{ at } t = t_{k+1}\}f(t_{k+1})$$
$$= \{\text{jump of } (\partial/\partial t)^{4q-1}g(t) \text{ at } t = t_{k+1}\}f(t_{k+1}).$$

The proof follows after making $2q$ integrations by parts, on the intervals $(-\infty, t_{k+1})$ and (t_{k+1}, ∞), and uses the fact that $(\partial^j/\partial t^j)g(t)$ is continuous for $j = 0, \ldots, 4q-2$.

7. We put $f = g$ and obtain

$$\int_{\mathbb{R}} \{(\partial/\partial t- \mid \xi \mid)^{2q} g(t)\}^2 dt$$

$$= \{\text{jump of } (\partial/\partial t)^{4q-1} g(t) \text{ at } t = t_{k+1}\} g(t_{k+1}).$$

Since $g(t_{k+1}) = \gamma_{k+1}$, from $\gamma_{k+1} = 0$ it follows that

$$\int_{\mathbb{R}} \{(\partial/\partial t- \mid \xi \mid)^{2q} g(t)\}^2 dt = 0,$$

hence $(\partial/\partial t- \mid \xi \mid)^{2q} g(t) = 0$, $t \in\mathbb{R}$. For $t \geq t_{k+1}$ this implies

$$(\partial/\partial t- \mid \xi \mid)^{2q} [Q_{k+1}(t) e^{-|\xi| t}] = e^{|\xi| t} (\partial/\partial t)^{2q} [Q_{k+1}(t) e^{-2|\xi| t}] = 0,$$

which is possible only if $Q_{k+1}(t) \equiv 0$.

This gives

$$(\partial/\partial t)^j (P_k(t) e^{|\xi| t}) =_{|t=t_{k+1}} 0, \quad j = 0, \ldots, 4q-2,$$

or, equivalently, $(\partial/\partial t+ \mid \xi \mid)^j P_k(t)_{|t=t_{k+1}} = 0$. By induction on j this implies $(\partial/\partial t)^j P_k(t)_{|t=t_{k+1}} = 0$, $j = 0, \ldots, 4q-2$, hence $P_k(t) \equiv 0$.

This proves that the system $(\Delta_k), k = 2, \ldots, N-1$ is solvable, i.e. the determinant $\Delta_k \neq 0$.

It is simpler to prove that $\Delta_1 \neq 0$, and $\Delta_N \neq 0$.

Thus we obtain that $\tilde{\Delta}_1(\xi)$ is a polynomial of $\mid \xi \mid$ which is not zero, and hence $\tilde{\Delta}(\xi) \neq 0$ for $\mid \xi \mid \geq \xi_0$.

8. Now let a be a coefficient of $P_k(t)$. Thanks to the theorem of Cramer in linear algebra we have

$$a = \Delta_a / \Delta$$

where Δ_a is the matrix of the system (Δ), in which the column of a is replaced by the right-hand side containing the data γ_{k+1}. Following the manipulations of p. 3, we see that

$$a = (1/e^{|\xi| t_{k+1}})(\tilde{\Delta}_a / \tilde{\Delta}),$$

where $\tilde{\Delta}_a = \tilde{\Delta}_{1,a}(\xi) + O(e^{-|\xi| T}) \tilde{\Delta}_{2,a}(\xi)$ and $\tilde{\Delta}_{1,a}(\xi)$ is a polynomial in $\mid \xi \mid$ of degree $< (1 + 2 + \ldots + (4q-2))N = (2q-1)(4q-1)N$.

In a similar way, for the coefficients b of Q_{k+1} we obtain the representation

$$b = e^{|\xi| t_{k+1}} (\tilde{\Delta}_b / \tilde{\Delta}).$$

9. Thanks to the above, we have the representation

$$P_k(t) = P_k(t, \xi) = \{\gamma_{k+1} + \sum_{i=1}^{2q-1} (t_{k+1} - t)^i \alpha_i(\xi) + \parallel \gamma \parallel O(e^{-|\xi| T})\} e^{-|\xi| t_{k+1}},$$

where $\alpha_i(\xi)$ has at most polynomial growth in $\mid \xi \mid$ of degree $< (2q-1)(4q-1)N$. In order to prove that

$$\| P_k(t,\xi)e^{|\xi|t} \|_C \leq C \| \gamma \|$$

it suffices to see that

$$\tau^i \mid \xi \mid^j e^{-|\xi|\tau} \leq C_{ij}$$

for $0 \leq \tau \leq t_{k+1}-t_k$, $0 \leq \mid \xi \mid < \infty$, and $1 \leq i \leq 2q-1$, $0 \leq j < (2q-1)(4q-1)N$, where we have put $\tau = t_{k+1} - t$.

Indeed, $C_{ij} = (i+j)!$ satisfies the above. Notice that it is very essential that $i \geq 1$!

This establishes the estimate of $\| P_k(t,\xi)e^{|\xi|t} \|$. In a similar way we estimate $\| Q_k(t,\xi)e^{-|\xi|t} \|$, and finally $\| v_k \|$. ■

5 Existence and Approximating Power of Polysplines

Let us denote by $F^{-1} = F_{\xi \to y}$ the inverse Fourier transform.

Throughout the rest of the paper we assume that $N \geq 2q$.

Theorem 4 *Let the data $f(t_j, y) \in L_2\left[\mathbb{R}_y^{n-1}\right]$, $j = 1, \ldots, N$. Then there exists a unique solution u to problem (2.4-7)&(2.2), i.e. interpolation polyspline, such that $u_j \in H^\infty(D_j)$.*

Proof. Let us denote by $\chi_j(t,\xi), j = 1, \ldots, N$, the L_ξ-spline such that $\chi_j(t_j,\xi) = 1$ and $\chi_j(t_i,\xi) = 0$ for $i \neq$ j.

Let us put

$$Fu(t,\xi) = \sum_{j=1}^{N} Ff(t_j,\xi)\chi_j(t,\xi).$$

We have

$$u(t,y) = F_{\xi \to y}[\sum_{j=1}^{N} Ff(t_j,\xi)\chi_j(t,\xi)].$$

The last expression makes sense since $\chi_j(t,\xi)$ is bounded by Lemma 3. The rest follows from Proposition 1 and Proposition 2. ■

Let us remark that the existence Theorem 4 shows that Lemma 3 is implicit in the method of apriori estimates for elliptic boundary value problems.

By $L_\infty[t_1,t_N]$ we denote, as usually, the space of bounded functions in $[t_1,t_N]$ (cf. [11, p. 14]).

Further we make use of the results on Tchebycheffian splines proved in [11, Ch. 9]. In particular, we use the operator Q introduced in §9.7 there, which maps the space of data $L_\infty[t_1,t_N]$ into the space of L_ξ-splines with knots at $\delta = \{t_1, \ldots t_N\}$. Put $\overline{\delta} = \max_{1 \leq j \leq N-1}(t_{j+1} - t_j)$. We extend the operator Q in a natural way to functions f defined in D by putting

$$Fu(t,\xi) = Q_\xi(t) = Q_\xi[Ff(t_j,\xi), j = 1, \ldots, N], \quad \xi \in \mathbb{R}^{n-1},$$

where $Q_\xi(t)$ is a L_ξ-spline on $[t_1, t_N]$, and we choose the same points for the extended grid for every $\xi \in \mathbb{R}^{n-1}$. The resulting extension is now given by

$$u(t, y) = \tilde{Q}[f] = F^{-1} Q_\xi [Ff(t_j, \xi), j = 1, \ldots, N].$$

In order that the last expression make sense it is sufficient to have

$$\| Q_\xi \| \| Ff(t, \xi) \|_{L_\infty[t_1, t_N]} \in L_1(\mathbb{R}^{n-1}_\xi).$$

Indeed, we have

$$\| Q_\xi [Ff(t_j, \xi), j = 1, \ldots, N](t) \|_{C[t_1, t_N]} \leq \| Q_\xi \| \max_{1 \leq j \leq N} | Ff(t_j, \xi) |,$$

for every $\xi \in \mathbb{R}^{n-1}$, which implies that $Q_\xi(t) \in L_1(\mathbb{R}^{n-1}_\xi)$.

In [11, §9.7] the approximating power of Tchebycheffian splines is studied using the operator Q. On the right-hand side there appears a constant C_1, cf. Theorems 9.37, 9.38. In our case, for the operators Q_ξ we have the constant $C_1 = C_1(\xi)$.

In the present paper we will avoid a thorough study of the asymptotics of the constants $C(\xi) = \| Q_\xi \|$ and $C_1(\xi)$, by implementing them in the definition of the necessary space of functions.

For every two integers $p \geq 0$ and $q \geq 1$ we introduce the space $FL_{p,q}(D)$ of measurable functions $g(t, y)$ such that the following integrals are bounded:

$$N(g) = \sum_{j=0}^{2q} \int_{\mathbb{R}^{n-1}} (1 + | \xi |)^{p+2q-j} \| D_t^j Fg(t, \xi) \|_{L_\infty[t_1, t_N]} C_1(\xi) d\xi < \infty,$$

and

$$\int_{\mathbb{R}^{n-1}} \| D_t^j Fg(t, \xi) \|_{L_\infty[t_1, t_N]} C(\xi) d\xi < \infty, \quad j = 0, \ldots, 4q - 1.$$

The following result on the approximating power of the polysplines holds:

Theorem 5 *Let $f \in FL_{p,q}(D)$. Then for every $j = 0, 1, ..., 4q - 1$, and multi-index $\alpha \in \mathbb{Z}_+^{n-1}, | \alpha | \leq p$, the following inequality holds:*

$$\| D_y^\alpha D_t^j (f - \tilde{Q} f)(t, y) \|_{L_\infty[t_1, t_N]} \leq C \overline{\delta}^{4q-j} N(f).$$

Proof. Since $f \in FL_{p,q}(D)$, according to the definition of the operator $\tilde{Q}$ and the properties of the Fourier transform (cf. [12]) we have that $D_y^\alpha D_t^j \tilde{Q} f(t, y)$ makes sense and

$$\begin{aligned} I &= D_y^\alpha D_t^j f(t, y) - D_y^\alpha D_t^j \tilde{Q} f(t, y) \\ &= \int_{\mathbb{R}^{n-1}} (-i\xi)^\alpha e^{i\xi y} (D_t^j Ff(t, \xi) - D_t^j Q_\xi Ff(t, \xi)) d\xi < \infty. \end{aligned}$$

The estimate of Theorem 9.38 in [11] gives

$$\begin{aligned}
&\| I \|_{L_\infty[t_1,t_N]} \leq \\
&\leq C_1 \overline{\delta}^{4q-j} \int_{\mathbb{R}^{n-1}} (1+ \mid \xi \mid)^{|\alpha|} \| L_\xi F f(t,\xi) \|_{L_\infty[t_1,t_N]} C_1(\xi) d\xi \\
&\leq C \overline{\delta}^{4q-j} \sum_{j=0}^{2q} \int_{\mathbb{R}^{n-1}} (1+ \mid \xi \mid)^{p+2q-j} \| D_t^j F f(t,\xi) \|_{L_\infty[t_1,t_N]} C_1(\xi) d\xi \\
&= C \overline{\delta}^{4q-j} N(f) < \infty.
\end{aligned}$$

■

Acknowledgement 6 *The present research was supported by the Alexander von Humboldt Foundation. Thanks are due to Prof. L. Schumaker for the up to date references on L-splines. The discussions with Prof. K. Jetter on the subject of the paper were very stimulating for the author.*

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