# Peano kernels for mean value properties of polyharmonic functions

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**Abstract**

We analyze the Peano kernels arising from different mean value properties for polyharmonic functions in the ball and in the annulus. We show examples where the Peano kernel arising from a mean value property of Bramble-Payne is not positive. To the contrary, the mean value properties which we prove in the annulus generate Peano kernels which are positive. All such Peano kernels have compact support and are polysplines.

We will work in the space $R^d$, $d \geq 2$.

For a domain $D \subset R^d$, $d \geq 2$, for $f \in C(D)$, $x \in D$ and $\rho > 0$ such that the open ball $B_\rho(x) := \{y \in R^d \,:\, |x-y| < \rho\}$ is strictly contained in $D$, i. e. $\overline{B_\rho(x)} \subset D$, we denote by

$$M(f,x;\rho) \;:= \frac{1}{\sigma_d}\int_{S_{d-1}} f(x+\rho\xi)d\sigma_\xi \;= \frac{1}{\sigma_d \rho^{d-1}}\int_{S_{d-1}(\rho)} f(x+y)d\sigma(y)$$

the surface mean of $f$ over $\partial B_\rho(x)$. Here $d\sigma_\xi$ denotes the area element of the $(d-1)$–sphere $S_{d-1}$, $d\sigma(y)$ is the area element of the sphere $S_{d-1}(\rho)$, and $\sigma_d := 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ denotes the area of $S_{d-1}$. For $\rho = 0$ we have $M(f,x;0) = f(x)$.

Denote by $R_p$ the fundamental solution for the operator $\Delta^p$ in $R^d$, $p \geq 1$, which is given by (cf. Aronszajn-Creese-Lipkin [1])

$$R_p(x) = R_p(|x|) = r^{2p-d}(A_{p,d}\log(r) + B_{p,d}), \tag{1}$$

where $r := |x|$, and $A_{p,d}$ and $B_{p,d}$ are appropriate constants with $A_{p,d} \neq 0$ only for even $d$ and $p \geq \frac{d}{2}$.

The function $R_p$ has the property that

$$\begin{aligned} \Delta^k R_p &= R_{p-k} && (0 \leq k < p), && (2) \\ \Delta^p R_p(x) &= 0 && (|x| > 0,\ p \in N). && (3) \end{aligned}$$

For details we refer to [1].

We will work in the annulus $A(a,b)$ where $0 < a < b$, which is given by

$$A(a,b) = \left\{ x \in R^d : a < |x| < b \right\}.$$

First we prove the following

**Theorem 1.** Every function $h$ which is polyharmonic of order $p$ and radially symmetric in the annulus $A(a,b)$, $0 < a < b$, is uniquely representable in the form

$$h(x) = h(r) = \sum_{k=0}^{p-1} c_k r^{2k} + \sum_{k=1}^{p} \rho_k R_k(r) \qquad (r = |x|), \tag{4}$$

where $R_k$ is the fundamental solution for the operator $\Delta^k$ given above.

**Proof.** We will provide a proof whose elements will also be used later. Since the radial part of the Laplacian is

$$\begin{aligned} \Delta_r &= \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} \\ &= \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left( r^{d-1}\frac{\partial}{\partial r} u \right) \end{aligned}$$

the radial symmetry of $h$ implies

$$\Delta^p h(x) = \Delta_r^p h(r) = 0 \qquad (x \in A(a,b))$$

and the operator $\Delta_r^p$ is given by

$$\Delta_r^p = \left[ \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left( r^{d-1}\frac{\partial}{\partial r} \right) \right]^p \tag{5}$$

and has continuous coefficients in $A(a,b)$. Note that $0 < a \leq r$.

By the general theory of ordinary differential equations, the space

$$N := \left\{ f \in C^{2p}([a,b]) : \Delta_r^p f(r) = 0, \quad a \leq r \leq b \right\} \tag{6}$$

has dimension $2p$. We will show that the system of functions $\{u_1, ..., u_p, u_{p+1}, ..., u_{2p}\}$ where

$$\begin{aligned} u_k(r) &:= r^{2k-2} \qquad (k = 1, ..., p), \\ u_{p+k}(r) &:= R_k(r) \qquad (k = 1, ..., p) \end{aligned} \tag{7}$$

is a linearly independent system of solutions on the interval $[a, b]$.

Indeed, by (3) we see that all functions $u_j \in N$. Let us prove that they are linearly independent.

For odd $d$ that is evident since then by (1) we have

$$R_k(r) = B_{k,d} r^{2k-d} \qquad (k = 1, ..., p)$$

and the exponents $2k - d$ are odd, so they do not coincide with the exponents of the functions $r^{2k},\ k = 0, ..., p - 1$.

For even $d$ we have that the functions $\{u_{p+1}, ..., u_{2p}\} = \{R_1, ..., R_p\}$ belong to two groups, namely

$$\begin{aligned} &\left\{B_{k,d} r^{2k-d}\right\} && \left(k = 0, ..., \frac{d}{2} - 1\right), \\ &\left\{A_{k,d} r^{2k-d} \log(r) + B_{k,d} r^{2k-d}\right\} && \left(k = \frac{d}{2}, ..., p\right). \end{aligned}$$

Since in the second group the terms $r^{2k-d} B_{k,d},\ k = \frac{d}{2}, ..., p$ are multiples of some of the functions $\{u_1, ..., u_p\} = \left\{r^{2k},\ k = 0, ..., p - 1\right\}$, it follows that the linear independence of $\{u_1, ..., u_{2p}\}$ is equivalent to the linear independence of the sytem

$$\left\{1, r^2, ..., r^{2p-2};\ r^{-d+2}, ..., r^{-2};\ \log r, r^2 \log r, ..., r^{2p-d} \log r\right\}.$$

We make the change of variables

$$v = 2 \log r,$$

i.e., $r = e^{\frac{v}{2}}$, and obtain the equivalent system of functions in the variable $v$ :

$$\left\{1, e^v, ..., e^{(p-1)v};\ e^{-\frac{d}{2}v}, e^{\left(-\frac{d}{2}+1\right)v} ..., e^{-v};\ v, ve^v, ..., ve^{\left(p-\frac{d}{2}\right)v}\right\}.$$

The last system is well-known to be linearly independent from the theory of linear ordinary differential equations, [8]. Hence, the functions $\{u_1, ..., u_{2p}\}$ are linearly independent as well.

Now we have to justify the use of radially symmetric polyharmonic functions only.

**Theorem 2.** Let $h$ be polyharmonic of order $p$ in the annulus $A(a,b)$. Then its symmetrization $\mathcal{M}(x) = \mathcal{M}(r) := M(h,0;|x|) = M(h,0;r)$ is polyharmonic of order $p$ and satisfies

$$\Delta^p \mathcal{M}(x) = \Delta_r^p \mathcal{M}(r) = 0 \qquad (x \in A(a,b)).$$

**Proof.** From John [5, end of Chapter 4, p. 88] it follows that

$$\Delta_x M(h,x;r) = M(\Delta h,x;r) = \Delta_r M(h,x;r), \tag{8}$$

hence

$$M(\Delta^p h,x;r) = \Delta_r^p M(h,x;r).$$

Now put $x=0$ in order to end the proof.

Combining Theorem 1 and Theorem 2 we get

**Theorem 3.** For every function $h$ which is polyharmonic of order $p$ in the annulus $A(a,b)$, we have the following equality

$$M(h,0;r) = \sum_{k=0}^{p-1} c_k r^{2k} + \sum_{k=1}^{p} \rho_k R_k(r).$$

for some appropriate constants $c_k, \rho_k$.

Indeed, by Theorem 2 we get that $M(h,0;r)$ satisfies $\Delta_r^p M(h,0;r) = 0$, and, hence, by Theorem 1 it has the representation (4).

Before stating our main result we introduce the notion of *generalized divided differences*, cf. Karlin-Studden [6], Schumaker [9].

In the proof of Theorem 1 we have seen that the system of functions $\{u_k\}_{k=1}^{2p}$ is a solution in $[a,b]$ to the following ordinary differential equation

$$\begin{aligned}\Delta_r^p f &= \left[\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right)\right]^p f \\ &= \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right)\cdots\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right) f = 0,\end{aligned}$$

which is equivalent to

$$Lf(r) := r^{d-1}\Delta_r^p f(r) = 0.$$

The operator $L$ has the form (see [6, p. 19])

$$\begin{aligned} L &= r^{d-1}\Delta_r^p \\ &= D_{2p}...D_1D_0 \\ &= \frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right)...\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right) \end{aligned}$$

where (in the notations of Karlin-Studden [6, p. 19]) we have

$$\begin{aligned} D_0 f &= f, \\ D_j f(r) &= \frac{\partial}{\partial r}\frac{f(r)}{w_j(r)} \qquad (j \geq 1) \end{aligned}$$

with the weights given by

$$\begin{aligned} w_1 &= 1; \\ w_3 &= ... = w_{2p-1} = r^{d-1}; \\ w_2 &= w_4 = ... = w_{2p} = \frac{1}{r^{d-1}}. \end{aligned}$$

Consequently, the system $\{u_k\}_{k=1}^{2p}$ is an *extended complete Chebyshev system (ECT)* following the terminology in [6, p. 379, Theorem 11.1.2].

It is easy to see that the operator $L$ is self-adjoint. Indeed, take the operators (see Schumaker [9, p. 373, (9.27)])

$$\begin{aligned} D_0^* f &= f \\ D_i^* f(r) &= \frac{1}{w_{2p-i+1}}\frac{\partial}{\partial r}f(r) \qquad i = 1, ..., 2p \end{aligned}$$

then the adjoint operator is (see Schumaker [9, p. 372, (9.26)])

$$L^* = D_{2p}^* \cdots D_1^* D_0^*$$

which is obviously identical with $L$. It follows that the dual system (see [9, p. 372, (9.25)]) coincides with the system $\{u_k\}_{k=1}^{2p}$ .

Let the points $a < r_1 < ... < r_{2p+1} < b$ be given. For an arbitrary function $f$ in $[a, b]$ we define the determinant

$$D\begin{pmatrix} r_1, ..., r_{2p+1} \\ u_1, ..., u_{2p}, f \end{pmatrix} = \det\begin{pmatrix} u_1(r_1) & ... & u_{2p}(r_1) & f(r_1) \\ ... & ... & ... & ... \\ u_1(r_{2p+1}) & ... & u_{2p}(r_{2p+1}) & f(r_{2p+1}) \end{pmatrix}.$$

Now we extend the system $\{u_k\}_{k=1}^{2p}$ by putting $u_{2p+1}(r) = r^{2p}$. The obtained system $\{u_k\}_{k=1}^{2p+1}$ will be again ECT, since if we put $w_{2p+1} = \frac{1}{r^{d-1}}$, we have the differential operator

$$\frac{\partial}{\partial r}\frac{1}{r^{d-1}}L = \frac{\partial}{\partial r}\Delta_r^p$$

and $u_{2p+1}(r) = r^{2p}$ satisfies

$$\frac{\partial}{\partial r}\frac{1}{r^{d-1}}Lr^{2p} = \frac{\partial}{\partial r}\Delta_r^p r^{2p} = \frac{\partial}{\partial r}C_{p,d} = 0$$

since $\Delta_r^p r^{2p} = C_{p,d}$ for come constant $C_{p,d}$. The linear independence of the system $\{u_k\}_{k=1}^{2p+1}$ follows like in Theorem 1.

Following Schumaker [9, p. 368], we put for the *generalized divided difference* of a function $f$ with respect to the system of functions $\{u_k\}_{k=1}^{2p}$ :

$$[r_1, ..., r_{2p+1}]\, f = \frac{D\begin{pmatrix} r_1, ..., r_{2p+1} \\ u_1, ..., u_{2p}, f \end{pmatrix}}{D\begin{pmatrix} r_1, ..., r_{2p+1} \\ u_1, ..., u_{2p}, u_{2p+1} \end{pmatrix}} =: \sum_{k=1}^{2p+1} \mu_k f(r_j)\,.$$

Now we prove the following

**Theorem 4.** Let the points $0 < a < r_1 < ... < r_{2p+1} < b$ be given. Then every function $h$ which is polyharmonic in the annulus $A(a, b)$ satisfies the mean value property

$$\mathcal{L}(h) := \sum_{j=1}^{2p+1} \mu_j M(h, 0; r_j) = 0.$$

The Peano kernel $K$ of the functional $\mathcal{L}$ (which exists by [3]) is positive for $x \in A(a, b)$ and is zero for $x \notin A(a, b)$.

**Proof.** By Theorem 3 we have that the function $u(r) = M(h, 0; r)$ satisfies

$$\Delta_r^p M(h, 0; r) = 0 \qquad (a \le r \le b)\,,$$

i.e. $u \in N$, and by Theorem 1 is a linear combination of the system $\{u_k\}_{k=1}^{2p}$. By Theorem 9.7 in Schumaker [9] we have that

$$[r_1, ..., r_{2p+1}]\, u = \sum_{j=1}^{2p+1} \mu_j M(h, 0; r_j) = 0.$$

According to [3, Theorem 1] for the Peano kernel of the linear functional $\mathcal{L}$ which vanishes on polyharmonic functions of order $p$ we have that for an arbitrary function $f \in C^{2p}\left(\overline{A\left(a,b\right)}\right)$ the following equality holds

$$\mathcal{L}\left(f\right) = \sum_{j=1}^{2p+1} \mu_j M\left(f, 0; r_j\right) = \int\limits_{\overline{A(a,b)}} K\left(x\right) \Delta^p f\left(x\right) dx,$$

where

$$K\left(x\right) = -\Omega_d \left[r_1, ..., r_{2p+1}\right]_y R_p\left(x-y\right),$$

with

$$\Omega_d = \begin{cases} \frac{1}{\sigma_2} & (d=2) \\ \frac{1}{(d-2)\sigma_d} & (d \geq 3). \end{cases}$$

Now we turn to the one-dimensional functional

$$l\left(h\right) := \sum_{j=1}^{2p+1} \mu_j h\left(r_j\right)$$

which is zero for the solutions of the ordinary differential equation

$$Lh\left(r\right) = 0 \qquad\qquad (a < r < b).$$

The Peano kernel of $l$ is given by, cf. Schumaker [9, Theorem 9.24],

$$\begin{aligned} \sum_{j=1}^{2p+1} \mu_j h\left(r_j\right) &= \int\limits_a^b K_1\left(r\right) Lh\left(r\right) dr \\ &= \int\limits_a^b K_1\left(r\right) r^{d-1} \Delta_r^p h\left(r\right) dr. \end{aligned}$$

Note that $L = L^*$, i.e. $L$ is self-adjoint as noted above.

Now what concerns the multivariate kernel $K$, since the spherical mean $M\left(R_p\left(x-\cdot\right), 0; r\right)$ is radially symmetric as a function of $x$ (which is easy to prove like in Haußmann-Kounchev [4, Lemma 4]) we see that $K\left(x\right)$ is radially symmetric as well.

On the other hand using equation (8) we obtain

$$\begin{aligned}
&\int\limits_{A(a,b)} K(x)\,\Delta^{p} f(x)\,dx \\
&= \int\limits_{a}^{b}\int\limits_{S^{d-1}} r^{d-1} K(r)\,\Delta^{p} f(r\xi)\,dr d\sigma_{\xi} \\
&= \int\limits_{a}^{b} \sigma_{d} M\left(r^{d-1} K(r)\,\Delta^{p} f(r\xi);0;r\right) dr \\
&= \sigma_{d} \int\limits_{a}^{b} r^{d-1} K(r)\,M\left(\Delta^{p} f(r\xi);0;r\right) dr \\
&= \sigma_{d} \int\limits_{a}^{b} r^{d-1} K(r)\,\Delta_{r}^{p} M(f;0;r)\,dr
\end{aligned}$$

The last expression is precisely the remainder of the Peano kernel in the one-dimensional case, i.e. we have that $K_1(r) = \sigma_d K(r)$. According to Theorem 9.22 in [9, p. 380], the kernel $K_1$ is positive inside $(a,b)$ and zero outside . Hence, $K$ is positive inside $A(a,b)$ and vanishes outside. The proof is finished.

**Remark 1.** Let us remark that the kernel $K$ is polyspline in the sense of [7]. Indeed, due to the fact that $K_1$ is the Peano kernel for the operator $L$, which has order $2p$, it follows that $K_1$ is a spline and has continuous derivatives up to order $2p-2$, i.e.

$$\left(\frac{\partial}{\partial r}\right)^{j} K_1 \in C([a,b]) \qquad (j = 0, ..., 2p-2).$$

On the other hand, by the definition of a polyspline, the kernel $K$ has to satisfy

$$\begin{aligned}
\Delta^{j} K &\in C\left(R^{d}\right) \qquad (j = 0, ..., p-1), \\
\frac{\partial}{\partial r}\Delta^{j} K &\in C\left(R^{d}\right) \qquad (j = 0, ..., p-2);
\end{aligned}$$

here we use the fact that $\frac{\partial}{\partial r}$ is the normal derivative to the spheres $S\left(0;r_k\right)$. Due to the radial symmetry of $K\left(x\right)$ we obtain that the above is equivalent to the following

$$\begin{aligned} \Delta^j K &= \left[\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right)\right]^j K\in C\left(R^d\right) \qquad \left(j=0,...,p-1\right), \\ \frac{\partial}{\partial n}\Delta^j K &= \frac{\partial}{\partial r}\left[\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right)\right]^j K\in C\left(R^d\right) \qquad \left(j=0,...,p-2\right). \end{aligned}$$

Since $a>0$ the last implies inductively that

$$\left(\frac{\partial}{\partial r}\right)^j K\in C\left(R^d\right) \qquad \left(j=0,...,2p-2\right).$$

But that is obviously true by the above since $K_1\left(r\right)=\sigma_d K\left(r\right)$.

**Remark 2.** In [3] we considered mean value properties for the polyharmonic functions in a ball, i.e. $a=0$, given by

$$h\left(0\right)=\sum_{j=1}^{p}\kappa_j M\left(h,0;r_j\right)$$

where $0<r_1<...<r_p$. They arise from a mean-value property of Bramble-Payne, [2].

The system of functions which are polyharmonic and radially symmetric in the ball are $u_k=r^{2k}$, $k=0,...,p-1$. They are solutions to the following ordinary differential operator

$$L=D_p\cdots D_1$$

where

$$\begin{aligned} D_1 &= \frac{\partial}{\partial r} \\ D_j &= \frac{\partial}{\partial r}\frac{1}{r} \qquad j\geq 2. \end{aligned}$$

So far, the coefficients $\frac{1}{r}$ are singular at 0. The Peano kernel will be given by

$$\mu\left(h\right)=h\left(0\right)-\sum_{j=1}^{p}\kappa_j h\left(r_j\right)=\int K\left(x\right)\Delta^p h\left(x\right).$$

But the Peano kernel has a definite sign.

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