Positivity of the Peano Kernel for the ball

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Abstract

A main corollary of our results is that if a function f is zero on p concentric spheres with center at 0 and $(-1)^p \Delta^p f > 0$ inside the largest sphere, then $f(0) > 0$. Thus we generalize the characteristic property of subharmonic functions, the case $p = 1$. The proofs are related to the positivity (or negativity) of the Peano Kernel for the functional arising from the Mean Value Property of Picone and Bramble-Payne for polyharmonic functions in the ball.

1 Notations.

In [13] Nicolescu generalized the Gauss mean value property for polyharmonic functions of order p in a bounded domain D. Led by this he introduced in a natural way the notion of *subharmonic function* of order p. He proved that they satisfy the following property

$$(-1)^p \Delta^p u(x) \leq 0 \qquad (x \in D).$$

He proved also a generalization of the M. Riesz property. Namely, he proved that if

$$\Delta^j u(x) = \Delta^j U(x) \qquad (x \in \partial D,\ j = 0, ..., p-1)$$

where the function U satisfies $\Delta^p U(x),\ x \in D$, then

$$u(x) \leq U(x) \qquad (x \in D).$$

This property justifies the usage of the term "subharmonic" of order p.

On the other hand the functions U polyharmonic of order p enjoy interesting qualitative properties related to the set where they are zero. In particular, let

$$U(x) = 0 \qquad |x| = r_1, ..., r_p$$

where the constants $0 < r_1 < ... < r_p$. Let the spheres $S(0, r_j)$, $j = 1, ..., p$ be all contained in D and the complement of D in $\mathbb{R}^n$ be connected. Then it follows that

$$U \equiv 0.$$

In the present paper we generalize the property of M. Riesz of the subharmonic function in another direction. We prove that if

$$u(x) = U(x) \qquad |x| = r_1, ..., r_p$$

then it follows that

$$u(0) \leq U(0).$$

We will work in the space $\mathbb{R}^d$, $d \geq 2$.For a domain $D \subset \mathbb{R}^d$, $d \geq 2$, for $f \in C(D)$, $x \in D$ and $\rho > 0$ such that the open ball $B(x;\rho) := \{y \in \mathbb{R}^d : |x - y| < \rho\}$ is strictly contained in D, i. e. $\overline{B(x;\rho)} \subset D$, we denote by

$$M(f, x; \rho) := \frac{1}{\sigma_d} \int_{S_{d-1}} f(x + \rho\xi) d\sigma_\xi = \frac{1}{\sigma_d \rho^{d-1}} \int_{S_{d-1}(\rho)} f(x + y) d\sigma(y) \quad (1)$$

the surface mean of f over $\partial B(x;\rho)$. Here $d\sigma_\xi$ denotes the area element of the $(d-1)$–sphere S_{d-1}, $d\sigma(y)$ is the area element of the sphere $S_{d-1}(\rho)$, and $\sigma_d := 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ denotes the area of S_{d-1}. For $\rho = 0$ we have $M(f, x; 0) = f(x)$.

Denote by R_p the fundamental solution for the operator Δ^p in $\mathbb{R}^d$, $p \geq 1$, which is given by (cf. Aronszajn-Creese-Lipkin [1, p. 8])

$$R_p(x) = R_p(|x|) = r^{2p-d}(A_{p,d} \log(r) + B_{p,d}), \quad (2)$$

where $r := |x|$, and $A_{p,d}$ and $B_{p,d}$ are appropriate constants with $A_{p,d} \neq 0$ only for even d and $p \geq \frac{d}{2}$.

The function R_p has the property that

$$\Delta^k R_p = R_{p-k} \qquad (0 \leq k < p), \quad (3)$$

$$\Delta^p R_p(x) = 0 \qquad (|x| > 0, \ p \in N). \quad (4)$$

For details we refer to [1].

We define the space of polyharmonic functions of order p in D by putting

$$H_p(D) = \{f : \Delta^p f(x) = 0, \ x \in D\}.$$

2 Preliminaries

In the present paper we consider some special functionals which vanish on the space $H_p,\ p \geq 1$, namely, those which arise from the mean value property considered by Picone [14], and later by Bramble-Payne [2]. The mean value property of Picone-Bramble-Payne reads as follows:

Theorem 1 *Let the constants* $r_{1,...,}r_p$ *be given such that*

$$0 < r_p < ... < r_1.$$

Then there exist coefficients $\lambda_j, j = 1, ..., p$, *such that the following equality holds*

$$h(0) = \sum_{j=1}^{p} \lambda_j M(h, 0; r_j) \tag{5}$$

for every function $h \in H_p(B(0; r_1)) \cap C\left(\overline{B(0; r_1)}\right)$.

We have already discussed in [8] the analogy of the mean value property (5) with the one-dimensional divided difference operator of order p, since the last might be treated as a mean value property for the polynomials of degree p.

Furtheron, it is a beautiful result of the one-dimensional spline-theory that the so-called *Peano kernel* of the divided difference operator of order p is a compact spline of degree p (called $B-$spline) which is strictly positive inside the interval of the finite difference operator (cf. [3], [16, Chapter 4 and Theorem 4.17]). The main purpose of the present paper is to prove analogous result in the multivariate case. We will show that the *Peano kernel* generated by the mean value property (5) is compact and strictly positive in the ball $B(0, r_1)$. It is a *polyspline* in the sense of [9].

From the point of view of the technics used in the present paper, we have to say that the results are based on accurate study of the properties of spherical means of R_p, namely, $M(R_p(x - \cdot), 0; \rho),\ x \in \mathbb{R}^d$, which are in fact *simple layer polyharmonic potentials*. The last are of basic importance for the theory of polysplines (see [9], [10]) and deserve special interest. The polyharmonic potentials do not seem to be treated from the point of view of our research in other sources. In order to make the exposition closed and transparent we will provide their basic properties below in Lemmata 7, 8, 9.

For general properties of the simple layer Newtonian potentials (when $p = 1$) we will rely upon the reference [12, Chapter 18, Sec. 6.].

We will need the following theorem of *Pizzetti* (see [13]):

Theorem 2 *For every ball $B(x;h)$, $h > 0$, and every $f \in H_p(B(0;r_1)) \cap C\left(\overline{B(0;r_1)}\right)$ the following equation holds*

$$M(f,x;\rho) = f(x) + \sum_{j=1}^{p-1} a_{dj} h^{2j} \Delta^j f(x)$$

where the constants are given by

$$a_{dp} := \frac{1}{2^p p! d(d+2)\dots(d+2p-2)} \qquad (p \geq 1,\ d \geq 2)$$

$$a_{d0} := 1.$$

In particular, for $d = 2$ we have

$$a_{2p} = \frac{1}{4^p (p!)^2}.$$

3 Main result

As said above the main purpose of the present paper is to investigate the positivity (negativity) of the *Peano kernel* of the functional generated by the mean value property (5), namely

$$\ell(f) = f(0) - \sum_{j=1}^{p} \lambda_j M(f,0;r_j).$$

We have the following

Theorem 3 *For every $f \in C^{2p}$ on an open neighbourhood of $\overline{B(0;r_1)}$ the following representation holds true*

$$\ell(f) = \int\limits_{B(0;r_1)} K_p(x) \Delta^p f(x)\, dx. \qquad (6)$$

The function $K_p(x)$ *is called* Peano kernel *for the functional* ℓ *and is given by*

$$K_p(x) = -\Omega_d \cdot \ell_y(R_p(x-y)) \tag{7}$$

$$= -\Omega_d \cdot \left(R_p(x) - \sum_{j=1}^{p} \lambda_j M(R_p(x-\cdot), 0; r_j) \right)$$

where the constant $\Omega_d = 1/(d-2)\sigma_d$ *for* $d \geq 3$ *and* $\Omega_2 = 1/\sigma_2$. *The subscript* y *indicates that the functional* ℓ *is applied with respect to the variable* y.

Remark 4 *1. In view of the extension theorems we may assume that* $f \in C^{2p}\left(\overline{B(0;r_1)}\right)$.

2. In [6], [7], [8] we considered a general class of functionals for which the Peano kernel has a representation as (6), (7).

Now we can state the main result of the present paper:

Theorem 5 *The Peano kernel* K_p *obtained in Theorem 3 satisfies*

$$(-1)^p K_p(x) > 0 \qquad (x \in B(0;r_1)).$$

For the proof of this Theorem we will investigate the spherical means of the fundamental functions $R_p(x-y)$. Their explicit expressions which are provided below were first obtained in [6], [8].

As a an important consequence of Theorem 5 we obtain an interesting result which generalizes the characteristic property for subharmonic functions [5]:

Corollary 6 *Let the function* f *satisfy* $f(x) = 0$, *for* $|x| = r_j$, $j = 1, ..., p$. *Let* $(-1)^p \Delta^p f > 0$ *for* $|x| < r_1$. *Then* $f(0) > 0$.

This result has to be considered in the framework of the uniqueness properties of polyharmonic functions, see e.g. [4].

4 Lemmata on polyharmonic potentials and proof of Theorem 3

Before proving Theorem 3 we will provide a technical Lemma which will be used throughout the paper

Lemma 7 *Let D be a bounded domain and the multi-index α satisfies $|\alpha| \leq 2p-1$. Then for every $g \in C\left(\overline{D}\right)$ we have that the integrals below are finite and*

(i)

$$\int_D \left(\frac{1}{\sigma_d} \int_{S_{d-1}} D_x^\alpha R_p \left(x - \rho\xi \right) d\sigma_\xi \right) g\left(x \right) dx$$
$$= \frac{1}{\sigma_d} \int_{S_{d-1}} \left(D_x^\alpha \int_D R_p \left(x - \rho\xi \right) g\left(x \right) dx \right) d\sigma_\xi < \infty$$

(ii) For $|\alpha| \leq 2p-2$ and $g \in C\left(S_{d-1}\left(0;\rho \right) \right)$ the integral

$$\frac{1}{\sigma_d} \int_{S_{d-1}} D_x^\alpha R_p \left(x - \rho\xi \right) g\left(\rho\xi \right) d\sigma_\xi$$

is finite.

Proof. For the fundamental solution R_p we obtain by direct differentiation from (2) the following inequalities

$$\left| D^\alpha R_p \left(x \right) \right| \leq C_1 \left| x \right|^{2p-d-|\alpha|} \cdot \left(\left| \log |x| \right| + C_2 \right) \qquad \left(x \in \mathbb{R}^d \setminus \{0\} \right) \tag{8}$$

where the log term appears only in the even dimensions d and for $p \geq d/2$. In order to be able to apply the theorem of Fubini and the dominated convergence theorem of Lebesque the following inequalities are sufficient

$$
\begin{aligned}
I &= \left| \frac{1}{\sigma_d} \int_{S_{d-1}} \left(\int_D D_x^\alpha R_p\left(x-\rho\xi\right) g\left(x\right) dx \right) d\sigma_\xi \right| \\
&\leq C_3 \frac{1}{\sigma_d} \int\limits_{S_{d-1}} \int\limits_{D} \left(C_1 \left|x-\rho\xi\right|^{2p-d-|\alpha|} \cdot \left(\left|\log\left|x-\rho\xi\right|\right| + C_2\right)\right) dx d\sigma_\xi \\
&\leq C_4 \frac{1}{\sigma_d} \int\limits_{S_{d-1}} \int\limits_{B(0;\rho+\gamma)} \left(C_1 \left|z\right|^{2p-d-|\alpha|} \cdot \left(\left|\log\left|z\right|\right| + C_2\right)\right) dz d\sigma_\xi \\
&= C_4 \int\limits_{B(0;\rho+\gamma)} \left(C_1 \left|z\right|^{2p-d-|\alpha|} \cdot \left(\left|\log\left|z\right|\right| + C_2\right)\right) dz
\end{aligned}
$$

where we have put $z = x - \rho\xi$ and $\gamma = diam\left(D\right)$. Passing to spherical coordiantes in z we obtain further

$$
I \leq C_5 \int\limits_0^{\rho+\gamma} \left(C_1 r^{2p-1-|\alpha|} \cdot \left(\left|\log r\right| + C_2\right)\right) dr
$$

which is finite since $2p-1-|\alpha| > -1$ by the condition of the Theorem.

The second equality is proved by a similar argument.

The finiteness of the third integral follows by (8) and by passing to local coordinates on the sphere S_{d-1}.

■

Proof. of Theorem 3. We may assume that the function f is C^{2p} on $B\left(0;r_1+2\delta\right)$ for some $\delta > 0$. For every function $f \in C^{2p}\left(\overline{B\left(0;r_1+\delta\right)}\right)$ we have the representation (cf. [1, p. 10])

$$
f\left(y\right) = h_p\left(y\right) - \Omega_d \cdot \int\limits_{B(0;r_1+\delta)} R_p\left(x-y\right) \Delta^p f\left(x\right) dx \qquad \left(y \in B\left(0;r_1\right)\right) \quad (9)
$$

where

$$
h_p\left(y\right) = \Omega_d \cdot \sum_{l=0}^{p-1} \int\limits_{S(0;r_1+\delta)} \left(\Delta^l f\left(x\right) \frac{\partial R_{l+1}\left(x-y\right)}{\partial n_x} - R_{l+1}\left(x-y\right) \frac{\partial \Delta^l f\left(x\right)}{\partial n} \right) d\sigma\left(x\right) \quad (10)
$$

The function h_p is polyharmonic in $B(0; r_1+\delta)$. Indeed, since $f \in C^{2p}\left(\overline{B(0; r_1+\delta)}\right)$ and $S(0; r_1+\delta)$ is a compact set we see in a standard way that we may differentiate under the sign of the integrals, and by (4) it follows that $\Delta^p h_p(y) = 0,\ y \in B(0; r_1+\delta)$, i.e. $h_p \in H_p(B(0; r_1))$. By the real analyticity of the polyharmonic functions in the interior it follows that $h_p \in C\left(\left(\overline{B(0; r_1)}\right)\right)$, and, consequently, $\ell(h_p) = 0$. We apply the functional ℓ to both sides of (9). We are in the state to apply Lemma 7, (i), and to interchange the two integrals, that of (9) and the one of (1), since $|\Delta^p f(x)|$ is bounded. This ends the proof.

■

For spherically symmetric functions f we will consider the spherical part of the Laplacian operator

$$\Delta_r = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} = \frac{1}{r^{d-1}}\frac{\partial}{\partial r} r^{d-1}\frac{\partial}{\partial r} \tag{11}$$

for which

$$\Delta f = \Delta_r f.$$

For the proof of Theorem 5 we will need basic properties of spherical means of translates of the functions R_p.

Lemma 8 *Let $\tau > 0$. For $p \geq 1$ the mean value integral*

$$I_p(y,\tau) := \mu(R_p, y; \tau) = \frac{1}{\sigma_d}\int_{S_{d-1}} R_p(\tau\xi - y)\, d\sigma_\xi$$

exists and is continuous. It is a spherically symmetric function of y, and has the following properties:

(i) $\Delta^p I_p(r,\tau) = 0$ for $|y| = r \neq \tau$;

(ii) $\Delta^k I_p(r,\tau) = I_{p-k}(r,\tau)$ for every r and for $0 \leq k \leq p-1$;

(iii) The functions $\Delta_r^k I_p(r,\tau)$, $k = 0, ..., p-1$, $\frac{\partial}{\partial r}\Delta_r^k I_p(r,\tau)$, $k = 0, ..., p-2$, $\frac{\partial^j}{\partial r^j} I_p(r,\tau)$, $j = 0, ..., 2p-2$, are continuous in the variable r for every $r \geq 0$.

Proof. The existence of the integrals follows from Lemma 7, (ii). The continuity of $I_1(y;\tau)$ is in [12, Theorem 18.6.1].

The spherical symmetry follows since $R_p(x) = R_p(|x|),\ x \in \mathbb{R}^d \backslash \{0\}$. Let $|y| = |y_1|$ and A be an orthogonal transformation such that $Ay_1 = y$.

Then we have $d\sigma_\xi = d\sigma_\eta$, where $A\eta = \xi$, and $|\tau\xi - y| = |\tau A\eta - Ay_1| = |A(\tau\eta - y_1)| = |\tau\eta - y_1|$. Hence $I_p(y_1, \tau) := I_p(y, \tau)$.

Property (i) follows from (4), $\Delta^p_{(y)} R_p(x-y) = 0$ for $x \neq y$.

What concerns (ii), the possibility to differentiate under the sign follows from Lemma 7, (ii), and by (3). In order to avoid direct computations, we refer to the local regularity theorem for solutions of elliptic equations, cf. e.g. [11, p. 125] through which the continuity of the integrals $I_p\,(y;\tau)\,,\ p \geq 2$, in y follows from the continuity of $I_1\,(y;\tau)$ and (ii).

Property (iii) follows directly from Lemma 7, (ii) and 11.

■

We can write the function I_p in an explicit way:

Lemma 9 *The integrals $I_p(y,\tau)$ satisfy (for $\tau > 0,\ p \geq 1$)*

$$I_p(y,\tau) \quad = \begin{cases} \sum\limits_{k=0}^{p-1} d_{pk}\,|y|^{2k} & (|y| \leq \tau) \\ \sum\limits_{k=0}^{p-1} a_{dk}\tau^{2k} R_{p-k}\,(|y|) & (|y| \geq \tau) \end{cases} \tag{12}$$

where

$$\begin{aligned} d_{pk} = d_{pk}\,(\tau) &:= \frac{1}{\Delta^k |y|^{2k}} R_{p-k}(\tau) \\ &= \frac{\Gamma(\frac{d}{2})}{2^{2k}\Gamma(k+1)\Gamma(k+\frac{d}{2})} R_{p-k}(\tau), \end{aligned}$$

for $k = 0, \ldots, p$, and the coefficients a_{dk} are those of Theorem 2.

Proof. Since $\Delta^p I_p(y,\tau) = 0$ for $|y| < \tau$ (see Lemma 8,(i)), and $I_p(y,\tau)$ is spherically symmetric, we shall see that $I_p(y,\tau)$ is a polynomial in $|y|^2$, i.e.

$$I_p(y,\tau) = \sum_{k=0}^{p-1} d_{pk}|y|^{2k} \qquad (|y| < \tau)\,. \tag{13}$$

In order to prove (13), we proceed by induction: For $p = 1$, I_1 is a harmonic function by Lemma 8, (i). Since I_1 is spherically symmetric, it is constant by the maximum principle.

Now assume that the statement is true for $p-1$. Since $\Delta I_p(y) = I_{p-1}(y)$, by induction hypothesis we have

$$\Delta I_p(y) = \sum_{k=0}^{p-2} c_k |y|^{2k} \qquad (c_k \in \mathbb{R}).$$

Since $\Delta |y|^{2l} = (2l)(2l-2+d)|y|^{2l-2}$, we get that the function

$$\phi(y) := \sum_{k=1}^{p-1} \frac{c_{k-1}}{2k(2k-2+d)} |y|^{2k}$$

satisfies $\Delta\phi = \Delta I_p$. Hence $\Delta(\phi - I_p) = 0$. Since $\phi - I_p$ is spherically symmetric, it follows that $\phi - I_p$ is equal to a constant, hence I_p is of the desired form.

Due to the continuity of $I_p(y,\tau)$ the formula holds for $|y| = \tau$ as well. In order to compute the constants d_{pk} consider

$$\Delta^k \Big(\sum_{l=0}^{p-1} d_{pl} |y|^{2l} \Big)\Big|_{y=0} = d_{pk} \Delta^k (|y|^{2k}) = I_{p-k}(0,\tau)$$

$$= \frac{1}{\sigma_d} \int_{S_{d-1}} R_{p-k}(\tau\xi) d\sigma_\xi = R_{p-k}(\tau) \cdot \frac{1}{\sigma_d} \int_{S_{d-1}} d\sigma_\xi = R_{p-k}(\tau).$$

The explicit value of d_{pk} follows from [1, p. 2].

For $|y| \geq \tau$ we compute $I_p(y,\tau)$ by the Pizzetti formula in Theorem 2.

■

We have finally

Theorem 10 *The Peano kernel K_p is spherically symmetric, i.e. $K_p(y) = K_p(r)$ for $r = |y|$, has a compact support, supp $(K_p) \subset \overline{B(0;r_1)}$, and satisfies*

(i) $\Delta^p K_p(y) = \Delta_r^p K_p(r) = 0$ *for* $r \neq 0,\ r \neq r_j,\ j = 0, ..., p.$

(ii) The functions $\Delta_r^k I_p(r,\tau)$, $k = 0, ..., p-1$, $\frac{\partial}{\partial r}\Delta_r^k I_p(r,\tau)$, $k = 0, ..., p-2$, $\frac{d^j}{dr^j} K_p(r)$, $j = 0, ..., 2p-2$ *are continuous for every* $r > 0$.

Proof. For the compactness of *supp* (K_p) we use the fact that for $|x| > r_1$ the function $R_p(x-y)$ as a function of y is polyharmonic of order p in $\overline{B(0;r_1)}$. Hence, it follows that $K_p(x) = 0$ for $|x| > r_1$ directly from Theorem 1. The rest follows easily from Lemma 8.

■

5 Positivity (negativity) of the Peano kernel

Now we are ready to prove the positivity (negativity) of the Peano kernel K_p.

Proof. of Theorem 5.

1. First we assume that $p = 1$. In this case we have only one interval $(0, r_1)$ and by Lemma 9 we obtain the representation

$$K_1(r) = -\Omega_d (R_1(r) + \lambda_1 v_0) \qquad r \in [0, r_1)$$

for some appropriate constant v_0. By Theorem 10 we have that $K_1(r_1) = 0$. It follows that the function $R_1(r) + \lambda_1 v_0$ may not have other zero in $[0, r_1)$ since for arbitrary $a, b > 0$

$$R_1(a) = R_1(b) \Leftrightarrow a = b.$$

2. Further, let $p \geq 2$. Let us assume that $K_p(r)$ has a zero $\alpha_0 \in [0, r_1)$.
Due to the representation

$$K_p(r) = -\Omega_d \left(R_p(r) + \sum_{j=1}^{p} \lambda_j I_p(r; r_j) \right) \tag{14}$$

and Lemma 9 we see that for $r \in [0, r_p)$

$$K_p(r) = -\Omega_d \left(R_p(r) + \sum_{k=0}^{p-1} v_k r^{2k} \right) \tag{15}$$

for some appropriate constants v_k.

On the other hand by Theorem 10 we have that

$$\frac{d^j}{dr^j} K_p(r_1) = 0 \qquad (j = 0, 1, ..., 2p-2);$$
$$\frac{d^j}{dr^j} K_p(r) \in C(0, \infty) \qquad (j = 0, 1, ..., 2p-2).$$

By the continuous differentiability of K_p it follows immediately that there is a point $\alpha_1 \in (\alpha_0, r_1)$ such that

$$\frac{d}{dr} K_p(\alpha_1) = 0.$$

Now we consider the function

$$u(r) = r^{d-1} \frac{d}{dr} K_p(r).$$

It satisfes

$$u(r_1) = u(\alpha_1) = u(0) = 0$$

where the last follows by representation (15). Indeed, for $0 \leq r \leq r_p$ we have

$$r^{d-1} \frac{d}{dr} K_p(r) = -\Omega_d r^{d-1} \left(\frac{d}{dr} R_p(r) + \sum_{k=1}^{p-1} v_k' r^{2k-1} \right)$$

for appropriate constants v_k' and by formula (2) we see that

$$r^{d-1} \frac{d}{dr} R_p(r) \Big|_{r|=0} = 0 \qquad\qquad (p \geq 2).$$

Hence, follows the above $u(0) = 0$.

Using further the differentiability of K_p we obtain that there exist *two different* points $\alpha_2, \alpha_3 \in (0, r_1)$ such that

$$\frac{d}{dr} u(\alpha_2) = \frac{d}{dr} u(\alpha_3) = 0.$$

By (11) they are evidently zeros of the function

$$\frac{1}{r^{d-1}} \frac{d}{dr} u(r) = \Delta_r K_p = \Delta K_p.$$

In order to continue inductively, we note that due to (14) and Lemma 9 we have

$$\Delta_r K_p(r) = -\Omega_d \left(R_{p-1}(r) + \sum_{j=1}^{p} \lambda_j I_{p-1}(r; r_j) \right).$$

3. Applying once again the above arguments we obtain that the function $\Delta K_p(r)$ has *three different* zeros in $(0, r_1)$, etc. Finally, for the function

$$\Delta^{p-1} K_p(r)$$

we obtain that it has p different zeros in the interval $(0, r_1)$. By formula (14) we have

$$\Delta^{p-1} K_p(r) = -\Omega_d \left(R_1(r) + \sum_{j=1}^{p} \lambda_j I_1(r; r_j) \right) \qquad\qquad 0 \leq r \leq r_1.$$

By the expression for I_1 in Lemma 9 it follows that $\Delta^{p-1}K_p(r)$ is a continuous function which is piecewise of the form

$$C_{1,p}R_1(r)+C_{2,p}$$

for some constants C_1, C_2 in every of the p intervals $[0,r_p]$, $[r_{p-1},r_{p-2}]$, ..., $[r_2,r_1]$. By the same argument like in point 1. every such function has no more that one zero in $[0,\infty)$. On the other hand, by Theorem 10

$$\Delta^{p-1}K_p(r_1)=0$$

which means that $\Delta^{p-1}K_p(r)$ has no more zeros in the interval $[r_2,r_1]$. And in every of the rest $p-1$ intervals there could be no more than $p-1$ zeros. This contradicts the above conclusion that there exist p different zeros in $(0,r_1)$.

This contradiction proves that K_p does not have zeros in $[0,r_1)$.

4. In order to determine exactly the sign of K_p let us remark that from the above it is clear that a continuous varying of the radii r_j does not change the sign of K_p. Hence, we may assume that we have the special choice $r_j=j$. Then it is easy to compute $\lambda_j=2(-1)^{j+1}\binom{2p}{p+j}\cdot\binom{2p}{p}^{-1}$.

Indeed, for the one-dimensional divided difference operator δ_h^k of order k and step $h>0$ we have (cf. [15, p. 11], [**?**, p. 47])

$$\begin{aligned}\left(\delta_h^2\right)^p f(x) &= \delta_h^{2p}f(x) \\ &= \sum_{j=0}^{2p}(-1)^j\binom{2p}{j}f\left(x+(p-j)h\right) \\ &= \sum_{j=1}^{p}(-1)^{p+j}\binom{2p}{p+j}\delta_{jh}^2 f(x)\end{aligned} \tag{16}$$

If we put $f(x)=x^{2k}$, $k=1,...,p-1$, and $h=1$ we obtain

$$2\binom{2p}{p}^{-1}\sum_{j=1}^{p}(-1)^{p+j}\binom{2p}{p+j}j^{2k}=0$$

since f is a polynomial of degree less tha $2p$. When $k=0$ we obtain directly using the properties of the Newton binom

$$
\begin{aligned}
&2\binom{2p}{p}^{-1}\sum_{j=1}^{p}(-1)^{p+j}\binom{2p}{p+j}\\
&=2\binom{2p}{p}^{-1}\frac{1}{2}\left\{\sum_{j=0}^{2p}(-1)^{j}\binom{2p}{j}-(-1)^{p}\binom{2p}{p}\right\}\\
&=(-1)^{p+1}
\end{aligned}
$$

(see [7, (2.1), (2.2), (2.3)]).

Next we put $f(x)=|x|^{2p}$ in (6). For the functional we have $\ell\left(|x|^{2p}\right)=-\sum_{j=1}^{p}\lambda_j j^{2p}$, and since $\Delta^p\left(|x|^{2p}\right)=C>0$, on the right-hand side of (6) we obtain $C\cdot\int K_p(x)\,dx$, i.e. $sign\left(K_p(x)\right)=sign\left(\ell\left(|x|^{2p}\right)\right).$ We see from (16) that $-\sum_{j=1}^{p}\lambda_j j^{2p}=(-1)^p\,\delta_1^{2p}\left(t^{2p}\right)=(-1)^p\,C_1$, where δ_1^{2p} is the symmetric difference operator with step 1 and the constant $C_1>0$, cf. [15, p. 11], [**?**, p. 47]. Hence, $sign\left(K_p(x)\right)=(-1)^p.$ ■

Remark 11 *The proof follows in general the idea and the application of the Rolle theorem well-known in the theory of $B-$splines, cf. [3]. So far, a direct application of the arguments in [3] is not applicable since K_p is a piecewise-linear combination of the functions*

$$|x|^{2k}\quad(k=0,...,p-1);\qquad R_k(x)\quad(k=1,...,p)\tag{17}$$

on every of the intervals $[r_{p-1},r_{p-2}],...,$ $[r_2,r_1]$, but on the interval $[0,r_p]$ it has a singularity of the type of $R_p(r)$. Let us remark that the system of functions (17) is a Cebyshev (Tchebysheff) one for $r>0$, cf. [16].

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