On a new multivariate sampling paradigm and a polyspline Shannon function

Ognyan Kounchev and Hermann Render

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Abstract

In [9] and [12] we have introduced and studied a new paradigm for cardinal interpolation which is related to the theory of multivariate polysplines. In the present paper we show that this is related to a new sampling paradigm in the multivariate case, whereas we obtain a Shannon type function $S(x)$ and the following Shannon type formula:

$$f(r\theta) = \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} S(e^{-j\rho} r\theta) f(e^j\theta) d\theta.$$ 

This formula relies upon infinitely many Shannon type formulas for the exponential splines arising from the radial part of the polyharmonic operator $\Delta^p$ for fixed $p \geq 1$.

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1 Introduction

The classical Shannon-Kotelnikov-Whittaker formula represents a function by means of the following series

$$f(t) = \sum_{j=-\infty}^{\infty} f(jT) \frac{\sin \sigma (t-jT)}{\sigma (t-jT)}$$

where $T = \frac{\pi}{\sigma}$ and $f$ is a continuous function in $L_2(\mathbb{R})$ having a Fourier transform with support in $[-\pi, \pi]$.

There have been many generalizations of this formula, see [7], [3], and references therein. There exists intimate relation between sampling theorems and Wavelet Analysis which have been studied exhaustively in [22], see also the references therein.
In the present paper we will consider a multivariate generalization of the Shannon theory which is based on a semi-orthogonal Wavelet Analysis using polysplines, and which has been recently developed in [9], [10], [1], [12], [11]. In the case of what we call "spherical sampling" our approach provides a formula for the recovery of a function \( f \) from a "band-limited class \( PV_0 \)" by taking its values over the spheres centered at the origin of \( \mathbb{R}^n \) and having radii \( e^j \) for all \( j \in \mathbb{Z} \).

In the second case, of what we call "parallel sampling" we recover a function \( f \) from a "band-limited class \( PV_0 \)" by taking its values on all hyperplanes in \( \mathbb{R}^n \) defined by \( x_1 = j \) for \( j \in \mathbb{Z} \).

An interesting feature of our results is that they use essentially all advances in the one-dimensional sampling theorems for Riesz basis as developed by Gilbert Walter in his book [22], see also the more recent paper [23].

2 Shannon-Walter sampling with exponential splines (\( L \)-splines)

In the present Section we will construct a formula of Shannon type which is based on exponential splines (these splines are called sometimes \( L \)-splines), cf. [9].

2.1 Shannon sampling for Riesz basis according to Gilbert Walter

First, we will provide the construction of Gilbert Walter of sampling and the corresponding Shannon-Walter functions for semi-orthogonal Wavelet Analysis generated by a scaling function from a Riesz basis; the proofs follow the proofs outlined in [22] for the case of orthogonal scaling functions.

Let us assume that the real-valued function \( \phi(t) \) is continuous and has shifts \( \{ \phi(t-j) \}_{j \in \mathbb{Z}} \) which represent a Riesz basis for the Hilbert subspace \( V_0 \) in \( L_2(\mathbb{R}) \), i.e.

\[
V_0 := \text{clos}_{L_2(\mathbb{R})} \{ \phi(t-j) : j \in \mathbb{Z} \},
\]

and there exist two constants \( A, B > 0 \) such that for arbitrary constants \( c_j \) holds

\[
A \, \sum_{j=-\infty}^{\infty} |c_j|^2 \leq \left\| \sum_{j=-\infty}^{\infty} c_j \phi(t-j) \right\|_{L_2(\mathbb{R})}^2 \leq B \, \sum_{j=-\infty}^{\infty} |c_j|^2 ;
\]

following the tradition we call \( \phi \) scaling function of \( V_0 \).

For every element \( f \in V_0 \) we have

\[
f(t) = \sum_{j=-\infty}^{\infty} f_j \phi(t-j) .
\]

Assume that the space \( V_0 \) has the property that for every \( f \in V_0 \) with

\[
f(j) = 0 \quad \text{for } j \in \mathbb{Z}
\]
follows \( f \equiv 0 \). Then there should exist a basis \( \{ \psi_j(t) \}_{j \in \mathbb{Z}} \) of \( V_0 \) such that a formula of Shannon type holds, i.e., for every \( f \in V_0 \)

\[
f(t) = \sum_{j=-\infty}^{\infty} f(j) \psi_j(t) \quad \text{for } t \in \mathbb{R}.
\]

The problem of finding the proper conditions on \( \phi \) and the basis \( \{ \psi_j(t) \}_{j \in \mathbb{Z}} \) has been resolved by Gilbert Walter, [22]. Below we provide in detail his construction.

It is well-known that to the Riesz basis \( \{ \phi(t - j) \}_{j \in \mathbb{Z}} \) there exists unique dual Riesz basis \( \{ \bar{\phi}(t - j) \}_{j \in \mathbb{Z}} \) where duality means

\[
\langle \phi(x - j), \bar{\phi}(x - \ell) \rangle = \delta_{j\ell}, \tag{2}
\]

cf. [21], [22]. Let us assume that the functions \( \phi \) and \( \bar{\phi} \) satisfy the following asymptotic conditions:

\[
\phi(t) = O \left( |t|^{-1-\varepsilon} \right), \quad \bar{\phi}(t) = O \left( |t|^{-1-\varepsilon} \right) \tag{3}
\]

as \( t \to \pm \infty, \ t \in \mathbb{R} \),

or, which is equivalent, there exist positive constants \( C_1 > 0 \) and \( C_2 > 0 \) such that for all \( t \in \mathbb{R} \) holds

\[
|\phi(t)| \leq \frac{C_1}{(1 + |t|)^{1+\varepsilon}} \quad \text{and} \quad |\bar{\phi}(t)| \leq \frac{C_2}{(1 + |t|)^{1+\varepsilon}}.
\]

Then a direct estimate shows that the function

\[
q(x, y) = \sum_{j \in \mathbb{Z}} \bar{\phi}(x - j) \phi(y - j) \quad \text{(4)}
\]

is uniformly convergent on every compact set in \( \mathbb{R}^2 \).

On the other hand for every fixed \( x_1 \in \mathbb{R} \) (or uniformly for \( x_1 \) from a compact subset in \( \mathbb{R} ) \) we have the estimate

\[
|q(x_1, y)|^2 \leq \sum_{j, k \in \mathbb{Z}} |\phi(y - j) \phi(y - k)| \frac{C_2^2}{(1 + |x_1 - j|)^{1+\varepsilon} (1 + |x_1 - k|)^{1+\varepsilon}}.
\]

We have also

\[
\int_{\mathbb{R}} \left| \phi(y - j) \phi(y - k) \right| dy \leq \int_{\mathbb{R}} \frac{C_1}{(1 + |y - j|)^{1+\varepsilon}} \frac{C_1}{(1 + |y - k|)^{1+\varepsilon}} dy
\]

\[
\leq C_1^2 \int_{\mathbb{R}} \frac{1}{(1 + |y - j|)^{1+\varepsilon}} dy
\]

\[
= C_1^2 \int_{\mathbb{R}} \frac{1}{(1 + |y|)^{1+\varepsilon}} dy < \infty.
\]
The last implies the convergence of the integral
\[ \int_{\mathbb{R}} |q(x_1, y)|^2 \, dy. \]
Thus for every element \( f \in L_2(\mathbb{R}) \) the integrals
\[ \int_{\mathbb{R}} q(x_1, y) f(y) \, dy \]
make sense. In a similar way follows that \( q(x, y_1) \in L_2(\mathbb{R}) \) uniformly for \( y_1 \) from a compact subset in \( \mathbb{R} \).
Thus, from (1) and the representation for the dual basis we obtain the following reproduction property for every \( f \in V_0 \),
\[ \int q(x, y) f(y) \, dy = f(x), \quad (5) \]
\[ \int q(x, y) f(x) \, dx = f(y). \quad (6) \]
Let us note that by the definition it is not clear whether the function \( q \) is symmetric.
Let us remark that from (4) it is obvious that
\[ q(t - j, 0) = q(t, j). \quad (7) \]
We summarize the main results of Gilbert Walter's approach from [22] in the following Proposition.

**Proposition 1** Let \( \phi(t) \) be a scaling function satisfying (3), i.e. \( \phi(t) \) and its dual defined by (2) satisfy \( \widetilde{\phi}(t) = O\left(|t|^{-1-\varepsilon}\right) \) as \( t \to \pm \infty \) for \( t \in \mathbb{R} \). Let us define the function \( \phi^* \) by
\[ \phi^*(\xi) := \sum_{j \in \mathbb{Z}} \phi(j) e^{-i\xi j}. \quad (8) \]
We assume that the function \( \phi^*(\xi) \) satisfies the **non-zero condition**
\[ \phi^*(\xi) \neq 0 \quad \text{for } \xi \in \mathbb{R}. \quad (9) \]
Then the following hold:
1. The system of functions \( \{q(t, j)\}_{j \in \mathbb{Z}} = \{q(t - j, 0)\}_{j \in \mathbb{Z}} \) represents a **Riesz basis** of \( V_0 \), where \( q \) is defined in (4).
2. The unique **dual Riesz basis** \( \{S_j(t)\}_{j \in \mathbb{Z}} \) corresponding to \( \{q(t - j, 0)\}_{j \in \mathbb{Z}} \) satisfies
\[ S_j(t) = S_0(t - j). \]
Hence \( \{S_0(t - j)\}_{j \in \mathbb{Z}} \) is a Riesz basis as well.
3. The following Shannon type formula holds for all $f$ in the space $V_0$:

$$f(t) = \sum_{j \in \mathbb{Z}} f(j) S_0(t - j) \quad \text{for } t \in \mathbb{R}. \quad (10)$$

4. The function $S_0(t)$ has a Fourier transform which satisfies

$$\widehat{S}_0(\xi) = \frac{\delta(\xi)}{\phi^*(\xi)} \quad \text{for } \xi \in \mathbb{R}. \quad (11)$$

**Remark 2** We will call the function $S_0$ Shannon-Walter function. Formula (10) may be considered as the analog to the Shannon-Kotelnikov-Whittaker formula whereby the condition for band-limitedness of $f$ (i.e. the compactness of the support of the Fourier transform $\hat{f}(\xi)$) is replaced by $f \in V_0$. In the context of Wavelet Analysis the frequency variable $\xi$ of Fourier Analysis is replaced by the index $j$.

**Proof.** 1. We will prove that the linear operator $T$ which is defined by the map

$$T \left[ \hat{\phi} (\cdot - j) \right](x) = q(x, j)$$

has a continuous extension to $V_0$. For this purpose we will consider the space $\widehat{V}_0$ of the Fourier transforms of the functions in $V_0$.

Let us consider the operator $T_1$ defined on the space $\widehat{V}_0$ as the operator of multiplication by $\phi^*(\xi)$, i.e.

$$T_1[f](\xi) := f(\xi) \cdot \phi^*(\xi).$$

Since by (3) the function $\phi$ has a fast decay it follows that $\phi^*(\xi)$ is a bounded, continuous and periodic function and satisfies

$$|\phi^*(\xi)| \geq c_0 > 0 \quad \text{for } \xi \in \mathbb{R}.$$

Hence, the linear operator $T_1$ is bounded and its inverse defined by multiplication with $\left(\phi^*(\xi)\right)^{-1}$ is also bounded.

Let us see that

$$T = F^{-1}T_1F$$

where $F$ denotes the Fourier transform and $F^{-1}$ its inverse. Indeed, it is enough to check this on the elements $\hat{\phi}(x - j)$. We have

$$F \left[ \hat{\phi} (\cdot - j) \right](\xi) = e^{-ij\xi} \hat{\phi}(\xi)$$

$$T_1F \left[ \hat{\phi} (\cdot - j) \right](\xi) = \phi^*(\xi) e^{-ij\xi} \hat{\phi}(\xi)$$

$$= \left( \sum_{\ell \in \mathbb{Z}} \phi(\ell) e^{i\ell\xi} \right) e^{-ij\xi} \hat{\phi}(\xi)$$

$$= \hat{\phi}(\xi) \cdot \sum_{\ell \in \mathbb{Z}} \phi(\ell) e^{i\ell(\xi - j)},$$
and taking $F^{-1}$ shows finally that $F^{-1}T_1F\left[\tilde{\phi}(\cdot - j)\right](t) = q(t - j, 0)$.

So we have proved that $T$ is a bounded invertible operator, and by a basic result in [21] (p. 30) it follows that the system $\{q(t, j)\}_{j \in \mathbb{Z}}$ is a Riesz basis of $V_0$.

2. By a theorem in [21] there is a unique Riesz basis $S_j(t)$ which is biorthogonal to $\{q(t, j, 0)\}_{j \in \mathbb{Z}}$, i.e.

$$\langle q(t, j, 0), S_k(t) \rangle = \delta_{jk}.$$ 

On the other hand due to a change of variable $\tau = t - k$ we have

$$\langle q(t, j, 0), S_0(t - k) \rangle = \int q(t, j, 0) \cdot S_0(t - k) \, dt$$

$$= \int q(\tau + k - j, 0) \cdot S_0(\tau) \, d\tau$$

$$= \delta_{k - j, 0}$$

which shows that the family of functions $\{S_0(t - j)\}_{j \in \mathbb{Z}}$ is biorthogonal to $\{q(t, j, 0)\}_{j \in \mathbb{Z}}$. Hence, by the uniqueness it follows $S_j(t) = S_0(t - j)$.

3. Equation (10) is obtained by the expansion of $f(t)$ in the basis $\{S_0(t - j)\}_{j \in \mathbb{Z}}$. Indeed, from

$$f(t) = \sum_j f_j S_0(t - j)$$

and by the biorthogonality relation (6) and (7) follows

$$f(j) = \langle f, q(t, j, 0) \rangle = f_j.$$ 

4. For proving (11) we need to apply the Shannon expansion (10) to the function $\phi$ and to take the Fourier transform.

2.2 The exponential splines ($L$-splines)

2.2.1 Preliminaries on exponential splines

At the beginning of Wavelet Analysis the splines played and important role (see some history in [4] and [13]). In 1991 Chui and Wang have constructed the compactly supported spline wavelets, cf. [2]. A characteristic feature of this Wavelet Analysis is that it is semi-orthogonal. Further Wavelet Analysis using exponential splines has been initiated in [6] and has been discussed in detail in [9], [10]. We will refer to the monograph [9] for all details of the outline on exponential splines following below.

The exponential splines are defined by means of ordinary differential operators with constant coefficients given by polynomials

$$L(z) := \prod_{j=1}^N (z - \lambda_j),$$

(12)
where \( \lambda_j \) are some real constants. For simplicity sake we will denote by \( \Lambda \) the non-ordered vector of the numbers \( \lambda_j \)

\[
\Lambda := [\lambda_1, \lambda_2, ..., \lambda_N],
\]

where some of the \( \lambda_j \)'s may repeat (the number of repetitions of \( \lambda_j \) is the \textit{multiplicity} \( \mu_j \) of this \( \lambda_j \)); this notation is a convenient way to avoid every time explicitly writing the multiplicities of the \( \lambda_j \)'s. We will have the operator

\[
L \left( \frac{d}{dt} \right) := L_\Lambda \left( \frac{d}{dt} \right) := \prod_{j=1}^{N} \left( \frac{d}{dt} - \lambda_j \right)
\]

and the space \( U_N \) of analytic solutions defined by

\[
U_N = U_N (\Lambda) := \left\{ f(t) : L_\Lambda \left( \frac{d}{dt} \right) f(t) = 0 \quad \text{for} \quad t \in \mathbb{R} \right\}
\]

which is well known to be of dimension \( N \), cf. \[15\]. The space \( U_N \) is generated by the exponents

\[
t^s e^{\lambda t} \quad \text{for} \quad s = 0, 1, ..., \mu_\lambda - 1
\]

where \( \lambda \in \Lambda \) and \( \mu_\lambda \) is its multiplicity, where obviously we have

\[
N = \sum \mu_\lambda,
\]

and the sum is over all \textit{different} values \( \lambda \) in \( \Lambda \).

The classical polynomial case is obtained when \( \lambda_j = 0 \) for all \( j = 1, 2, ..., N \).

The basic objects which we will consider are the so-called \textbf{cardinal exponential splines} (\( L \)-splines ) generated by the operator \( L_\Lambda \left( \frac{d}{dt} \right) \). Namely, we define the space \( S_\Lambda \) by putting

\[
S_\Lambda := \{ f : f(t) \in C^{N-2} (\mathbb{R}) \cap C^\infty (\mathbb{R} \setminus \mathbb{Z}), \quad f_{|_{(j,j+1)}} \in U_N \},
\]

i.e. they are piecewise analytic solutions to the equation \( L_\Lambda \left( \frac{d}{dt} \right) f(t) = 0 \).

We will consider the Wavelet Analysis generated by the \( B \)-splines arising from the operator \( L_\Lambda \left( \frac{d}{dt} \right) \) which are called sometimes \( TB \)-splines (cf. \[17\] – these are cardinal exponential splines having minimal support). As in the classical polynomial spline theory there exists up to a factor only one such \( TB \)-spline, i.e. a cardinal exponential spline \( Q_N [\Lambda] (t) = Q_N (t) \in S_\Lambda \) supported in the interval \( [0, N] \) and satisfying

\[
Q_N (t) > 0 \quad \text{for} \quad t \in (0, N),
\]

cf. \[5\], \[2\], \[17\].

It is convenient to define \( Q_N (t) \) by means of its Fourier transform (cf. p. 274 in \[9\]), namely we have

\[
\widehat{Q_N} (\xi) = \frac{\prod_{j=1}^{N} (e^{-\lambda_j} - e^{-i\xi})}{\prod_{j=1}^{N} (i\xi - \lambda_j)} \quad \text{for} \quad \xi \in \mathbb{R}.
\]

\[
\lambda_j \text{ are some real constants. For simplicity sake we will denote by } \Lambda \text{ the non-ordered vector of the numbers } \lambda_j
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and the space \( U_N \) of analytic solutions defined by

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U_N = U_N (\Lambda) := \left\{ f(t) : L_\Lambda \left( \frac{d}{dt} \right) f(t) = 0 \quad \text{for} \quad t \in \mathbb{R} \right\}
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which is well known to be of dimension \( N \), cf. \[15\]. The space \( U_N \) is generated by the exponents

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where \( \lambda \in \Lambda \) and \( \mu_\lambda \) is its multiplicity, where obviously we have

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N = \sum \mu_\lambda,
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and the sum is over all \textit{different} values \( \lambda \) in \( \Lambda \).

The classical polynomial case is obtained when \( \lambda_j = 0 \) for all \( j = 1, 2, ..., N \).

The basic objects which we will consider are the so-called \textbf{cardinal exponential splines} (\( L \)-splines ) generated by the operator \( L_\Lambda \left( \frac{d}{dt} \right) \). Namely, we define the space \( S_\Lambda \) by putting

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S_\Lambda := \{ f : f(t) \in C^{N-2} (\mathbb{R}) \cap C^\infty (\mathbb{R} \setminus \mathbb{Z}), \quad f_{|_{(j,j+1)}} \in U_N \},
\]

i.e. they are piecewise analytic solutions to the equation \( L_\Lambda \left( \frac{d}{dt} \right) f(t) = 0 \).

We will consider the Wavelet Analysis generated by the \( B \)-splines arising from the operator \( L_\Lambda \left( \frac{d}{dt} \right) \) which are called sometimes \( TB \)-splines (cf. \[17\] – these are cardinal exponential splines having minimal support). As in the classical polynomial spline theory there exists up to a factor only one such \( TB \)-spline, i.e. a cardinal exponential spline \( Q_N [\Lambda] (t) = Q_N (t) \in S_\Lambda \) supported in the interval \( [0, N] \) and satisfying

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Q_N (t) > 0 \quad \text{for} \quad t \in (0, N),
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cf. \[5\], \[2\], \[17\].

It is convenient to define \( Q_N (t) \) by means of its Fourier transform (cf. p. 274 in \[9\]), namely we have

\[
\widehat{Q_N} (\xi) = \frac{\prod_{j=1}^{N} (e^{-\lambda_j} - e^{-i\xi})}{\prod_{j=1}^{N} (i\xi - \lambda_j)} \quad \text{for} \quad \xi \in \mathbb{R}.
\]
Many important quantities which we will need are defined by means of the $TB$--spline $Q_N (t)$.

In the case of the classical polynomial splines Schoenberg introduced the so--called Euler-Frobenius polynomial. In order to define a generalization of the Euler-Frobenius polynomials introduced by Schoenberg we will consider the function (see Corollary 13.24, p. 235 in [9], Micchelli [14])

$$A_{N-1} (x; \lambda) := \frac{1}{2\pi i} \int_\Gamma \frac{1}{L_\lambda (z)} e^{\frac{xz}{z-\lambda}} dz,$$  \hspace{1cm} (18)

where the closed contour $\Gamma$ surrounds all points $\lambda_j \in \Lambda$ but excludes all zeros of the function $\frac{e^{xz}}{e^z-\lambda}$. If we put

$$r (\lambda) := \prod_{j=1}^N (e^{\lambda_j} - \lambda), \quad s (\lambda) := \prod_{j=1}^N (e^{-\lambda_j} - \lambda)$$  \hspace{1cm} (19)

then we obtain the following representation (see Corollary 13.25 p. 235 in [9])

$$\Pi_{N-1} (x; \lambda) = r (\lambda) A_{N-1} (x; \lambda)$$  \hspace{1cm} (20)

where $\Pi_{N-1} (x; \lambda)$ is a polynomial of degree $\leq N - 1$ of $\lambda$. The so-called Euler-Frobenius polynomial

$$\Pi_{N-1} (\lambda) := \Pi_{N-1} (0; \lambda)$$  \hspace{1cm} (21)

has degree $\leq N - 2$, cf. Corollary 13.25 in [9].

We have the following important Proposition (cf. [14], [16], or Theorem 13.31 on p. 237 and Corollary 13.53 on p. 253 in [9]).

**Proposition 3** The polynomial $\Pi_{N-1} (\lambda)$ has exactly $N - 2$ negative zeros.

By means of the $TB$--spline we may define the Euler-Schoenberg exponential spline, (see p. 254 in [9]):

$$\Phi_{N-1} (x; \lambda) = \sum_{j=-\infty}^{\infty} \lambda^j Q_N (x - j).$$  \hspace{1cm} (22)

Note that the word "exponential" has nothing to do with the exponential meant in the present paper, but it is related to the following easy to check property

$$\Phi_{N-1} (x + 1; \lambda) = \lambda \Phi_{N-1} (x; \lambda).$$  \hspace{1cm} (23)

We have the following important (cf. p. 255, Proposition 13.55 and Theorem 13.56 on p. 256, Corollary 13.57 p. 256 in [9]).

**Proposition 4 1.** The Euler-Schoenberg exponential spline in (22) satisfies the following relation for all $x$ with $0 \leq x \leq 1$,

$$\Phi_{N-1} (x; \lambda) = \frac{(-1)^{N-1}}{\lambda^{N-1}} e^{-\sum_{j=1}^\infty \lambda_j} \Pi_{N-1} (x; \lambda).$$  \hspace{1cm} (24)
2. If the vector $\Lambda$ is symmetric, i.e. satisfies $\Lambda = -\Lambda$, then
\[
\Phi_{N-1} \left( \frac{N}{2}; \frac{1}{z} \right) = \Phi_{N-1} \left( \frac{N}{2}; z \right) \quad \text{for all } z \in \mathbb{C},
\]
and the polynomial $\Phi_{N-1} (0; z) \neq 0$ satisfies
\[
\Phi_{N-1} (0; z) \neq 0 \quad \text{for all } z \in \mathbb{C} \text{ with } |z| = 1.
\]

In particular, we see that from (23) and (24) the following representation holds:
\[
\Phi_{N-1} \left( \frac{N}{2}; \lambda \right) = \lambda^N \Phi_{N-1} (0; \lambda) = (-1)^{N-1} \lambda^{-\frac{N}{2}+1} \exp \left( - \sum_{j} \lambda_j \right) \Pi_{N-1} (\lambda).
\]

We will use further the following basic Lemma 5

Lemma 5 Let the vector $\Lambda$ be symmetric, i.e. satisfies $\Lambda = -\Lambda$. Then the following holds
\[
|\Pi_{N-1} (-1)| \leq |\Pi_{N-1} (e^{i\xi})| \leq |\Pi_{N-1} (1)| \quad \text{for } \xi \in \mathbb{R}.
\]

Proof. By Proposition 3 the polynomial $\Pi_{N-1} (\lambda)$ has precisely $N-2$ zeros $v_j < 0$, and by Proposition 4, 1) and 2) they satisfy
\[
v_j v_{N-1-j} = 1 \quad \text{for } j = 1, 2, ..., N-2.
\]

Hence, we have the representation
\[
\Pi_{N-1} (\lambda) = D \prod_{j=1}^{N-2} (\lambda - v_j),
\]
where $D$ is a non-zero coefficient. By the reality of the zeros $v_j < 0$ it follows
\[
|\Pi_{N-1} (e^{-i\xi})| = |D| \prod_{j=1}^{N-2} |e^{-i\xi} - v_j| = |D| \prod_{j=1}^{N-2} |e^{-i\xi} - v_j| \left| e^{-i\xi} - \frac{1}{v_j} \right|
\]
\[
= |D| \prod_{j=1}^{N-2} \left| \frac{e^{-i\xi} - v_j}{v_j} \right|^2 = |D| \prod_{j=1}^{N-2} \frac{1 - 2v_j \cos \xi + v_j^2}{|v_j|}
\]
\[
\leq |D| \prod_{j=1}^{N-2} \frac{1 - 2v_j + v_j^2}{|v_j|} = |\Pi_{N-1} (1)|.
\]
In a similar way we obtain
\[ |\Pi_{N-1}(e^{-i\xi})| \geq |D| \prod_{j=1}^{\frac{N-2}{2}} \frac{1 + 2v_j + v_j^2}{|v_j|} \]
\[ = |\Pi_{N-1}(-1)|. \]

### 2.3 Checking the conditions of Gilbert Walter

Now we put for the scaling function
\[ \phi(t) = Q_N(t). \quad (28) \]

A fundamental result in [9] (Theorem 14.6) says that for arbitrary non-ordered vector \( \Lambda \) the system \( \{\phi(t - j)\}_{j \in \mathbb{Z}} \) represents a Riesz basis for the space \( V_0 \) defined by means of the cardinal splines \( S_\Lambda \) as
\[ V_0 = S_\Lambda \cap L_2(\mathbb{R}); \quad (29) \]
the explicit values of the Riesz constants is found in [9].

According to (17) the Fourier transform of the basic \( TB \)-spline satisfies the asymptotic
\[ \hat{\phi}(\xi) = \hat{Q}_N(\xi) = \prod_{j=1}^{N} \frac{e^{-\lambda_j} - e^{-i\xi}}{i\xi - \lambda_j} = O\left(\frac{1}{|\xi|^N}\right) \quad \text{for} \quad \xi \to \infty, \]
and it is clear that the derivatives of \( \hat{\phi}(\xi) \) satisfy similar asymptotic.

As we mentioned before formula (2), the basis \( \{\phi(t - j)\}_{j \in \mathbb{Z}} \) has a dual Riesz basis generated by a unique function \( \widetilde{\phi} \), i.e. the set of functions \( \{\widetilde{\phi}(t - j)\}_{j \in \mathbb{Z}} \) generates a Riesz basis of \( V_0 \). For the dual function \( \widetilde{\phi} \) one has the Fourier transform (see p. 356 in [9])
\[ \widehat{\widetilde{\phi}}(\xi) = C_S^2 \frac{\hat{\phi}(\xi)}{\Phi_{2N-1} \Lambda(N; e^{i\xi})} \quad \text{for} \quad \xi \in \mathbb{R} \quad (30) \]
with \( C_S = \exp\left(\frac{1}{2} \sum_{j=1}^{N} \lambda_j\right) \),
where \( \Lambda = [\Lambda, -\Lambda] \), i.e. it is the symmetrized vector of \( \Lambda \).

**Proposition 6** For arbitrary integer \( N \geq 1 \) and arbitrary non-ordered vector \( \Lambda \) of order \( N \), the functions \( \phi \) and \( \widetilde{\phi} \) defined by (28) and (30) satisfy condition (3).
Proof. The asymptotic of the function $\phi$ is clear since $Q_N(t)$ is a compactly supported function, in particular, for every integer $m \geq 0$ the following asymptotic holds:

$$\phi(t) = O\left(\frac{1}{|t|^m}\right) \quad \text{for} \quad t \to \pm \infty.$$ 

Now from Lemma 5, applied to the vector $\Lambda$ and to the corresponding function $\Phi_{2N-1}[\Lambda](N; e^{i\xi})$ (whereby we use essentially the symmetry of the vector $\Lambda$), and from formula (27), it follows that the function $\Phi_{2N-1}[\Lambda](N; e^{i\xi})$ is bounded, namely

$$|\Pi_{2N-1}(-1)| \leq \left|\Phi_{2N-1}[\Lambda](N; e^{i\xi})\right| \leq |\Pi_{2N-1}(1)| \quad \text{for} \quad \xi \in \mathbb{R};$$

here the Euler polynomial $\Pi_{2N-1}(\lambda)$ corresponds to the vector $\Lambda$.

On the other hand, we have the following equality proved by integration by parts:

$$\tilde{\phi}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi t} \tilde{\phi}(\xi) d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + t^2} \left(1 - \frac{d^2}{d\xi^2}\right) e^{i\xi t} \cdot \tilde{\phi}(\xi) d\xi$$

$$= \frac{1}{2\pi} \frac{1}{1 + t^2} \int_{-\infty}^{\infty} e^{i\xi t} \left(1 - \frac{d^2}{d\xi^2}\right) \tilde{\phi}(\xi) d\xi.$$ 

Let us estimate the last integral. Since $\Phi_{2N-1}[\Lambda](N; \lambda)$ is a polynomial in $\lambda$, it follows that the derivatives of $\Phi_{2N-1}[\Lambda](N; e^{i\xi})$ with respect to $\xi$ have nice asymptotic for $\xi \to \pm \infty$. Now by the above remarks about the asymptotic of the derivatives of the function $\phi(\xi)$ and the representation formula (30) follows the inequality

$$\left|\tilde{\phi}(t) (1 + t^2)\right| \leq C \quad \text{for} \quad t \in \mathbb{R}.$$ 

This justifies the asymptotic condition for $\tilde{\phi}$ in (3).

For proving the asymptotic of $\phi$ and $\tilde{\phi}$ we needed no restrictions on the vector $\Lambda$. However the non-zero condition (9) is more restrictive. In fact, G. Walter has shown that the classical splines (this corresponds to the case of all $\lambda_j = 0$ in $\Lambda$) generate a simple and natural Shannon type formula only in the case of odd degree cardinal splines, [22]. Similar is the situation with the exponential splines. For that reason we will restrict our attention to the case

$$N = 2p.$$  
(31)
Proposition 7 1. We have the equality

\[ \phi^*(\xi) := \sum_{j \in \mathbb{Z}} \phi(j) e^{-i\xi j} = \Phi_{N-1}[\Lambda](0, e^{i\xi}) \quad \text{for } \xi \in \mathbb{R}; \]

hence the non-zero condition (9) is equivalent to the condition

\[ \Phi_{N-1}[\Lambda](0, \lambda) \neq 0 \quad \text{for } |\lambda| = 1. \tag{32} \]

2. For arbitrary symmetric non-ordered vector \( \Lambda \) of order \( N = 2p \) and for \( \phi \) given by (28) condition (9) holds, i.e.

\[ \phi^*(\xi) \neq 0 \quad \text{for } \xi \in \mathbb{R}. \]

Proof. By the definition of \( \Phi_{N-1} \) in (22) and by (24) follows

\[
\sum_{j \in \mathbb{Z}} \phi(j) \lambda^{-j} = \sum_{j \in \mathbb{Z}} \phi(-j) \lambda^j = \Phi_{N-1}[\Lambda](0, \lambda)
= (-1)^{N-1} \lambda^{N-1} \exp \left( -\sum_{j=1}^{N} \lambda_j \right) \Pi_{N-1}(\lambda).
\]

(Please note the difference between \( \Phi_{N-1}[\Lambda] \) and \( \Phi_{2N-1}[\Lambda] \) since they correspond to different vectors). Now Lemma 5 gives us the lower and upper bounds of \( |\Pi_{N-1}(e^{i\xi})| \) from which we obtain immediately (9), i.e.

\[ \sum_{j \in \mathbb{Z}} \phi(j) e^{-i\xi j} \neq 0 \quad \text{for } \xi \in \mathbb{R}. \]

Note that by means of Lemma 5 we have used essentially the symmetry of the vector \( \Lambda \) and the fact that \( N = 2p \). We will apply the above Proposition 7 to the case of non-ordered vectors \( \Lambda = \Lambda_k = [\lambda_1, ..., \lambda_N] \) given by

\[ \Lambda_k := \begin{cases} 
\lambda_j = -k & \text{for } j = 1, ..., p \\
\lambda_j = k & \text{for } j = p + 1, ..., 2p 
\end{cases} \tag{33} \]

where \( k \) is some integer.

On the other hand there are other vectors \( \Lambda \) which are important for our multivariate theory, which are not symmetric but close to being symmetric, and for which the non-zero condition (9) holds; they have been considered in the paper [12] and called nearly symmetric there. For us the following vectors \( \Lambda = \Lambda_k = [\lambda_1, ..., \lambda_N] \) will be important:

\[ \Lambda_k := \begin{cases} 
\lambda_{1+j} = k + 2j & \text{for } j = 0, ..., p - 1, \\
\lambda_{1+p+j} = -n - k + 2 + 2j & \text{for } j = 0, ..., p - 1; 
\end{cases} \tag{34} \]

here \( k \) is some integer.

We have the following analog to Proposition 7.
Proposition 8 Let $\Lambda = \Lambda_k$ be defined by (34). Then $\phi^*$ satisfies the non-zero condition (9).

Proof. In the proof of Proposition 13 in the paper [12], we proved that
\[ A_{N-1} (0; -1) \neq 0 \]
for every $k \geq 0$. On the other hand by Proposition 3 the polynomial $\Pi_{N-1} (\lambda)$ can have only real zeros with $\lambda < 0$. Since by (20) and Proposition 4 we have
\[ \Phi_{N-1} [\Lambda_k] (0; \lambda) = -\lambda \cdot s (\lambda^{-1}) A_{N-1} (0; \lambda) \]
it follows that the function $\Phi_{N-1} [\Lambda_k] (0; \lambda)$ has no zeros for $|\lambda| = 1$.

Hence, in the above cases we may apply the results of G. Walter as presented in the previous section. By Proposition 1, 5) it follows that there exists a Shannon-Walter exponential spline $S_0 \in V_0 = S_\Lambda \cap L_2 (\mathbb{R})$ such that its Fourier transform satisfies
\[ \hat{S}_0 (\xi) = \frac{\hat{\phi} (\xi)}{\phi^* (\xi)} = \frac{\hat{\phi} (\xi)}{\Phi_{N-1} [\Lambda] (0, e^{i\xi})}. \]

From the above Propositions we obtain the following:

Theorem 9 Let $N = 2p$ and $\Lambda$ be symmetric non-ordered vector of order $N$, or $\Lambda$ be a vector given by (34). Then for every cardinal exponential spline $f \in V_0$ there holds a Shannon type formula:
\[ f (t) = \sum_{j=-\infty}^{\infty} S_0 (t - j) f (j). \]

3 The multivariate case

Our main purpose is to find a multivariate generalization of the Shannon-type formula (10). For that purpose we will use a Wavelet Analysis which is genuinely multivariate. We will work in the framework of the polyharmonic Wavelet Analysis introduced and studied in [9]. It uses piecewise polyharmonic functions (polysplines) which generalize the piecewise polynomial functions (and splines) in the one-dimensional spline theory.

We assume that the integer $p \geq 1$ is fixed.

We will consider the solutions to the polyharmonic equation
\[ \Delta^p u (x) = 0 \]
in annular domains, i.e. in annuli $A_{a,b} = \{ x \in \mathbb{R}^n : a < |x| < b \}$ given by two numbers $a, b > 0$. Here
\[ \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \]
is the Laplace operator and $\Delta^p$ is its $p$-th iterate.

For every integer $k \geq 0$ we will assume that we are given an orthonormal basis of the spherical harmonics of homogeneity degree $k$, which we will denote as usually by $\{Y_{k,\ell} (x)\}_{\ell=1}^{a_k}$, cf. [20], [9]. Thus we have

$$
\int_{S^{n-1}} Y_{k,\ell_1} (\theta) Y_{k,\ell_2} (\theta) d\theta = \delta_{\ell_1 - \ell_2};
$$

here $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$, $d\theta$ is the spherical area measure, and $\delta_j$ is the Kronecker symbol taking on $1$ for $j = 0$ and $0$ elsewhere. We will work with the spherical variables $x = r \theta$ where $r = |x|$, $\theta \in S^{n-1}$. It is known that for special functions of the form $f (r) Y_{k,\ell} (\theta)$ one has (see [9], p. 152)

$$
\Delta^p (f (r) Y_{k,\ell} (\theta)) = L_{(k)}^p f (r) \cdot Y_{k,\ell} (\theta)
$$

where the operator $L_{(k)}$ is given by

$$
L_{(k)} := \frac{\partial^2}{\partial r^2} + \frac{n - 1}{r} \frac{\partial}{\partial r} - \frac{k (n + k - 2)}{r^2}.
$$

On the other hand by the change $v = \log r$ we obtain another operator $M_{k,p}$ satisfying

$$
L_{(k)}^p \left( \frac{d}{dv} \right) = e^{-2pv} M_{k,p} \left( \frac{d}{dv} \right)
$$

where

$$
M_{k,p} (z) := \prod_{j=0}^{p-1} (z - k - 2j) (z + n + k - 2 - 2j),
$$

cf. [9], Theorem 10.34. Here it is important that unlike $L_{(k)}^p \left( \frac{d}{dr} \right)$ the operator $M_{k,p} \left( \frac{d}{dv} \right)$ has constant coefficients. Thus we are within the framework of the exponential polynomials and we may consider exponential splines for the operator $L = M_{k,p}$ defined by the vector $\Lambda = \Lambda_k = [\lambda_1, \ldots, \lambda_N]$ given by (34).

The following representation for polyharmonic functions in annulus is a fundamental result (cf. Theorem 10.39 in [9]):

**Proposition 10** Every function $u \in C^\infty (A_{a,b})$ satisfying $\Delta^p u (x) = 0$ in the annulus $A_{a,b}$ permits the representation

$$
u (x) = \sum_{k,\ell} u_{k,\ell} (r) Y_{k,\ell} (\theta), \quad \text{for } x = r \theta \quad (37)
$$

where $u_{k,\ell} (r)$ is a solution to

$$
L_{(k)}^p \left( \frac{d}{dr} \right) u_{k,\ell} (r) = 0 \quad \text{for } a < r < b.
$$
Let us assume that the integer \( k \geq 0 \) is fixed.

Further we consider the space of exponential splines generated by the operator \( L = M_{k,p} (\frac{d}{dv}) \). We define the space of cardinal exponential splines and the Wavelet Analysis corresponding to the operator \( L = M_{k,p} (\frac{d}{dv}) \) as in (29), namely we put

\[
\tilde{V}_0^k = \left\{ \tilde{f} \in L_2(\mathbb{R}) : M_{k,p} \left( \frac{d}{dv} \right) \tilde{f}(v) \mid_{(j,j+1)} = 0, \quad \text{and} \quad \tilde{f} \in C^{p-2}(\mathbb{R}) \right\}. \tag{38}
\]

Let us recall that in the case of odd dimension \( n \) (since there are no repeating frequencies in \( \Lambda \) of (34)) the analytic solutions of \( M_{k,p} (\frac{d}{dv}) f = 0 \) are linear combinations of the exponentials

\[
\left\{ e^{(k+2j)v}, e^{(-n-k+2+2j)v} \right\}_{j=0}^{p-1}.
\]

Respectively, by the change of the variable \( v = \log r \) we obtain the space

\[
V_0^k := \left\{ f : f(r) = \tilde{f}(\log r), \quad \text{for} \quad \tilde{f} \in \tilde{V}_0^k \right\}. \tag{39}
\]

Finally, we may apply Theorem 9, and we will obtain the Shannon-Walter exponential spline given by (35) which we denote further by \( S_0^{(k)}(v) \). By (36) \( S_0^{(k)}(v) \) satisfies a Shannon-type formula, i.e. for every element \( f \in \tilde{V}_0^k \) we have the equality

\[
\tilde{f}(v) = \sum_{j=-\infty}^{\infty} \tilde{f}(j) \tilde{S}_0^{(k)}(v-j). \tag{40}
\]

By means of the change \( r = e^v \) we have \( f(e^v) := \tilde{f}(v) \) and we put

\[
S_0^{(k)}(r) := \tilde{S}_0^{(k)}(\log r).
\]

Hence, for every \( f \in \tilde{V}_0^k \) we have the Shannon-type formula

\[
f(r) = \sum_{j=-\infty}^{\infty} f(e^j) S_0^{(k)}(re^{-j}). \tag{41}
\]

### 3.1 Asymptotic of \( S_0^{(k)} \) for \( k \to \infty \)

It is essential to see what is the asymptotic of \( \tilde{S}_0^{(k)}(v) \) and of \( \tilde{S}_0^{(k)}(\xi) \) as \( k \to \infty \).

This will be important for the existence and the regularity of the multivariate Shannon type function.

We assume as above that

\[
\phi(t) = Q_N[\Lambda](t) \tag{42}
\]

for the nonordered vector \( \Lambda = \Lambda_k = [\lambda_1, ..., \lambda_N] \) given by (34) (or (33) below), and that the non-zero condition (9) holds.
By formula (35) we have

$$\hat{S}_0^{(k)}(\xi) = \frac{\hat{\phi}_N(\xi)}{\Phi_{N-1}[A_k](0; e^{i\xi})}.$$  \hspace{1cm} (43)

On the other hand by formula (24) we have the representation

$$\Phi_{N-1}[A_k](0; \lambda) = \frac{(-1)^{N-1}}{\lambda^{N-1}} \exp \left(-\sum_j \lambda_j \right) \Pi_{N-1}(\lambda).$$  \hspace{1cm} (44)

**Lemma 11** The Fourier transform of the Shannon-Walter function $S_0^{(k)}$ and its Fourier transform satisfy the following asymptotic:

1. uniformly for $\xi \in \mathbb{R}$

$$\hat{S}_0^{(k)}(\xi) = O \left( \frac{1}{k} \right) \quad \text{for } k \to \infty;$$

2. uniformly for all $t \in \mathbb{R}$

$$\hat{S}_0^{(k)}(t) = O(1) \quad \text{for } k \to \infty.$$

**Proof.** By (42) it follows directly that for appropriate constants $k_0 \geq 0$ and $C > 0$ and for all $\xi \in \mathbb{R}$ and $k \geq k_0$ we have

$$\left| \hat{\phi}_N(\xi) \right| = \left| \prod_{j=1}^N (e^{-\lambda_j} - e^{-i\xi}) \prod_{j=1}^N (i\xi - \lambda_j) \right| \leq C \frac{e^p k}{(\xi^2 + k^2)^p}.$$

Now we will take care of the denominator in (43). We will apply a subtle estimate of Theorem 11 in the paper [12] (please note the conventions in the paper, where the size of the vector $A$ is numbered through $N + 1$ and here we use $N$ ). It says that there exist two constants $C_1, C_2 > 0$ and an integer $k_0 \geq 1$ such that for $k \geq k_0$ holds

$$\frac{C_1}{k^{N-1}} \leq |A_{N-1}(0; \lambda)| \leq \frac{C_2}{k^{N-1}} \quad \text{for } |\lambda| = 1.$$  

From relation (20) we obtain

$$\Pi_{N-1}(\lambda) = r(\lambda) A_{N-1}(0; \lambda)$$

and from (44) we obtain

$$\Phi_{N-1}[A_k](0; \lambda) = -\lambda \cdot s(\lambda^{-1}) A_{N-1}(0; \lambda).$$

From (19) it follows that the asymptotic of the polynomial $s(\lambda)$ for $k \to \infty$ with $\lambda = e^{i\xi}$ is

$$s(\lambda) \approx e^{kp}.$$
Hence, for $k \to \infty$ and for appropriate constant $C_3 > 0$ we have
\[
\tilde{S}_0^{(k)}(\xi) \approx C_3 \frac{e^{\rho k}}{(\xi^2 + k^2)^{\rho \pi}} = \frac{1}{k}.
\]

2. By definition we have
\[
\tilde{S}_0^{(k)}(t) = \int_{-\infty}^{\infty} e^{i\xi t} \tilde{S}_0^{(k)}(\xi) d\xi;
\]
hence by a direct estimate and a change of variable $\xi' = \xi/k$ we obtain
\[
\left| \tilde{S}_0^{(k)}(t) \right| \leq C_3 k^{2\rho - 1} \int_{-\infty}^{\infty} \frac{1}{(\xi'^2 + k^2)^{\rho \pi}} d\xi' = C'_0.
\]

Obviously the results of Lemma 11 hold for the Shannon function $S_0^{(k)}(r) := \tilde{S}_0^{(k)}(\log r)$ in the variable $r$.

### 3.2 The Shannon polyspline

We have seen that for the vector $A_k$ defined by (34) the non-zero condition (32) is satisfied and we may define the Shannon-Walter exponential splines $S_0^{(k)}$.

We will call the following function (actually it is understood in a distributional sense) **Shannon polyspline kernel**:

\[
S_0(r, \theta, \psi) := \sum_{k=0}^{\infty} \sum_{\ell=1}^{n_k} a_k S_0^{(k)}(r) Y_{k,\ell}(\theta) Y_{k,\ell}(\psi)
\]
\[
= \sum_{k=0}^{\infty} S_0^{(k)}(r) Z_k(\theta, \psi)
\]
\[
= \sum_{k=0}^{\infty} \tilde{S}_0^{(k)}(\log r) Z_k(\theta, \psi),
\]

where $Z_k(\theta, \psi)$ is the zonal harmonic of degree $k$, cf. [20]. Let us note that for every fixed $\psi$ the function $S_0(r, \theta, \psi)$ is a polyspline of $r\theta$, and for every fixed $\theta$ the function $S_0(r, \theta, \psi)$ is a polyspline of $r\psi$.

We will characterize the smoothness-convergence properties of this series in terms of Sobolev spaces. Let us recall that the function $h$ defined by the series
\[
h(x) := \sum_{k,\ell} h_{k,\ell} Y_{k,\ell}(\theta)
\]
belongs to the Sobolev space on the sphere $H^s(\mathbb{S}^{n-1})$ for a real number $s$ if and only if
\[
\sum_{k,\ell} (1 + k^2)^s h_{k,\ell}^2 < \infty;
\]
cf. section 7 and section 22 in [19].

**Lemma 12** As a function of \( \theta \) (or \( \psi \)) the Shannon polyspline \( S_0 \) satisfies

\[
S(r, \cdot, \psi) \in H^s (\mathbb{S}^{n-1}), \quad S(r, \theta, \cdot) \in H^s (\mathbb{S}^{n-1})
\]

where

\[
s = -n + \frac{3}{2} - \varepsilon
\]

for every number \( \varepsilon > 0 \).

**Proof.** We know that for some constant \( C > 0 \) and for all \( k \geq 0 \), and \( \ell = 1, 2, \ldots, a_k \) holds

\[
|Y_{k, \ell} (\theta)| \leq Ck^{\frac{n}{2} - 1},
\]

see e.g. [18], [8]. Also, and one has the estimate

\[
a_k \leq C_1 k^{n-2},
\]

cf. e.g. [20].

Since the coefficients of the series (45) are \( h_{k, \ell} = S_0^{(k)} (r) Y_{k, \ell} (\psi) \) we have the following inequalities

\[
\sum_{k, \ell} (1 + k^2)^s h_{k, \ell}^2 \leq \sum_{k, \ell} (1 + k^2)^s C^2 k^{n-2} \leq C_2 \sum_{k=0}^{\infty} (1 + k^2)^s k^{2n-4}
\]

which proves the theorem.

\[
\Box
\]

### 3.3 The polyharmonic Wavelet Analysis

In [9] we have introduced a **polyharmonic Wavelet Analysis** by considering the following basic space

\[
PV_0 := \text{clos}_{L_2 (\mathbb{R}^n)} \left\{ f (x) : \Delta^p f (x) = 0 \quad \text{on} \ e^j < r < e^{j+1}, \quad x = r\theta, \quad f \in C^{2p-2} (\mathbb{R}^n) \right\}.
\]

Let us note that the radii \( e^j \) appear in a natural way since by the representation (37) and Proposition 10 every function \( u \in PV_0 \) has coefficients \( u_{k, \ell} (r) \) satisfying

\[
\begin{align*}
\end{align*}
\]

where the spaces \( V_0^k \) are those defined in (39). Hence, we have the following decomposition in the sense of \( L_2 \) metric (cf. [9])

\[
PV_0 = \bigoplus_{k=0}^{\infty} \bigoplus_{\ell=1}^{a_k} V_0^k
\]

which is generated by the expansion (37).
As we have seen above, the Shannon polyspline kernel (45) is a function in some Sobolev space $H^s$ and might be a distribution in the case of a negative exponent. Hence, the formulas which we will write will be understood in a distributional sense. For two functions (distributions) $f \in H^s (\mathbb{S}^{n-1})$ and $g \in H^{-s} (\mathbb{S}^{n-1})$ with expansions

$$f (\theta) = \sum_{k,\ell} f_{k,\ell}Y_{k,\ell} (\theta), \quad g (\theta) = \sum_{k,\ell} g_{k,\ell}Y_{k,\ell} (\theta) \quad \text{for } \theta \in \mathbb{S}^{n-1}$$

we have the scalar product on $\mathbb{S}^{n-1}$ defined by an integral understood in a distributional sense

$$\langle f, g \rangle := \int_{\mathbb{S}^{n-1}} f (\theta) g (\theta) d\theta = \sum_{k,\ell} f_{k,\ell} g_{k,\ell}.$$

Note that the infinite sum which gives sense to the above integral is absolutely convergent for $f \in H^s (\mathbb{S}^{n-1})$ and $g \in H^{-s} (\mathbb{S}^{n-1})$.

In our main Theorem below we will use the spaces $\text{PV}_0$ to mimic the one-dimensional cardinal spline space $V_0$, and to formulate an analog to the Shannon type formula.

**Theorem 13** For every function $f \in \text{PV}_0 \cap C (\mathbb{R}^n)$ the following Shannon type formula holds:

$$f (r \psi) = \sum_{j=-\infty}^{\infty} \int_{S^{n-1}} S_0 \left( \frac{r}{e^{j}}, \theta, \psi \right) f (e^{j} \theta) d\theta,$$

where $S_0 (r, \theta, \psi)$ is the Shannon polyspline function defined in (45), and the integral is understood in distributional sense.

**Proof.** Since by definition $f \in C (\mathbb{R}^n)$ it follows that for all $j \in \mathbb{Z}$ we have $f (e^{j} \theta) \in L_2$, hence the following $L_2$-convergent expansion for every fixed $r > 0$ holds true

$$f (x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \tilde{f}_{k,\ell} (r) Y_{k,\ell} (\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \bar{f}_{k,\ell} (\log r) Y_{k,\ell} (\theta),$$

where $x = r \theta$.

We obtain in distributional sense the following equalities, for $v = \log r$,

\[
\sum_{j=-\infty}^{\infty} \int_{S^{n-1}} S_0 \left( \frac{r}{e^{j}}, \theta, \psi \right) f (e^{j} \theta) d\theta = \sum_{j=-\infty}^{\infty} \left( \sum_{k=0}^{a_k} \sum_{\ell=1}^{a_k} S_0^{(k)} (re^{-j}) f_{k,\ell} (e^{j}) Y_{k,\ell} (\psi) \right) \\
= \sum_{k=0}^{a_k} \sum_{\ell=1}^{a_k} \left( \sum_{j=-\infty}^{\infty} S_0^{(k)} (re^{-j}) f_{k,\ell} (e^{j}) Y_{k,\ell} (\psi) \right) \\
= \sum_{k=0}^{a_k} \sum_{\ell=1}^{a_k} \left( \sum_{j=-\infty}^{\infty} S_0^{(k)} (v - j) f_{k,\ell} (e^{j}) Y_{k,\ell} (\psi) \right).
\]

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On the other hand since for every $k \geq 0$ we have $f_{k,t} (e^v) \in V_0^k$, by the Shannon formula in (40) we obtain
\[ \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \left( \sum_{j=-\infty}^{\infty} \overline{S}_0^{(k)} (v-j) f_{k,t} (e^j) \right) Y_{k,\ell} (\psi) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} f_{k,t} (e^v) Y_{k,\ell} (\psi) = f (e^v \psi) = f (r \psi) \]

This ends the proof.

Let us remark that the sampling
\[ \{ f (\kappa) : \kappa \in \mathbb{Z}^n \} \]

is the one used in the usual multivariate generalizations of the Shannon type formula. In Theorem 13 we use though the sampling
\[ \{ f_{k,\ell} (e^j) : k \geq 0, \ \ell = 1, 2, ..., a_k, \ j \in \mathbb{Z} \} \]

which defines a new multivariate sampling paradigm.

### 4 The polysplines on parallel strips

We consider very briefly for every integer $p \geq 1$ the Wavelet Analysis generated by the polysplines on strips (with periodic conditions), and the corresponding Shannon polyspline function and Shannon formula.

We will introduce some convenient notations:

\[ x = (t, y) \in \mathbb{R}^n \]

\[ t \in \mathbb{R} \quad \text{and} \quad y \in \mathbb{T}^{n-1}, \]

\[ \mathbb{T}^{n-1} := [0, 2\pi]^{n-1}. \]

The polyspline Wavelet Analysis on strips is defined by means of the following space, cf. [9],

\[ PV_0 := \text{clos}_{L_2 (\mathbb{R}^n)} \{ f (t, y) : \Delta^p f (t, y) = 0 \quad \text{for} \ t \notin \mathbb{Z}, \quad f \in C^{2p-2} (\mathbb{R} \times \mathbb{T}^{n-1}) \}; \]

these are functions which are piecewise polyharmonic on every strip $(j, j+1) \times \mathbb{T}^{n-1}_y$, for all $j \in \mathbb{Z}$, and which are $2\pi$–periodic in the variables $y$. The functions $f \in PV_0$ permit the representation

\[ f (t, y) = \sum_{\kappa \in \mathbb{Z}^{n-1}} f_{\kappa} (t) e^{i (\kappa, y)}, \]
where for every $\kappa \in \mathbb{Z}^{n-1}$ the function $f_\kappa (t)$ is a solution to the equation

$$\left( \frac{d^2}{dt^2} - |\kappa|^2 \right)^p f_\kappa (t) = 0 \quad \text{for } t \notin \mathbb{Z},$$

$$|\kappa|^2 = \sum_j \kappa_j^2.$$

In the present case we have the vector $\Lambda = \Lambda_k = [\lambda_1, ..., \lambda_N]$ given by (33) for $k = |\kappa|$. For this $\Lambda = \Lambda_k$ and for every integer $p \geq 1$ we obtain the operator

$$L^p_{(k)} := \left( \frac{d^2}{dt^2} - k^2 \right)^p.$$

Now as in (29) we have similar definition of the cardinal exponential spline spaces, namely we put

$$V_k^0 := \left\{ f \in L_2 (\mathbb{R}) : L^p_{(k)} \left( \frac{d}{dt} \right) f (t)_{(j,j+1)} = 0, \quad \text{and } f \in C^{2p-2} (\mathbb{R}) \right\}.$$

For the vector $\Lambda_k$ defined in (33), by Proposition 7 condition (32) always holds. Hence, for every $k \geq 0$ there always exists the Shannon-Walter exponential spline $S_0^{(k)}$ defined in (35).

**Remark 14** The case $k = 0$ corresponds to the classical cardinal splines of odd degree considered by G. Walter in [22].

Since all basic lemmata have been proved above also for this type of vectors, we obtain the following Shannon type formula:

**Theorem 15** For every $f \in PV_0 \cap C (\mathbb{R}^n)$ holds the Shannon type formula

$$f (t, y') = \sum_{j=-\infty}^{\infty} \int_{\mathbb{T}^{n-1}} S_0 (t-j, y', y) f (j, y) \, dy,$$

where $S_0 (t, y, y')$ is the Shannon polyspline distribution defined by

$$S_0 (t, y, y') = \sum_{\kappa \in \mathbb{Z}^{n-1}} S_0^{(\kappa)} (t) e^{-i(\kappa, y)} e^{i(\kappa, y')}.$$

Again we may compare the usual multivariate sampling $\{ f (\kappa) : \kappa \in \mathbb{Z}^n \}$ with our sampling paradigm

$$\{ f_\kappa (j) : \kappa \in \mathbb{Z}^{n-1}, j \in \mathbb{Z} \}.$$

Finally, an idea for future research might be the error analysis of the Shannon type formulas of Theorem 13 and Theorem 15.
References


Corresponding author:
1. Ognyan Kounchev: Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 8 Acad. G. Bonchev St., 1113 Sofia, Bulgaria
   kounchev@math.bas.bg; kounchev@gmx.de
2. Hermann Render: Departamento de Matemáticas y Computación, Universidad de La Rioja, Edificio Vives, Luis de Ulloa s/n., 26004 Logroño, Spain
   render@gmx.de