The approximation order of polysplines

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Abstract

We show that the scaling spaces defined by the polysplines of order $p$ provide approximation order $2p$. For that purpose we refine the results on one-dimensional approximation order by $L$—splines obtained in [2].

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1 Introduction

In the last decade the approximation order of shift-invariant subspaces of the space $L^2 (\mathbb{R}^n)$ of all square-integrable functions on the euclidean space $\mathbb{R}^n$ has been investigated extensively, e.g. in the survey paper [10] approximately 100 references are given. The problem can be formulated in a rather general way: suppose that $(V_h)_{h \in I}$ is a family of subspaces of $L^2 (\mathbb{R}^n)$ (not necessarily shift-invariant) where $I$ is subset of $(0, \infty)$ having 0 as an accumulation point. One has to estimate the rates of decay of the approximation error

$$E (f, V_h) := \inf \\left\{ \| f - s \|_{L^2 (\mathbb{R}^n)} : s \in V_h \right\}$$  \hspace{1cm} (1)
for \( h \) tending to 0. If \( W \) is a subspace of \( L^2(\mathbb{R}^n) \) endowed with a norm \( \| \cdot \|_W \), we say that \((V_h)_{h \in I}\) provides approximation order \( m \) with respect to the norm \( \| \cdot \|_W \) if there exists a constant \( c_W \) such that for every \( f \in W \) and for every \( h \in I \)

\[
E(f, V_h) \leq c_W \cdot h^m \cdot \| f \|_W .
\]

(2)

Usually \( W \) is the potential space \( \mathcal{W}^m(\mathbb{R}^n) \) for \( m \in (0, \infty) \) defined as the subspace of those \( f \in L^2(\mathbb{R}^n) \) such that

\[
\| f \|_{\mathcal{W}^m(\mathbb{R}^n)} := (2\pi)^{-\frac{n}{2}} \left\| (1 + |\xi|^2)^{\frac{m}{2}} \hat{f}(\xi) \right\|_{L^2(\mathbb{R}^n)} < \infty .
\]

(3)

In this note we want to prove that cardinal polysplines of order \( p \) provide approximation order \( 2p \).

The motivation for the present work comes from the fact that polysplines are useful for solving multivariate interpolation problems [4], [5], [6] and they are of importance for the multivariate Wavelet Analysis, cf. the monograph [9]. Recall that a function \( S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C} \) is called a cardinal polyspline\(^1\) (on annuli) of order \( p \) if \( S \) is a \((2p-2)\)-times continuously differentiable and the restriction of \( S \) to each open annulus \( \{ x \in \mathbb{R}^n : e^l < |x| < e^{l+1} \} \) is a polyharmonic function\(^2\) of order \( p \) for \( l \in \mathbb{Z} \). The reason for calling such polysplines ”cardinal” is found in Theorem 3 where it is seen that after expanding \( S \) in a Fourier–Laplace series of spherical harmonics the coefficients \( S_{k,l}(\log r) \) are cardinal \( L \)-splines in the usual sense of the word, cf. Micchelli’s paper of 1976 [8].

Introducing a parameter \( h > 0 \), by \( P_h \) we denote the set of all functions \( S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C} \) which are \((2p-2)\)-times continuously differentiable and whose restriction to each open annulus \( A_{h,l} := \{ x \in \mathbb{R}^n : e^l < |x| < e^{h(l+1)} \} \) is a polyharmonic function of order \( p \) for \( l \in \mathbb{Z} \). Then the scaling spaces of polysplines of order \( p \), shortly \( PV_h \), are defined as the \( L^2 \)-closure of \( P_h \cap L^2(\mathbb{R}^n) \), \( h > 0 \).

The main result is the following:

**Theorem 1** The sequence \((PV_h)_{h > 0}\) provides approximation order \( 2p \) where \( p \) denotes the order of the polysplines. More precisely, there exists a constant

\(^1\) The first author introduced in 1991 polysplines in a more general setting with arbitrary interfaces, see [3] and [9].

\(^2\) Recall that a function \( f \) defined on an open set \( U \) in the euclidean space \( \mathbb{R}^n \) is polyharmonic of order \( p \) if \( f \) is \( 2p \)-times continuously differentiable and \( \Delta^p f(x) = 0 \) for all \( x \in U \) where \( \Delta \) is the Laplace operator and \( \Delta^p \) its \( p \)-th iterate.
\( C > 0 \) such that for all \( h \) with \( 0 < h < 1 \) and \( f \in L^2(\mathbb{R}^n) \) the following inequality holds

\[
\inf \left\{ \| f - g \|_{L^2(\mathbb{R}^n)} : g \in PV_h \right\} \leq C \cdot h^{2p} \cdot \left( \int_{\mathbb{R}^n} |x|^{2p} \cdot |\Delta^p f(x)|^2 \, dx \right)^{1/2}.
\]

Note that in place of the norm \((3)\) we have a semi-norm on the right-hand side which is zero on the polyharmonic functions of order \( p \).

The paper is organized as follows: in the next Section we discuss the approximation order of cardinal \( L \)-splines by using important results from [2]. In Section 3 the main result will be proven.

## 2 Approximation order of cardinal \( L \)-splines

Let us recall Theorem 4.3 in the fundamental paper [2]: Suppose that for every \( h > 0 \), the space \( S_h \) is the \( L^2(\mathbb{R}^n) \)-closure of the linear space generated by the shifts \( \varphi_h \cdot (-m), m \in \mathbb{Z}^n \) of the functions \( \varphi_h \in L^2(\mathbb{R}^n) \) (so \( S_h \) is the shift-invariant space generated by \( \varphi_h \)) and that \( V_h = \{ s \left( \frac{x}{h} \right) : s \in S_h \} \). Then the family \((V_h)_{h \in I}\) provides approximation order \( m \) with respect to the norm \( \| \cdot \|_{H^m(\mathbb{R}^n)} \) defined in \((3)\) if and only if there exists \( D > 0 \) such that for all \( h \in I \) and for almost all \( x \in C := [-\pi, \pi]^n \)

\[
|\Lambda_{\varphi_h}(x)| \leq D \cdot (h + |x|^m),
\]

where

\[
(\Lambda_{\varphi_h}(\xi))^2 := \frac{\sum_{\alpha \in \mathbb{Z}^n, \alpha \neq 0} |\hat{\varphi_h}(\xi + 2\pi \alpha)|^2}{\sum_{\beta \in \mathbb{Z}^n} |\hat{\varphi_h}(\xi + 2\pi \beta)|^2} \leq 1.
\]

We will need a refinement of that result. For our purposes it will be useful to consider instead of \((3)\) different norms. In the following we replace the function \((1 + |x|^m)\) by a measurable function \( Q(x) \) with the following properties: (i) the zero set \( Q^{-1}(0) \) of \( Q \) is a set of Lebesgue measure zero and (ii) there exists a constant \( D_1 > 0 \) such that

\[
\left| Q \left( \frac{x}{h} \right) \right| \geq D_1 \frac{1}{h^m} \quad \text{for all } x \notin C := [-\pi, \pi]^n.
\]

Suppose further that there exists a constant \( D_2 > 0 \) such that for all \( x \in C \) and for all \( 0 < h < 1 \)

\[
|\Lambda_{\varphi_h}(hx)| \leq h^m D_2 \left| Q \left( x \right) \right|.
\]
An analysis of the proof in [2] shows that then the following inequality holds
(for us the constants $D_1$ and $D_2$ defined in the formula will be very important!)

$$E(f, V_h) \leq \left( D_2 \left( \frac{2\pi}{2} \right)^{\frac{n}{2}} + \frac{1}{D_1 (2\pi)^{\frac{n}{2}}} \right) \cdot h^m \cdot \left\| Q(\xi) \tilde{f}(\xi) \right\|_{L_2(\mathbb{R}^n)}. \quad (7)$$

Let us recall some facts about $L$–splines: Let $L$ be a linear differential
operator with constant coefficients of order $N + 1$, say

$$L = M_{\Lambda} := \prod_{j=1}^{N+1} \left( \frac{d}{du} - \lambda_j \right) \text{ where } \Lambda := (\lambda_1, ..., \lambda_{N+1}). \quad (8)$$

Then a function $u : \mathbb{R} \to \mathbb{R}$ is called cardinal $L$–spline on the mesh $h\mathbb{Z}$
($h > 0$) if $u$ is $(N - 1)$-times continuously differentiable and if for every
$l \in \mathbb{Z}$ there exists $f_l \in U_L := \{ f \in C^\infty (\mathbb{R}) : Lf = 0 \}$ such that $u(t) = f_l(t)$
for all $t \in (lh, (l + 1)h)$. The set of all cardinal $L$–splines for the operator
$L = M_{\Lambda}$ on $h\mathbb{Z}$ will be denoted by $S_{h\mathbb{Z}}(\Lambda)$. The scaling spaces $V_h(\Lambda)$ are
defined by

$$V_h(\Lambda) = \left. L^2(\mathbb{R}) \right\text{-closure of } S_{h\mathbb{Z}}(\Lambda) \cap L^2(\mathbb{R}). \quad (9)$$

Let $Q_{\Lambda}$ be the basic spline which can be defined by its Fourier transform by

$$\hat{Q}_{\Lambda}(\xi) = \prod_{j=1}^{N+1} \left( \frac{e^{-\lambda_j} - e^{-i\xi}}{i\xi - \lambda_j} \right). \quad (10)$$

**Theorem 2** Let $N \in \mathbb{N}$ be fixed. Then there exists a constant $D > 0$ such
that for all $\Lambda = (\lambda_1, ..., \lambda_{N+1}) \in \mathbb{R}^{N+1}$ and for all $f \in L_2(\mathbb{R})$ the following
inequality holds

$$E(f, V_h(\Lambda)) \leq h^{N+1} \cdot D \left\| P_{\Lambda}(\xi) \tilde{f}(\xi) \right\|_{L_2(\mathbb{R})}, \quad (11)$$

where the polynomial $P_{\Lambda}(x) = \prod_{j=1}^{N+1} (ix - \lambda_j)$.

**Remark 3** Note that if we used the usual Sobolev norm (3) then we could not be able to obtain the sharp constant $D$ of inequality (11); the last is the main virtue of Theorem 2.
Proof. By the above we have to check (5) and (6). Note that for $Q := P_A$ we have the estimate

$$|P_A \left( \frac{x}{h} \right)|^2 = \prod_{j=1}^{N+1} \left( \frac{x_j}{h} \right)^2 + \lambda_j^2 \geq \pi^{2(N+1)} \frac{1}{h^{2(N+1)}}$$  \hspace{1cm} (12)

for all $|x| \geq \pi$ and for all $h > 0$. Hence it suffices to show that

$$|\Lambda_{\varphi_h} (h\xi)|^2 \leq h^{2(N+1)} \left| P_A (\xi) \right|^2 \sum_{\alpha \in \mathbb{Z}, \alpha \neq 0} \frac{1}{\pi |\alpha|^2(N+1)}$$  \hspace{1cm} (13)

The trivial inequality $(\Lambda_{\varphi_h} (\xi))^2 \leq \frac{\sum_{\alpha \in \mathbb{Z}, \alpha \neq 0} |\varphi_h (\xi + 2\pi \alpha)|^2}{|\varphi_h (\xi)|^2}$ and the estimate

$$\frac{|\varphi_h (\xi + 2\pi \alpha)|^2}{|\varphi_h (\xi)|^2} = \frac{|\tilde{Q}_{hA} (\xi + 2\pi \alpha)|^2}{|\tilde{Q}_{hA} (\xi)|^2} = \prod_{j=1}^{N+1} \left| i \frac{\lambda_j}{(\xi + 2\pi \alpha) - h \lambda_j} \right|^2$$

yields

$$|\Lambda_{\varphi_h} (h\xi)|^2 \leq h^{2(N+1)} \prod_{j=1}^{N+1} (\xi^2 + \lambda_j^2) \sum_{\alpha \in \mathbb{Z}, \alpha \neq 0} \prod_{j=1}^{N+1} \frac{1}{(h\xi + 2\pi \alpha)^2 + h^2 \lambda_j^2}.$$  \hspace{1cm} (13)

Since $(h\xi + 2\pi \alpha)^2 + h^2 \lambda_j^2 \geq (h\xi + 2\pi \alpha)^2 \geq (2\pi |\alpha| - |h\xi|)^2$ we obtain for $0 < h < 1$ and $|\xi| \leq \pi$ the estimate $2\pi |\alpha| - |h\xi| \geq \pi |\alpha|$ (since $\alpha \neq 0$) arriving at (13). \textcircled{QED}

3 The approximation order of Polysplines

Let $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$ be the unit sphere. Each $x \in \mathbb{R}^n$ will be written in spherical coordinates $x = r\theta$ with $r \geq 0$ and $\theta \in S^{n-1}$. Recall that a function $Y : S^{n-1} \to \mathbb{C}$ is a spherical harmonic of degree $k \in \mathbb{N}_0$ if there exists a homogeneous harmonic polynomial $P (x)$ of degree $k$ such that $P (\theta) = Y (\theta)$ for all $\theta \in S^{n-1}$. The set $\mathcal{H}_k$ of all spherical harmonics of degree exactly $k$ is a linear space of dimension $a_k := \dim \mathcal{H}_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$. We denote by $Y_{k,l}$ with $l = 1, 2, ..., a_k$ a base of $\mathcal{H}_k$. For a detailed account we refer to Stein–Weiss [12].
Let \( u : (R_1, R_2) \to \mathbb{C} \) be infinitely differentiable and \( Y_k \in \mathcal{H}_k \). Then it is well known that \( \Delta (u (r) Y_k (\theta)) = Y_k (\theta) L_{(k)} u (r) \) where we have put
\[
L_{(k)} = \frac{d^2}{dr^2} + \frac{n - 1}{r} \frac{d}{dr} - \frac{k (k + n - 2)}{r^2}.
\] (14)
By iteration we have \( \Delta^p u = Y_k (\theta) \cdot [L_{(k)}]^p u (r) \). Let us put for convenience
\[
\Lambda_+ (k, p) := \{k, k + 2, ..., k + 2p - 2\},
\]
\[
\Lambda_- (k, p) := \{-k + n + 2, -k - n + 4, ..., -k - n + 2p\}.
\]
The space of solutions of the equation \( L_{(k)}^p f (r) = 0 \) which are \( C^\infty \) for \( r > 0 \) is generated by a simple basis: for \( j \in \Lambda_+ (k, p) \cup \Lambda_- (k, p) \) the function \( r^j \) is clearly a solution, while for \( j \in \Lambda_+ (k, p) \cap \Lambda_- (k, p) \) we obtain a second solution \( r^j \log r \). It will be convenient to make a transform of the variable \( r \) to \( u = \log r \). Then a solution of the form \( r^j \) will be transformed to \( e^{jv} \) and a solution of the form \( r^j \log r \) is transformed to \( ve^{jv} \). We see immediately that all solutions to the equation \( L_{(k)}^p f (r) = 0 \) are transformed to solutions of the equation \( M_{\Lambda_+(k)} g (v) = 0 \) where \( M_{\Lambda_+(k)} \) is defined by (8) with respect to the vector
\[
\Lambda_k := (k, k + 2, ..., k + (p - 1), -(k + n) + 2, ..., -(k + n) + 2p) \). (15)
The dependence on the parameter \( p \) and \( n \) will be suppressed throughout the paper.

A proof of the following can be found in [6], [9, Theorem 9.7].

**Theorem 4** Let \( S : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) be a polyspline of order \( p \). Then the Laplace-Fourier coefficient \( S_{k, l} : \mathbb{R} \to \mathbb{R} \) defined by
\[
S_{k, l} (v) := \int_{S^{n-1}} S (e^{v \theta}) Y_{k, l} (\theta) d\theta
\] (16)
is a cardinal \( L \)-spline with respect to the linear differential operator \( M_{\Lambda_+(k)} \).

We want to characterize the \( L^2 (\mathbb{R}^n) \)-closure \( PV_h \). It is a temptation to assume that for \( S \in PV_h \) the Fourier-Laplace coefficient defined through formula (16) will be in \( V_h (\Lambda_k) \), i.e. in the closure of \( S_{k, l} (\Lambda_k) \cap L^2 (\mathbb{R}) \). This is not true since the transformation rule will give us an additional weight for \( f \in L^2 (\mathbb{R}^n) \):
\[
\int_{\mathbb{R}^n} |f (x)|^2 dx = \int_0^\infty \int_{S^{n-1}} |f (r \theta)|^2 r^{n-1} d\theta dr.
\] (17)
Fortunately, this problem can be easily solved, see e.g. [7].
Theorem 5 Define \( \overline{\Lambda}_k = \left( \frac{\alpha_1}{2}, \ldots, \frac{\alpha_p}{2} \right) + \Lambda_k \). Then for each \( k \in \mathbb{N}_0, l = 1, \ldots, a_k \), the following map, defined on \( P_h \cap L^2(\mathbb{R}^n) \) by
\[
S \mapsto S_{k,l}(v) := e^{2\pi iv} \int_{S^{n-1}} S(e^v \theta) Y_{k,l}(\theta) \, d\theta,
\]  
maps into \( S_{k,l}(\overline{\Lambda}_k) \cap L^2(\mathbb{R}^n, dv) \), and by continuity it can be extended to a map from \( PV_j \to V_j(\overline{\Lambda}_k) \). Further, \( PV_j \) is isomorphic to \( V_h := \bigoplus_{k \in \mathbb{N}_0, l = 1, \ldots, a_k} V_h(\overline{\Lambda}_k) \).

Proof of Theorem 1. Let \( f \in L^2(\mathbb{R}^n) \) and \( g \in PV_h \). Then by the transformation rule (17)
\[
\|f - g\|_{L^2(\mathbb{R}^n)}^2 = \int_0^\infty \int_{S^{n-1}} |f(r\theta) - g(r\theta)|^2 r^{n-1} \, d\theta \, dr.
\]  
Let \( f_{k,l} \) and \( g_{k,l} \) be the Laplace Fourier coefficients of \( f \) and \( g \) respectively as defined in (16). Note that \( v \mapsto g_{k,l}(v) := e^{2\pi iv} g_{k,l}(e^v) \) is in \( V_h(\overline{\Lambda}_k) \). Since \( Y_{k,l}(\theta) \) constitutes an orthonormal basis we obtain
\[
\|f - g\|_{L^2(\mathbb{R}^n)}^2 = \sum_{k=0}^\infty \sum_{l=1}^{a_k} \int_{-\infty}^\infty |f_{k,l}(e^v) - g_{k,l}(e^v)|^2 e^{2\pi iv} \, dv.
\]  
Taking \( g \in PV_h \) such that \( \|f - g\|_{L^2(\mathbb{R}^n)}^2 \) attains the infimum is equivalent to take for each \( k \in \mathbb{N}, l = 1, \ldots, a_k \) the expression
\[
\int_{-\infty}^\infty |e^{2\pi iv} f_{k,l}(e^v) - g_{k,l}(e^v)|^2 \, dv
\]  
to be minimal, where \( g_{k,l} \in V_j(\overline{\Lambda}_k) \). Theorem 2 applied to \( \Lambda = \overline{\Lambda}_k \) (hence \( N + 1 = 2p \)) shows that there exists a constant \( C_p > 0 \) which only depends on \( p \) (and not on the values \( \lambda_j \) in \( \overline{\Lambda}_k \)) such that
\[
E\left( e^{2\pi iv} f_{k,l}(e^v), V_h(\overline{\Lambda}_k) \right) \leq h^{2p} \cdot C_p \|P_{\overline{\Lambda}_k} \cdot e^{2\pi iv} f_{k,l}(e^v)\|_{L^2(\mathbb{R})}.
\]  
Put \( G_{k,l}(v) := e^{2\pi iv} f_{k,l}(e^v) \). A simple computation (using Parseval’s identity and the fact that differentiation becomes multiplication via Fourier transform) shows that
\[
\frac{1}{2\pi} \|P_{\overline{\Lambda}_k} \cdot G_{k,l}\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^\infty |M_{\overline{\Lambda}_k} G_{k,l}(v)|^2 \, dv.
\]  
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A calculation shows that \( M_{\Lambda_k} (e^{\frac{2\pi i}{n} f_{k,l}(e^v)}) = e^{\frac{2\pi i}{n} M_{\Lambda_k} (f(e^v))} \). Then (20) and (21) yield

\[
E(f, PV_k)^2 \leq h^{A_p} \cdot 2\pi C_p^{2} \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \left\| e^{\frac{2\pi i}{n} M_{\Lambda_k} (f(e^v))} \right\|_{L^2(\mathbb{R}^n)}^2.
\]

The next theorem applied to the case \( p = q \) finishes the proof. ■

**Theorem 6** Let \( p, q \in \mathbb{N}_0 \) and define \( \left\| f(x) \right\|_{q,p}^2 := \int \left\| x^{2q} \cdot \Delta^p f(x) \right\|^2 \, dx \) for \( f \in L^2_2(\mathbb{R}^n) \). Then

\[
\left\| f(x) \right\|_{q,p}^2 = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int \left| e^{(2q-2p+\frac{n}{2})} M_{\Lambda_k} (f_{k,l}(e^v)) \right|^2 \, dv
\]

where \( f_{k,l}(r) \) are the Laplace-Fourier coefficients of \( f \) defined as in equality (16).

**Proof.** Assume that \( f(r\theta) = f_{k,l}(r) Y_{k,l}(\theta) \). Since \( \Delta^p f(x) = L^p_k f_{k,l}(r) Y_{k,l}(\theta) \) we obtain

\[
\left\| f(x) \right\|_{q,p}^2 = \int_0^\infty \int_{S^{n-1}} \left| r^{2q} L^p_k f_{k,l}(r) Y_{k,l}(\theta) \right|^2 r^{n-1} \, dr \, d\theta.
\]

The integration over \( \theta \) only gives a factor 1. Now we change the variable \( r = e^v \) and apply the identity \((L^p_k f_{k,l})(e^v) = e^{-2\nu p} M_{\Lambda_k} (f_{k,l}(e^v))\), see e.g. Theorem 10.34 in [9]. Then

\[
\left\| f(x) \right\|_{q,p}^2 = \int \left| e^{2\nu} e^{-2\nu p} M_{\Lambda_k} (f_{k,l}(e^v)) \right|^2 e^{\nu v} \, dv.
\]

Finally we see that for arbitrary \( f \in L^2_2(\mathbb{R}^n) \) the result follows via the orthogonal decomposition of \( f \) in spherical harmonics. ■

**References**


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