The approximation order of polysplines

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Abstract

We show that the scaling spaces defined by the polysplines of order p provide approximation order $2p$. For that purpose we refine the results on one–dimensional approximation order by $L-$splines obtained in [2].

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1 Introduction

In the last decade the approximation order of shift-invariant subspaces of the space $L^2(\mathbb{R}^n)$ of all square-integrable functions on the euclidean space $\mathbb{R}^n$ has been investigated extensively, e.g. in the survey paper [10] approximately 100 references are given. The problem can be formulated in a rather general way: suppose that $(V_h)_{h\in I}$ is a family of subspaces of $L^2(\mathbb{R}^n)$ (not necessarily shift-invariant) where I is subset of $(0,\infty)$ having 0 as an accumulation point. One has to estimate the rates of decay of the approximation error

$$E(f, V_h) := \inf\left\{ \|f - s\|_{L^2(\mathbb{R}^n)} : s \in V_h \right\} \tag{1}$$

for h tending to 0. If W is a subspace of $L^2(\mathbb{R}^n)$ endowed with a norm $\|\cdot\|_W$ we say that $(V_h)_{h\in I}$ *provides approximation order* m with respect to the norm $\|\cdot\|_W$ if there exists a constant c_W such that for every $f \in W$ and for every $h \in I$

$$E(f, V_h) \leq c_W \cdot h^m \cdot \|f\|_W \ . \tag{2}$$

Usually W is the potential space $W_2^m(\mathbb{R}^n)$ for $m \in (0, \infty)$ defined as the subspace of those $f \in L^2(\mathbb{R}^n)$ such that

$$\|f\|_{W_2^m(\mathbb{R}^n)} := (2\pi)^{-\frac{n}{2}} \left\| (1 + |\xi|)^m \widehat{f}(\xi) \right\|_{L^2(\mathbb{R}^n)} < \infty. \tag{3}$$

In this note we want to prove that cardinal polysplines of order p provide approximation order $2p$.

The motivation for the present work comes from the fact that polysplines are useful for solving multivariate interpolation problems [4], [5], [6] and they are of importance for the multivariate Wavelet Analysis, cf. the monograph [9]. Recall that a function $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called a *cardinal polyspline*[1] *(on annuli) of order* p if S is a $(2p-2)$-times continuously differentiable and the restriction of S to each open annulus $\left\{x \in \mathbb{R}^n : e^l < |x| < e^{l+1}\right\}$ is a polyharmonic function[2] of order p for $l \in \mathbb{Z}$. The reason for calling such polysplines "cardinal" is found in Theorem 3 where it is seen that after expanding S in a Fourier–Laplace series of spherical harmonics the coefficients $S_{k,l}(\log r)$ are cardinal $L-$splines in the usual sense of the word, cf. Micchelli's paper of 1976 [8].

Introducing a parameter $h > 0$, by P_h we denote the set of all functions $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ which are $(2p-2)$-times continuously differentiable and whose restriction to each open annulus $A_{h,l} := \left\{x \in \mathbb{R}^n : e^{hl} < |x| < e^{h(l+1)}\right\}$ is a polyharmonic function of order p for $l \in \mathbb{Z}$. Then the *scaling spaces of polysplines of order* p, shortly PV_h, are defined as the L^2-closure of $P_h \cap L^2(\mathbb{R}^n)$, $h > 0$.

The main result is the following:

Theorem 1 *The sequence* $(PV_h)_{h>0}$ *provides approximation order* $2p$ *where* p *denotes the order of the polysplines. More precisely, there exists a constant*

[1] The first author introduced in 1991 polysplines in a more general setting with arbitrary interfaces, see [3] and [9].

[2] Recall that a function f defined on an open set U in the euclidean space $\mathbb{R}^n$ is *polyharmonic of order* p if f is $2p$–times continuously differentiable and $\Delta^p f(x) = 0$ for all $x \in U$ where Δ is the Laplace operator and Δ^p its $p-$th iterate.

$C > 0$ such that for all h with $0 < h < 1$ and $f \in L^2(\mathbb{R}^n)$ the following inequality holds

$$\inf\left\{\|f-g\|_{L^2(\mathbb{R}^n)} : g \in PV_h\right\} \leq C \cdot h^{2p} \cdot \left(\int_{\mathbb{R}^n} \left|\ |x|^{2p} \cdot \Delta^p f(x)\right|^2 dx\right)^{\frac{1}{2}}.$$

Note that in place of the norm (3) we have a semi–norm on the right–hand side which is zero on the polyharmonic functions of order p.

The paper is organized as follows: in the next Section we discuss the approximation order of cardinal $L-$splines by using important results from [2]. In Section 3 the main result will be proven.

2 Approximation order of cardinal $L-$splines

Let us recall Theorem 4.3 in the fundamental paper [2]: Suppose that for every $h > 0$, the space S_h is the $L^2(\mathbb{R}^n)$–closure of the linear space generated by the shifts $\varphi_h(\cdot - m), m \in \mathbb{Z}^n$ of the functions $\varphi_h \in L^2(\mathbb{R}^n)$ (so S_h is the shift-invariant space generated by φ_h) and that $V_h = \left\{s\left(\frac{\cdot}{h}\right) : s \in S_h\right\}$. Then the family $(V_h)_{h\in I}$ provides approximation order m with respect to the norm $\|\cdot\|_{W_2^m(\mathbb{R}^n)}$defined in (3) if and only if there exists $D > 0$ such that for all $h \in I$ and for almost all $x \in C := [-\pi, \pi]^n$

$$|\Lambda_{\varphi_h}(x)| \leq D \cdot (h + |x|^m), \tag{4}$$

where

$$(\Lambda_{\varphi_h}(\xi))^2 := \frac{\sum_{\alpha\in\mathbb{Z}^n,\alpha\neq 0} \left|\widehat{\varphi_h}(\xi + 2\pi\alpha)\right|^2}{\sum_{\beta\in\mathbb{Z}^n} \left|\widehat{\varphi_h}(\xi + 2\pi\beta)\right|^2} \leq 1.$$

We will need a refinement of that result. For our purposes it will be useful to consider instead of (3) different norms. In the following we replace the function $(1 + |x|)^m$ by a measruable function $Q(x)$ with the following properties: (i) the zero set $Q^{-1}(0)$ of Q is a set of Lebesgue measure zero and (ii) there exists a constant $D_1 > 0$ such that

$$\left|Q\left(\frac{x}{h}\right)\right| \geq D_1 \frac{1}{h^m} \qquad \text{for all } x \notin C := [-\pi, \pi]^n. \tag{5}$$

Suppose further that there exists a constant $D_2 > 0$ such that for all $x \in C$ and for all $0 < h < 1$

$$|\Lambda_{\varphi_h}(hx)| \leq h^m D_2 |Q(x)|. \tag{6}$$

An analysis of the proof in [2] shows that then the following inequality holds (for us the constants D_1 and D_2 defined in the formula will be very important!)

$$E(f,V_h) \leq \left(D_2 (2\pi)^{\frac{n}{2}} + \frac{1}{D_1 (2\pi)^{\frac{n}{2}}} \right) \cdot h^m \cdot \left\| Q(\xi) \widehat{f}(\xi) \right\|_{L_2(\mathbb{R}^n)}. \qquad (7)$$

Let us recall some facts about $L-$splines: Let L be a linear differential operator with constant coefficients of order $N+1$, say

$$L = M_\Lambda := \prod_{j=1}^{N+1} \left(\frac{d}{dv} - \lambda_j \right) \text{ where } \Lambda := (\lambda_1, ..., \lambda_{N+1}). \qquad (8)$$

Then a function $u : \mathbb{R} \to \mathbb{R}$ is called *cardinal $L-$spline on the mesh $h\mathbb{Z}$* ($h > 0$) if u is $(N-1)$-times continuously differentiable and if for every $l \in \mathbb{Z}$ there exists $f_l \in U_L := \{ f \in C^\infty(\mathbb{R}) : Lf = 0 \}$ such that $u(t) = f_l(t)$ for all $t \in (lh, (l+1)h)$. The set of all cardinal $L-$splines for the operator $L = M_\Lambda$ on $h\mathbb{Z}$ will be denoted by $\mathcal{S}_{h\mathbb{Z}}(\Lambda)$. The scaling spaces $V_h(\Lambda)$ are defined by

$$V_h(\Lambda) = L^2(\mathbb{R}) \text{ -closure of } \mathcal{S}_{h\mathbb{Z}}(\Lambda) \cap L^2(\mathbb{R}). \qquad (9)$$

Let Q_Λ be the basic spline which can be defined by its Fourier transform by

$$\widehat{Q_\Lambda}(\xi) = \frac{\prod_{j=1}^{N+1} \left(e^{-\lambda_j} - e^{-i\xi} \right)}{\prod_{j=1}^{N+1} (i\xi - \lambda_j)}. \qquad (10)$$

Theorem 2 *Let $N \in \mathbb{N}$ be fixed. Then there exists a constant $D > 0$ such that for all $\Lambda = (\lambda_1, ..., \lambda_{N+1}) \in \mathbb{R}^{N+1}$ and for all $f \in L_2(\mathbb{R})$ the following inequality holds*

$$E(f, V_h(\Lambda)) \leq h^{N+1} \cdot D \left\| P_\Lambda(\xi) \widehat{f}(\xi) \right\|_{L_2(\mathbb{R})}, \qquad (11)$$

where the polynomial $P_\Lambda(x) = \prod_{j=1}^{N+1} (ix - \lambda_j)$.

Remark 3 *Note that if we used the usual Sobolev norm (3) then we could not be able to obtain the sharp constant D of inequality (11); the last is the main virtue of Theorem 2.*

Proof. By the above we have to check (5) and (6). Note that for $Q := P_\Lambda$ we have the estimate

$$\left|P_\Lambda\left(\frac{x}{h}\right)\right|^2 = \prod_{j=1}^{N+1}\left(\left(\frac{x}{h}\right)^2 + \lambda_j^2\right) \geq \pi^{2(N+1)}\frac{1}{h^{2(N+1)}} \tag{12}$$

for all $|x| \geq \pi$ and for all $h > 0$. Hence it suffices to show that

$$\left|\Lambda_{\varphi_h}(h\xi)\right|^2 \leq h^{2(N+1)}\left|P_\Lambda(\xi)\right|^2 \sum_{\alpha\in\mathbb{Z},\alpha\neq 0}\frac{1}{(\pi|\alpha|)^{2(N+1)}} \tag{13}$$

The trivial inequality $\left(\Lambda_{\varphi_h}(\xi)\right)^2 \leq \frac{\sum_{\alpha\in\mathbb{Z},\alpha\neq 0}\left|\widehat{\varphi_h}(\xi+2\pi\alpha)\right|^2}{\left|\widehat{\varphi_h}(\xi)\right|^2}$ and the estimate

$$\frac{\left|\widehat{\varphi_h}(\xi+2\pi\alpha)\right|^2}{\left|\widehat{\varphi_h}(\xi)\right|^2} = \frac{\left|\widehat{Q_{h\Lambda}}(\xi+2\pi\alpha)\right|^2}{\left|\widehat{Q_{h\Lambda}}(\xi)\right|^2} = \prod_{j=1}^{N+1}\left|\frac{i\xi - h\lambda_j}{i(\xi+2\pi\alpha) - h\lambda_j}\right|^2$$

yields

$$\left|\Lambda_{\varphi_h}(h\xi)\right|^2 \leq h^{2(N+1)}\prod_{j=1}^{N+1}\left(\xi^2+\lambda_j^2\right)\sum_{\alpha\in\mathbb{Z},\alpha\neq 0}\prod_{j=1}^{N+1}\frac{1}{(h\xi+2\pi\alpha)^2 + h^2\lambda_j^2}.$$

Since $(h\xi+2\pi\alpha)^2 + h^2\lambda_j^2 \geq (h\xi+2\pi\alpha)^2 \geq (2\pi|\alpha| - |h\xi|)^2$ we obtain for $0 < h < 1$ and $|\xi| \leq \pi$ the estimate $2\pi|\alpha| - |h\xi| \geq \pi|\alpha|$ (since $\alpha \neq 0$) arriving at (13). ■

3 The approximation order of Polysplines

Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$ be the unit sphere. Each $x \in \mathbb{R}^n$ will be written in spherical coordinates $x = r\theta$ with $r \geq 0$ and $\theta \in \mathbb{S}^{n-1}$. Recall that a function $Y : \mathbb{S}^{n-1} \to \mathbb{C}$ is a *spherical harmonic* of degree $k \in \mathbb{N}_0$ if there exists a homogeneous harmonic polynomial $P(x)$ of degree k such that $P(\theta) = Y(\theta)$ for all $\theta \in \mathbb{S}^{n-1}$. The set $\mathfrak{H}_k$ of all spherical harmonics of degree exactly k is a linear space of dimension $a_k := \dim\mathfrak{H}_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$. We denote by $Y_{k,l}$ with $l = 1, 2, ..., a_k$ a base of $\mathfrak{H}_k$. For a detailed account we refer to *Stein–Weiss* [12].

Let $u : (R_1, R_2) \rightarrow \mathbb{C}$ be infinitely differentiable and $Y_k \in \mathfrak{H}_k$. Then it is well known that $\Delta\left(u\left(r\right) Y_k\left(\theta\right)\right) = Y_k\left(\theta\right) L_{(k)} u\left(r\right)$ where we have put

$$L_{(k)} = \frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} - \frac{k\left(k+n-2\right)}{r^2}. \tag{14}$$

By iteration we have $\Delta^p u = Y_k\left(\theta\right) \cdot \left[L_{(k)}\right]^p u\left(r\right)$. Let us put for convenience

$$\Lambda_+\left(k,p\right) := \left\{k, k+2, ..., k+2p-2\right\},$$
$$\Lambda_-\left(k,p\right) := \left\{-k-n+2, -k-n+4, ..., -k-n+2p\right\}.$$

The space of solutions of the equation $L_{(k)}^p f\left(r\right) = 0$ which are C^∞ for $r > 0$ is generated by a simple basis: for $j \in \Lambda_+\left(k,p\right) \cup \Lambda_-\left(k,p\right)$ the function r^j is clearly a solution, while for $j \in \Lambda_+(k,p) \cap \Lambda_-(k,p)$ we obtain a second solution $r^j \log r$. It will be convenient to make a transform of the variable r to $v = \log r$. Then a solution of the form r^j will be transformed to e^{jv} and a solution of the form $r^j \log r$ is transformed to ve^{jv}. We see immediately that all solutions to the equation $L_{(k)}^p f\left(r\right) = 0$ are transformed to solutions of the equation $M_{\Lambda(k)} g(v) = 0$ where $M_{\Lambda(k)}$ is defined by (8) with respect to the vector

$$\Lambda_k := \left(k, k+2, ..., k+2\left(p-1\right), -\left(k+n\right)+2, ..., -\left(k+n\right)+2p\right). \tag{15}$$

The dependence on the parameter p and n will be suppressed throughout the paper.

A proof of the following can be found in [6], [9, Theorem 9.7].

Theorem 4 *Let $S : \mathbb{R}^{\,n} \setminus \{0\} \rightarrow \mathbb{R}$ be a polyspline of order p. Then the Laplace-Fourier coefficient $S_{k,l} : \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$S_{k,l}\left(v\right) := \int_{\mathbb{S}^{n-1}} S\left(e^v \theta\right) Y_{k,l}\left(\theta\right) d\theta \tag{16}$$

is a cardinal $L-$spline with respect to the linear differential operator $M_{\Lambda(k)}$.

We want to characterize the $L^2\left(\mathbb{R}^n\right)$-closure PV_h. It is a temptation to assume that for $S \in PV_h$ the Fourier-Laplace coefficient defined through formula (16) will be in $V_h\left(\Lambda_k\right)$, i.e. in the closure of $\mathcal{S}_{h\mathbb{Z}}\left(\Lambda_k\right) \cap L_2\left(\mathbb{R}\right)$. This is *not true* since the transformation rule will give us an additional weight for $f \in L_2\left(\mathbb{R}^n\right)$:

$$\int_{\mathbb{R}^n} \left|f\left(x\right)\right|^2 dx = \int_0^\infty \int_{\mathbb{S}^{n-1}} \left|f\left(r\theta\right)\right|^2 r^{n-1} d\theta dr. \tag{17}$$

Fortunately, this problem can be easily solved, see e.g. [7].

Theorem 5 *Define* $\overline{\Lambda_k} = \left(\frac{n}{2}, ..., \frac{n}{2}\right) + \Lambda_k$. *Then for each* $k \in \mathbb{N}_0, l = 1, ..., a_k$, *the following map, defined on* $P_h \cap L^2(\mathbb{R}^n)$ *by*

$$S \longmapsto \overline{S}_{k,l}(v) := e^{\frac{n}{2}v} \int_{\mathbb{S}^{n-1}} S(e^v\theta) Y_{k,l}(\theta) d\theta, \tag{18}$$

maps into $\mathcal{S}_{h\mathbb{Z}}\left(\overline{\Lambda_k}\right) \cap L^2(\mathbb{R}, dv)$, *and by continuity it can be extended to a map from* $PV_j \to V_j\left(\overline{\Lambda_k}\right)$. *Further,* PV_h *is isomorphic to* $V_h := \bigoplus_{k\in\mathbb{N}_0, l=1,...,a_k} V_h\left(\overline{\Lambda_k}\right)$.

Proof of Theorem 1. Let $f \in L^2(\mathbb{R}^n)$ and $g \in PV_h$. Then by the tranformation rule (17)

$$\|f - g\|^2_{L^2(\mathbb{R}^n)} = \int_0^\infty \int_{\mathbb{S}^{n-1}} |f(r\theta) - g(r\theta)|^2 r^{n-1} d\theta dr. \tag{19}$$

Let $f_{k,l}$ and $g_{k,l}$ be the Laplace Fourier coefficients of f and g respectively as defined in (16). Note that $v \longmapsto \overline{g_{k,l}}(e^v) := e^{\frac{n}{2}v} g_{k,l}(e^v)$ is in $V_h\left(\overline{\Lambda_k}\right)$. Since $Y_{k,l}(\theta)$ constitutes an orthonormal basis we obtain

$$\|f - g\|^2_{L^2(\mathbb{R}^n)} = \sum_{k=0}^\infty \sum_{l=1}^{a_k} \int_{-\infty}^\infty |f_{k,l}(e^v) - g_{k,l}(e^v)|^2 e^{nv} dv. \tag{20}$$

Taking $g \in PV_h$ such that $\|f - g\|^2_{L^2(\mathbb{R}^n)}$ attains the ifimum is equivalent to take for each $k \in \mathbb{N}$, $l = 1, ..., a_k$ the expression

$$\int_{-\infty}^\infty \left|e^{\frac{n}{2}v} f_{k,l}(e^v) - \overline{g_{k,l}}(e^v)\right|^2 dv$$

to be minimal, where $\overline{g_{k,l}} \in V_j\left(\overline{\Lambda_k}\right)$. Theorem 2 applied to $\Lambda = \overline{\Lambda_k}$ (hence $N + 1 = 2p$) shows that there exists a constant $C_p > 0$ which only depends on p (and not on the values λ_j in $\overline{\Lambda_k}$) such that

$$E\left(e^{\frac{n}{2}v} f_{k,l}(e^v), V_h\left(\overline{\Lambda_k}\right)\right) \leq h^{2p} \cdot C_p \left\| P_{\overline{\Lambda_k}} \cdot e^{\frac{n}{2}v} \widehat{f_{k,l}(e^v)} \right\|_{L_2(\mathbb{R})}. \tag{21}$$

Put $G_{k,l}(v) := e^{\frac{n}{2}v} f_{k,l}(e^v)$. A simple computation (using Parseval's identity and the fact that differentiation becomes multiplication via Fourier transform) shows that

$$\frac{1}{2\pi} \left\| P_{\overline{\Lambda_k}} \cdot \widehat{G_{k,l}} \right\|^2_{L_2(\mathbb{R})} = \int_{-\infty}^\infty \left| M_{\overline{\Lambda_k}} G_{k,l}(v) \right|^2 dv.$$

A calculation shows that $M_{\overline{\Lambda_k}}\left(e^{\frac{n}{2}v} f_{k,l}\left(e^{v}\right)\right) = e^{\frac{n}{2}v} M_{\Lambda_k}\ \left(f\left(e^{v}\right)\right).$ Then (20) and (21) yield

$$E\left(f, PV_h\right)^2 \leq h^{4p} \cdot 2\pi C_p^2 \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \left\| \ e^{\frac{n}{2}v} M_{\Lambda_k}\ \left(f\left(e^{v}\right)\right) \right\|_{L_2(\mathbb{R}^n)}^2 .$$

The next theorem applied to the case $p = q$ finishes the proof. ■

Theorem 6 *Let $p, q \in N_0$ and define $\|f(x)\|_{q,p}^2 := \int \left| |x|^{2q} \cdot \Delta^p f(x) \right|^2 dx$ for $f \in L_2(\mathbb{R}^n)$. Then*

$$\|f(x)\|_{q,p}^2 = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int \left| e^{v\left(2q-2p+\frac{n}{2}\right)} M_{\Lambda_k}\ \left(f_{k,l}\left(e^{v}\right)\right) \right|^2 dv$$

where $f_{k,l}(r)$ are the Laplace-Fourier coefficients of f defined as in equality (16).

Proof. Assume that $f(r\theta) = f_{k,\ell}(r)\, Y_{k,\ell}(\theta)$. Since $\Delta^p f(x) = L_k^p f_{k,\ell}(r)\, Y_{k,\ell}(\theta)$ we obtain

$$\|f(x)\|_{q,p}^2 = \int_0^{\infty} \int_{\mathbb{S}^{n-1}} \left| r^{2q} L_{(k)}^p f_{k,\ell}(r)\, Y_{k,\ell}(\theta) \right|^2 r^{n-1} dr d\theta.$$

The integration over θ only gives a factor 1. Now we change the variable $r = e^v$ and apply the identity $\left(L_k^p f_{k,l}\right)\left(e^{v}\right) = e^{-2vp} M_{\Lambda_k}\ \left(f_{k,l}\left(e^{v}\right)\right)$, see e.g. Theorem 10.34 in [9]. Then

$$\|f(x)\|_{q,p}^2 \ = \int \left| e^{2vq} e^{-2vp} M_{\Lambda_k}\ \left(f_{k,l}\left(e^{v}\right)\right) \right|^2 e^{nv} dv.$$

Finally we see that for arbitrary $f \in L^2(\mathbb{R}^n)$ the result follows via the orthogonal decomposition of f in spherical harmonics. ■

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