Polyharmonically exact formula of Euler-Maclaurin, multivariate Bernoulli functions, and Poisson type formula

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ABSTRACT. In the present paper we find multivariate Bernoulli functions and two formulas of Euler-Maclaurin type.

RÉSUMÉ. Dans cet article nous introduisons fonctions multivariables de Bernoulli et deux formules de type d'Euler-Maclaurin.

Version française abrégée:

Nous donnons ici une généralisation multidimensionnelle de la formule d'Euler-Maclaurin qui est basée sur une généralisation en plusieurs variables des fonctions de Bernoulli en une variable.

L'idée centrale de cette Note est la généralisation de la propriété de *Reproduction par Rapport aux Domaines Voisins (RRDV)* qui est une propriété remarquable et charactéristique de la formule d'Euler-Maclaurin unidimensionnelle (1).

Dans la Note présente on approche les integrales $d-$dimensionelles sur le cube unitaire D_{uc} par les combinaisons linéaires d'intégrales sur des surfaces de dimensions plus petites, respectivement 0 (des valeurs fonctionnelles), $1, ..., d-1$.

La propriété *RRDV est conservée.* Les formules généralisées d'Euler-Maclaurin sont exactes pour les fonctions polyharmoniques d'un degré donné (c'est un espace de dimension infinie), ou pour les solutions d'une autre équation différentielle partielle.

C'est une manifestation du paradigme polyharmonique qui trouve beaucoup d'applications dans les théories d'approximation et splines multivariables, voir [7], [8], [9].

Nous avons deux formules d'intégration approximative de type d'Euler-Maclaurin. La première, (11),(12),(13),(14), se base sur les fonctions pairs de Bernoulli, $P_{2k,d}(x)$, $k = 1, ..., m$. Cette formule est exacte pour les solutions de l'équation polyharmonique $\Delta^m f(x) = 0,\ x \in D_{uc}$.

La seconde formule, (15),(16),(17),(18), se base sur les fonctions de Bernoulli d'ordre impair, $P_{2k+1,d}(x)$, $k = 1, ..., m$. Cette formule d'intégration approximative est exacte pour les solutions de l'équation $\Delta^m \frac{\partial^d}{\partial x_1 \ldots \partial x_d} f(x) = 0,\ x \in D_{uc}$.

Remarquons que les deux formules, (11) et (15), possèdent la propriété $RRDV$ pareillement àu cas unidimensionnel. Cette propriété assure une complexité minimale calculataire par analogie à la formule d'Euler-Maclaurin unidimensionnelle.

Les deux formules d'Euler-Maclaurin engendrent des formules du type de Poisson (d'ordre pair et impair). Nous donnons ici la formule qui correspond au cas impair, voir (19). La formule de Poisson (19) est exacte pour les polynômes trigonométriques de degré $n-1$, où n est l'ordre de la règle composée trapézoïdalle.

On peux trouver les résultats complets dans [2].

Main text (English).

One of the most beautiful devices of classical analysis which has found numerous applications in number theory and divergent series [1, p. 7 in Vol. 1, Chapters 7,8], [6, Chapter 13], and which plays profound role in numerical integration [3, Chapter 2.9], is the Euler-Maclaurin formula. In the present paper we provide a multivariate generalization of the Euler-Maclaurin formula which is based on an appropriate generalization of the Bernoulli functions.

1. The celebrated *quadrature formula of Euler-Maclaurin* is the following (cf. [3, p. 109], [6, p. 323]):

$$\int_a^b f(t)\,dt = TR(f) - BT(f) + R_{n,k}(f) \tag{1}$$

with remainder of *even order* or of *odd order,* for $f \in C^{2k+1}[a,b]$,

$$R_{n,k}(f) = h^{2k} \int_a^b P_{2k}\left(n\frac{t-a}{b-a}\right) f^{(2k)}(t)\,dt = h^{2k+1} \int_a^b P_{2k+1}\left(n\frac{t-a}{b-a}\right) f^{(2k+1)}(t)\,dt \quad , \tag{2}$$

compound trapezoidal rule $TR(f)$ and *boundary terms* $BT(f)$ given by

$$TR(f) = h\left[\frac{1}{2}f(a) + \sum_{i=1}^{n-1} f(a+ih) + \frac{1}{2}f(b)\right], \quad BT(f) = \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!} h^{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right].$$

Here the functions $P_k(t) = B_k(t)/k!$ are the normalized Bernoulli functions (see [3, p. 109], [6, p. 323], [10, Chapter 1]) and the Bernoulli numbers are related to them by $B_k/k! = P_k(0) = P_k(1)$. Note that $B_{2k+1} = 0$, $k \geq 1$.

Now let us explain the *RRND property.* Imagine that we write formula (1) (putting $n = 1$) for each subinterval $[a + jh, a + (j + 1) h]$, $j = 0, ..., n - 1$. After summing up over j we see that the terms with interior derivatives cancel each other and we obtain formula (1) itself for the interval $[a, b]$, i.e. it has reproduced itself!

Why is the *RRND property* interesting and important from computational point of view? To answer this question, let us make further subdivision into $2n$ equal parts in the interval $[a, b]$ and write (1) for $h = (b - a) / 2n$. We see that we do not have to compute anything else except the functional values by the trapezoidal rule, and put $h/2$ instead of h in (1). This shows the *minimal computational complexity* of formula (1) with respect to subdivisions which makes it very efficient from algorithmic point of view.

2. What concerns the interest in the *RRND property* in the multivariate case let us say that it has been alluded in [3, Section 5.8, p. 286, top of page] as a compound rule. An analog to the *RRND property* is considered in [15, Chapter 4] where it is understood as "extension of formulas" from a lower–dimensional to higher–dimensional domains.

In the known generalizations of the Euler-Maclaurin formula (see [4], [11], [12], [14], [15], and references there) the integral is approximated in terms of values of the function at the integer points (or the points of some lattice), and the formulas are *exact for polynomials* up to a certain degree. So far (to the best of our knowledge) none of the results in these references generalizes the *RRND* property.

On the other hand, the main idea of the present work is to represent the integral of the function $f(x)$ over a $d-$dimensional domain like an integral of the same function $f(x)$ over surfaces of lower dimension, $d - 1$, $d - 2$, etc., in such a way that the formula obtained would possess the property of *RRND* (at least for some special domains). An important feature of this approach is that we obtain formulas which are *exact for functions which are polyharmonic* of a certain degree or which are solutions to similar higher order partial differential equations. The last fact is a manifestation of what we call *polyharmonicity paradigm* which has proved to be very successful in multivariate approximation and spline theory (see [7], [8], [9]).

3. One of the interesting discoveries of the present research was the observation that the theory of the one-dimensional Bernoulli polynomials may be viewed in the framework of the *Hodge theory* [5]. Namely, the sequence P_k splits into odd and even order parts as solutions to the Poisson problems $u''(t) = P_k(t), k \geq 0$, in $1-$periodic functions u, starting with the $1-$periodic functions $P_0(t) = 1 - \sum_{j=-\infty}^{\infty} \delta(t - j)$ and $P_1(t) = t - \frac{1}{2}$, $0 < t < 1$; $P_1(j) = 0$, $j \in Z$. The existence and uniqueness follow from the orthogonality condition $\int_0^1 u(t)\, dt = 0$.

Now, for the multivariate generalization of the Bernoulli functions, it is of basic importance to choose *proper initial functions* $P_{0,d}(x)$ and $P_{1,d}(x)$, which are multidimensional analogs to P_0 and P_1. For the proper choice of such we are lead by the idea explained above to represent the integral $\int_D f dx$ like a sum of integrals over manifolds of lower dimension, not only zero-dimensional.

We choose as initial functions the following

$$P_{0,d}\left(x\right)=d-\sum\nolimits_{k=1}^{d}\sum\nolimits_{j=-\infty}^{\infty}\delta\left(x_k-j\right)\qquad x=\left(x_1,...,x_d\right) \tag{3}$$

and

$$P_{1,d}\left(x\right)=P_1\left(x_1\right)P_1\left(x_2\right)...P_1\left(x_d\right), \tag{4}$$

and construct two series of Bernoulli functions. The first one $P_{2k,d}\left(x\right)$ is the multidimensional generalization of the even order Bernoulli functions P_{2k} and the second series $P_{2k+1,d}\left(x\right)$ corresponds to the odd order Bernoulli functions P_{2k+1}. They are $1-$periodic in every variable and satisfy

$$\Delta P_{k,d}=P_{k-2,d}\qquad k\geq 2 \tag{5}$$

as well as the orthogonality condition

$$\int_{D_{uc}}P_{k,d}\left(x\right)dx=0\qquad k\geq 0, \tag{6}$$

where D_{uc} denotes the $d-$dimensional unit cube. The last condition is necessary for the solubility of the Poisson problem

$$\Delta u=P_{k,d}\left(x\right) \tag{7}$$

on the torus $T^d=\left(S^1\right)^d$ and represents the orthogonality to the harmonic space consisting of the constants only [5, p. 89]. Now by the same arguments the solution u is determined on the torus up to a constant. Hence, u is completely determined by putting

$$\int_{D_{uc}}u\left(x\right)dx=0. \tag{8}$$

One may prove that

$$P_{2k,d}\left(x\right)=\sum\nolimits_{j=1}^{d}P_{2k}\left(x_j\right), \tag{9}$$

where P_{2k} are the one-dimensional normalized Bernoulli functions. For the odd order we have only multiple Fourier series expression

$$P_{2k+1,d}\left(x\right)=\left(-1\right)^{d+k}\frac{1}{\left(2\pi\right)^{2k}}\sum_{l_1=1}^{\infty}...\sum_{l_d=1}^{\infty}\frac{1}{\left|l\right|^{2k}}\left\{\prod_{j=1}^{d}\frac{2\sin\left(2\pi l_j x_j\right)}{2\pi l_j}\right\}, \tag{10}$$

where $\left|l\right|^2=l_1^2+...+l_d^2$.

Let us remark that these functions are multivariate polysplines in the sense of [8].

4. Thus we have generated two series of multivariate Bernoulli functions. Respectively, we obtain two different Euler-Maclaurin type *cubature formulas*. The following one is based on the even order multivariate Bernoulli functions.

Theorem 1 *For every function $f \in C^{2m}\left(\overline{D_{uc}}\right)$ the following formula holds*

$$I(f) = \int_{D_{uc}} f(y)\,dy = TR_{n,d}(f) - BT_{n,m,d}(f) + R_{n,m,d}(f) \qquad (11)$$

where by using the notation $\widehat{dy}_j = dy_1...dy_{j-1}dy_{j+1}...dy_d,\ j = 1,...,d,$ the compound trapezoidal sum is given by

$$TR(f) = (nd)^{-1}\sum_{k=1}^{n-1}\sum_{j=1}^{d}\int_0^1 ... \int_0^1 f(y)_{|y_j=k/n}\,\widehat{dy}_j + (2nd)^{-1}\sum_{k=0,n}\sum\nolimits_{j=1}^{d}\int_0^1 ... \int_0^1 f(y)_{|y_j=k/n}\,\widehat{dy}_j, \qquad (12)$$

the boundary terms by

$$BT_{n,m,d}(f) = d^{-1}\sum_{j=1}^{d}\sum_{k=1}^{m} n^{-2k}\int_0^1 ... \int_0^1 \left\{\frac{\partial}{\partial y_j}\Delta_y^{k-1} f(y)_{|y_j=1} - \frac{\partial}{\partial y_j}\Delta_y^{k-1} f(y)_{|y_j=0}\right\}, \qquad (13)$$

$$\times P_{2k,d}(ny)_{|y_j=0}\,\widehat{dy}_j$$

and the remainder by

$$R_{n,m,d}(f) = d^{-1}n^{-2m}\int_{D_{uc}} \Delta^m f(y)\,P_{2m,d}(ny)\,dy\ . \qquad (14)$$

Let us remark that the above cubature formula (11) is exact for polyharmonic functions of order m.

In order to formulate the second multivariate Euler-Maclaurin type formula we introduce the following subdivision of D_{uc} into smaller cubes $D_{\alpha,n} = \prod_{i=1}^{d}[\alpha_i/n, \alpha_{i+1}/n]$, $\alpha = (\alpha_1, ..., \alpha_d) \in Z^d$. Evidently, we have $D_{uc} = \bigcup\left\{D_{\alpha,n} : \alpha \in Z^d,\ 0 \le \alpha_i \le n-1,\ i = 1,..,d\right\}$.

Let Ω_j denote the set of all $j-$dimensional edges of the cubes $D_{\alpha,n}$ for $0 \le \alpha_i \le n-1, i = 1,..,d$. For any $K \in \Omega_j$ let the number of the cubes $D_{\alpha,n}$ for which $K \subset D_{\alpha,n}$ be denoted by $ind_{D_{uc}}(K)$. It is easy to see that $ind_{D_{uc}}(K) = 2^{r-j}$ where K is contained in an $r-$dimensional edge of D_{uc}, $j \le r \le d$.

The odd order Bernoulli functions generate the following *cubature formula* of Euler-Maclaurin type.

Theorem 2 *For every function $f \in C^{2k+d}\left(\overline{D_{uc}}\right)$ the following formula holds*

$$I(f) = \int_{D_{uc}} f(y)\,dy = (-1)^d\left[TR_{n,d}(f) + BT_{n,m,d}(f) + R_{n,m,d}(f)\right], \qquad (15)$$

where the compound trapezoidal sum is

$$TR_{n,d}(f) = \sum_{k=0}^{d-1}(-1)^{k+1}\frac{1}{2^{d-k}}\frac{1}{n^{d-k}}\sum_{\dim(K_1)=k} ind_{D_{uc}}(K_1)\int_{K_1} f(y)\,d\sigma_{k,y}, \qquad (16)$$

the boundary terms are

$$BT_{n,m,d}(f) = \sum_{i=1}^{m} n^{-(2i+d)} \int_{\partial D_{uc}} \Delta_y^{i-1} \frac{\partial^d}{\partial y_1 ... \partial y_d} f(y) \frac{\partial}{\partial \nu_y} P_{2i+1,d}(ny)\, dy, \tag{17}$$

the remainder is

$$R_{n,m,d}(f) = n^{-(2m+d)} \int_{D_{uc}} \Delta_y^{m} \frac{\partial^d}{\partial y_1 ... \partial y_d} f(y) P_{2m+1,d}(ny)\, dy, \tag{18}$$

where $d\sigma_{k,y}$ *is the* $k-$*dimensional measure on* K_1 *and* $\frac{\partial}{\partial \nu_y}$ *denotes the exterior normal derivative to* ∂D_{uc}.

Let us remark that formulas (11) and (15) have the *RRND* property, namely, if we sum the formulas which are written for two neighboring cubes the interior terms containing derivatives cancel.

Let us remark that, as in the one-dimensional case (see [3, p. 109, f. (2.9.16)], [10, p. 216]), the boundary terms $BT_{n,m,d}$ in both Theorem 1 and 2 may be used for convergence acceleration of the compound trapezoidal rules $TR_{n,d}$. The following estimates of the error hold

Theorem 3 *1. Let the function* $f \in C^{2m}\left(\overline{D_{uc}}\right)$. *Let* $|\Delta^m f| \leq M$. *Then in the notations of Theorem 1 we have*

$$|I(f) - TR_{n,d}(f) + BT_{n,m,d}(f)| \leq MC_1 \zeta(2m)/n^{2m};$$

2. Let the function $f \in C^{2m+d}\left(\overline{D_{uc}}\right)$. *Let* $\left|\Delta^m \frac{\partial^d}{\partial x_1 ... \partial x_d} f(x)\right| \leq M$. *Then in the notations of Theorem 2 we have*

$$\left|I(f) - (-1)^d TR_{n,d}(f) - (-1)^d BT_{n,m,d}(f)\right| \leq MC_2 \left[\zeta\left(1 + \frac{2m}{d}\right)\right]^d / n^{2m+d};$$

Here ζ *denotes the Riemann zeta function, and* $C_1 = 2/(2\pi)^{2m}$, $C_2 = 2^d/\left[d^m (2\pi)^{2m+d}\right]$.

Both above Euler-Maclaurin formulas are related to Poisson type formulas. The one corresponding to Theorem 2 is:

$$\int_{D_{uc}} f(x)\, dx = (-1)^d \left[TR_{n,d}(f) + \sum_{k=1}^{d} \left\{\widehat{f}(nl) : l \in \mathbf{Z}^d,\ l_k \neq 0\right\}\right], \tag{19}$$

where $f \in C\left(\overline{D_{uc}}\right)$, $\widehat{f}(\xi) = \int_{D_{uc}} f(x) e^{-2\pi i x.\xi} dx$, and satisfies the asymptotic condition $\left|\widehat{f}(\xi)\right| \leq C(1 + |\xi|)^{-d-\delta}$ for $\xi \in \mathbf{Z}^d$.

The cubature formula (19) is exact for all $d-$dimensional trigonometric polynomials of degree $\leq n-1$ with respect to every coordinate variable. It is convenient for approximate integration of smooth periodic functions, that is for functions with fast decreasing Fourier coefficients.

The complete proofs are available in [2].

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References

[1] Berndt B. C., 1985. *Ramanujan's Notebooks,* Vol. 1 and 2, Springer-Verlag, N.Y., Berlin.

[2] Dryanov D. and Kounchev O., 1997. Multivariate formula of Euler-Maclaurin, Bernoulli functions, and Poisson type formulas, *preprint Institute of Applied Mathematics, University of Hamburg*, Reihe A, No. 129, October 1997, submitted.

[3] Davis Ph. J. and Rabinowitz Ph., 1975. *Methods of Numerical Integration,* Academic Press, N.Y., San Francisco, London.

[4] de Doncker E., 1987. Asymptotic expansions and their application in numerical integration, *Numerical Integration – Recent Developments, Software and Applications*, P. Keast and G. Fairweather eds., Reidel, Dordrecht, pp. 141–151.

[5] Griffiths Ph. and Harris J., 1978. *Principles of Algebraic Geometry*, J. Wiley and Sons, N.Y., Toronto.

[6] Hardy G. H., 1956. *Divergent Series*, Clarendon Press, Oxford.

[7] Kounchev O. I., 1992. Sharp estimate of the Laplacian of a polyharmonic function and applications. *Trans. Amer. Math. Soc.* **332**, no. 1, 121–133.

[8] Kounchev O. I., Minimizing the Laplacian of a function squared with prescribed values on interior boundaries—theory of polysplines. *Trans. Amer. Math. Soc.* **350** (1998), no. 5, 2105–2128.

[9] Kounchev O. I., *Multivariate Polysplines with Applications to Numerical Analysis*, Academic Press, in preparation.

[10] Krylov V. I., 1962. *Approximate Calculation of Integrals*, Macmillan, N.Y., London.

[11] Lyness J. N., 1986. Numerical integration, *Numerical Algorithms*, J.L. Mohamed and J. Walsh eds., Oxford University Press, Oxford, pp. 104–124.

[12] 1992. *Numerical Integration, Recent Developments, Software and Applications*, NATO ASI Series, Eds. T. O. Espelid and A. Genz, Kluwer, Dordrecht, Boston, London.

[13] Shaneson J. L., 1995. Characteristic classes, lattice points, and Euler-Maclaurin formulas, *Proceedings of the International Congress of Mathematicians,* Zürich, Switzerland 1994, Birkhäuser Verlag, Basel, Switzerland, pp. 612-624.

[14] Sobolev S. L., 1992. *Cubature Formulas and Modern Analysis: An introduction*, Gordon and Breach Science Publishers, Montreux.

[15] Stroud A. H., 1971. *Approximate Calculation of Multiple Integrals,* Prentice-Hall, Englewood Cliffs, New Jersey.

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