Extremizers for Multivariate Landau-Kolmogorov Inequality

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Abstract

In the present paper we prove the following sharp inequality:

$$\|\Delta f\|_\infty \leq 2\sqrt{\frac{d}{d+2}}\sqrt{\|f\|_\infty \|\Delta^2 f\|_\infty}$$

in $\mathbf{R}^d$ and we write the explicit solution (the extremizer) for the equality case. This result is an anlogue to the famous inequality of Landau in the one-dimensioanal case: $\|f'\|_\infty \leq \sqrt{2}\sqrt{\|f\|_\infty \|f''\|_\infty}$. In both cases $\|.\|_\infty$ means the L_∞ norm.

1 The inequality of Landau-Kolmogorov

We will give a general setting for inequalities of the type of Landau-Kolmogorov. For that purpose we will first recall the complete result of A.N. Kolmogorov, see [K], which includes the inequality of Landau as a special case. For Lipschitz functions $f \in C^{n-1}(\mathbf{R})$ we use the following notation:

$$M_n(f) = \sup_{x\in\mathbf{R}} \left| f^{(n)} \right|.$$

Theorem 1 *For every $f \in C^n(\mathbf{R})$ we have*

$$M_k \leq C_{nk} M_0^{1-\frac{k}{n}} M_n^{\frac{k}{n}} \tag{1}$$

where $C_{nk} = K_{n-k}/K_n^{1-\frac{k}{n}}$ for $0 < k < n$;

$$K_i = \begin{cases} \frac{4}{\pi}\left(1 - \frac{1}{3^{i+1}} + \frac{1}{5^{i+1}} - \frac{1}{7^{i+1}} + ...\right) & \text{for even } i \\ \frac{4}{\pi}\left(1 + \frac{1}{3^{i+1}} + \frac{1}{5^{i+1}} - \frac{1}{7^{i+1}} + ...\right) & \text{for odd } i \end{cases}$$

The extremizing function making (1) equality for all k is given by:

$$f_n(x) = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\sin\left((2m+1)x - \frac{\pi}{2}n\right)}{(2m+1)^{n+1}}.$$

Remark 2 *It is well known that the function f_n is a cardinal spline of degree n , i.e. $f_n(x)$ is a polynomial of degree n on the intervals $(k\pi, (k+1)\pi)$, $k \in \mathbf{Z}$, and $f_n(x) \in C^{(n-1)}(\mathbf{R})$. It is a perfect spline in the sense that $f_n^{(n)}(x) = \pm Const$, cf. [S]. Another equivalent formulation of Kolmogorov's inequality using the extremizing function f_n is: Let for some function f we have the following inequalities:*

$$\|f\|_\infty \leq 1, \left\|f^{(n)}\right\|_\infty \leq \left\|f_n^{(n)}\right\|_\infty .$$

Then it follows that

$$\left\|f^{(k)}\right\|_\infty \leq \left\|f_n^{(k)}\right\|_\infty, \text{ for } 1 \leq k \leq n.$$

That is the way of formulating the inequalities which appeared in the original paper of E. Landau, [L]. J. Hadamard was the first who formulated in [H] the inequality of Landau in the more compact form (1). Let us point to the paper of de Boor and Schoenberg [B-S] (see also [S]), where an original proof of (1) and the uniqueness of the extremizers is given.

2 General framework for inequalities of Landau-Kolmogorov type

Here we provide a general setting for inequalities of the Landau-Kolmogorov type which will be very useful for generalizations.

Let D_1, D_2 be homogeneous differential operators (of constant or variable coefficients), of orders $d_1 = \deg(D_1) < d_2 = \deg(D_2)$. Now the generalized inequality of Landau-Kolmogorov type may be formulated in the following way:

$$\|D_1 f\| \leq K \|f\|^{1-\frac{d_1}{d_2}} \|D_2 f\|^{\frac{d_1}{d_2}} . \qquad (2)$$

It is clear that in the case of inequality (1) we have $D_1 = \frac{d^k}{dx^k}$, $D_2 = \frac{d^n}{dx^n}$.

The sharp equality is obtained for a function f_0 which is a piecewise solution to $D_2 f_0(x) = \pm Const$, $x \in \mathbf{R}$. It is natural to call f_0 *perfect L-spline*, for the operator $L = D_2$, cf. [Sm].

Let us remark that the powers $1 - \frac{d_1}{d_2}$ and $\frac{d_1}{d_2}$ at which $\|D_1 f\|$ and $\|D_2 f\|$ appear, come in a natural way from the requirement that inequality (2) should hold for all functions of the type

$$af(bx) \qquad a, b \in \mathbf{R}$$

with the same constant K !

3 The multivariate case

In the space $\mathbf{R}^d$ the operator D_2 has to be elliptic. This follows, roughly speaking, from the general theory of partial differential operators, cf. [Ho].

The simplest elliptic differential operator in $\mathbf{R}^d$ is the Laplace:

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}.$$

The first result in the multivariate case belongs to Timofeev [T]. He proved the following inequality:

$$\left\| \frac{\partial f}{\partial \xi} \right\|_\infty \leq \sqrt{2} \sqrt{\|f\|_\infty \|\Delta f\|_\infty}.$$

Here $\frac{\partial f}{\partial \xi}$ is an arbitrary directional derivative.The expression Δf is taken in distributional (Sobolev) sense, and it is assumed that $\Delta f \in L_\infty \left(\mathbf{R}^d \right)$. In view of the general framework, we have in the Timofeev's inequality the operator $D_1 = \frac{\partial}{\partial x_j}, .D_2 = \Delta.$

The result of Timofeev was later generalized by Kh. Boyadziev [B] and finally by Z. Ditzian [Di]. The main result in [D], in Theorem 6.1, is the following inequality:

$$\left\| \frac{\partial^k f}{\partial \xi_1 ... \partial \xi_k} \right\|_\infty \leq C(n,k) \|f\|_\infty^{1-\frac{k}{2n}} \|\Delta^n f\|_\infty^{\frac{k}{2n}}, \qquad 0 < k < 2n \tag{3}$$

which holds for functions $f \in C \left(\mathbf{R}^d \right), \Delta^n f \in L_\infty \left(\mathbf{R}^d \right),$ where $\Delta^n f$ is taken in distributional (Sobolev) sense. It is also proved that if the right-hand side is bounded then $\frac{\partial^k f}{\partial \xi_1 ... \partial \xi_k} \in C \left(\mathbf{R}^d \right)$.

Clearly, from the point of view of the general framework in the above inequality we have $D_1 = \frac{\partial^k}{\partial \xi_1 ... \partial \xi_k},\ D_2 = \Delta^n$.

Let us remark that no question of finding extremizers or proving sharpness of the constants $C(n,k)$ has been solved in the above works.

4 Extremizers in the multivariate case

The main point in the present paper is to find extremizers and to find the sharp constant K_d for the following inequality in $\mathbf{R}^d$:

$$\|\Delta f\|_\infty \leq K_d \sqrt{\|f\|_\infty \|\Delta^2 f\|_\infty}.$$

From the point of view of the general framework in that case we have $D_1 = \Delta,\ D_2 = \Delta^2$.Let us remark that the existence of a constant $K_d > 0$ follows from (3).

We will need some preliminary definitions in order to formulate and prove the main Theorem of the paper.

Let us introduce the following 2−periodic function:

$$
\begin{aligned}
E_2(x) &= 1-4x^2 \quad \text{in } \left[-\frac{1}{2},\frac{1}{2}\right] \\
E_2(x) &= -E_2(x-1) \quad \text{in } \left[\frac{1}{2},\frac{3}{2}\right] \\
E_2(x) \text{ for } x &\notin \left[-\frac{1}{2},\frac{3}{2}\right], \text{ defined by periodicity}
\end{aligned}
$$

Following [S] we introduce the following function:

$$
g(x) = \begin{cases} 1-4x^2 & \text{if } -1/\sqrt{2} \le x \le 0 \\ E_2(x) & \text{if } \quad 0 \le x < \infty \end{cases}
$$

and we define

$$
f_0(x) = g\left(x - 1/\sqrt{2}\right) \qquad \text{for } x \ge 0.
$$

We will prove the following

Theorem 3 *For every function the following sharp inequality holds:*

$$
\|\Delta f\|_\infty \le 2\sqrt{\frac{d}{d+2}}\sqrt{\|f\|_\infty \|\Delta^2 f\|_\infty}. \tag{4}
$$

The function defined by

$$
F_0(x) = f_0\left(|x|^2\right).
$$

is extremizer

We will need some preparatory work for the proof.

We will introduce the mean value operator: For a domain $D \subset \mathbf{R}^d$, $d \ge 2$, for $f \in C(D)$, $x \in D$ and $h > 0$ such that the open ball $B_h(x) := \{y \in \mathbf{R}^d : |x-y| < h\}$ is strictly contained in D, i. e. $\overline{B_h(x)} \subset D$, we denote by

$$
\mu(f,x;h) := \frac{1}{\sigma_d}\int_{S_{d-1}} f(x+h\xi)d\sigma_\xi = \frac{1}{\sigma_d h^{d-1}}\int_{S_{d-1}(h)} f(x+y)d\sigma(y)
$$

the surface mean of f over $\partial B_h(x)$. Here $d\sigma_\xi$ denotes the area element of the $(d-1)$–sphere S_{d-1} and $d\sigma(y)$ is the area element of the sphere $S_{d-1}(h)$. For $h = 0$ we have $\mu(f,x;0) = f(x)$.

We introduce the linear linear integral operator J defined for a given $h > 0$ by

$$
J(\phi;t) := \int_0^t \left(r - \frac{r^{d-1}}{t^{d-2}}\right)\phi(r)dr \qquad \text{for } \phi \in C[0,h],\ 0 \le t \le h
$$

if $d \ge 3$. In the case $d = 2$, J is defined by

$$J(\phi;t) \;:=\; \int_0^t r\log\frac{t}{r}\phi(r)dr \qquad\qquad \text{for } \phi\in C[0,h],\;\; 0\le t\le h.$$

The powers of J are defined as usual by $J^1 := J$, and $J^{k+1} := J(J^k)$ for $k \geq 1$. When $\phi = 1$, then

$$J^p(1;t) := \begin{cases} a_{dp}\,(d-2)^p\,t^{2p} & \text{for } d\geq 3 \\ \frac{1}{4^p(p!)^2}t^{2p} & \text{for } d=2 \end{cases}$$

where $p \geq 1$ and

$$a_{dp} := \frac{1}{2^p p! d(d+2)...(d+2p-2)} \qquad\qquad (p\geq 1,\;\; d\geq 2).$$

Note that $a_{2p} = \frac{1}{4^p(p!)^2}$. We put $a_{d0} = 1$.

Now we may state the formula of Pizzetti-Nicolescu:

Proposition 4 *i) Suppose $f \in H^{2p}(D)$. Then for any ball $B_h(x)$ strictly contained in D, the following equation holds:*

$$\mu(f,x;h) = f(x)+\sum_{j=1}^{p-1} a_{dj}h^{2j}\Delta^j f(x)+J^p(\mu(\Delta^p f,x;\cdot);h)\cdot\begin{cases} \frac{1}{(d-2)^p} & \textit{for } d\geq 3 \\ 1 & \textit{for } d=2 \end{cases}$$

(ii) The remainder in the Pizzetti-Nicolescu formula can be written as:

$$\begin{aligned} J^p(\mu(\Delta^p f,x;\cdot);h) \;&=\; \mu(\Delta^p f,x;\vartheta_p h)\cdot J^p(1;h) = \\ &=\; \Delta^p f(\xi_{x,p})\begin{cases} a_{dp}h^{2p}\,(d-2)^p & \textit{for }\; d\geq 3 \\ a_{dp}h^{2p} & \textit{for }\; d=2 \end{cases} \end{aligned}$$

for suitable $\xi_{x,p} \in B_h(x)$.

For a concise proof of the above results see [H-K], cf. also the original paper of Nicolescu [Ni].

Proof of the Theorem.

We will prove that the function

$$F_0(x) = f_0\left(r^2\right), \quad r = |x| \tag{5}$$

is an extremizer.

Let us put $h = \left(\frac{1}{2}\right)^{\frac{1}{4}}$. Remark that $F_0(0) = -1,\; F_0(h) = 1$.

Applying the formula of Pizzetti-Nicolescu for $p = 2$ and $x = 0$ we obtain:

$$\mu(f,0;h) = f(0) + a_{d1}h^2\Delta f(0) + J^2(\mu(\Delta^2 f,0;\cdot);h)\cdot\begin{cases} \frac{1}{(d-2)^p} & \text{for } d\geq 3 \\ 1 & \text{for } d=2 \end{cases} \tag{6}$$

On the other hand we have

$$a_{d1} = \frac{1}{2d} \quad, a_{d2} = \frac{1}{8d(d+2)} \quad \text{for} \quad d\geq 2$$

and

$$J^2(1;t)=\begin{cases} a_{d2}\,(d-2)^2\,t^4 & \text{for } d\geq 3\\ \frac{1}{64}t^4 & \text{for } d=2\end{cases}$$

Taking into account that the radial part of the Laplace operator is

$$\Delta_r=\frac{\partial^2}{\partial r^2}+\frac{d-1}{r}\frac{\partial}{\partial r}$$

and that by definition we have

$$F_0\,(x)=f_0\left(|x|^2\right)=1-4\left(|x|^2-\frac{1}{\sqrt{2}}\right)^2 \qquad\qquad |x|=r\leq h$$

we obtain the following values:

$$\begin{aligned}
\Delta F_0\,(x) &= \Delta_r F_0\,(r)\\
&= \Delta_r f_0\left(r^2\right)\\
&= \left(f_0\left(r^2\right)\right)''+\frac{d-1}{r}\left(f_0\left(r^2\right)\right)'\\
&= \left(2rf_0'\left(r^2\right)\right)'+2\,(d-1)\,f_0'\left(r^2\right)\\
&= 2f_0'\left(r^2\right)+4r^2f_0''\left(r^2\right)+2\,(d-1)\,f_0'\left(r^2\right)\\
&= 4r^2f_0''\left(r^2\right)+2df_0'\left(r^2\right)\\
&= 4r^2\,(-8)+2d\left(-8\left(r^2-\frac{1}{\sqrt{2}}\right)\right)\\
&= 16\left(-2r^2-dr^2+\frac{d}{\sqrt{2}}\right).
\end{aligned}$$

It follows that the maximum of $|\Delta_r F_0\,(r)|$ in the interval $r\leq h$ is obtained for $r=0$, i.e. it is equal to

$$\Delta_r F_0\,(0)=8\sqrt{2}d.$$

Due to the symmetry of the function E_2, it follows that this is also the absolute maximum for every $r\geq 0$, i.e.

$$\|\Delta_r F_0\|_\infty=8\sqrt{2}d.$$

Also taking into account that for $r\leq h$ the function $F_0\,(x)$ is a polynomial of degree 4 in r, we obtain

$$\begin{aligned}
\Delta^2F_0\,(x) &= -4\Delta_r^2\left(r^4\right)\\
&= -32d\,(d+2) \quad \text{for} \quad r=|x|\leq h.
\end{aligned}$$

and due to the definition of the function f_0 we have

$$\left|\Delta^2F_0\,(x)\right|=32d\,(d+2) \quad \text{for every} \quad x\in\mathbf{R}^d,$$

i.e.

$$\left\|\Delta^2 F_0\left(x\right)\right\|_{\infty} = 32d\left(d+2\right).$$

In formula (4.3) we put $f = F_0$. Using the above formulas and the fact that the spherical mean of a radially symmetric function is the value of the function at that radius, we obtain:

$$\mu(F_0, 0; h) = F_0\left(h\right)$$

In a similar way since $\Delta^2 F_0\left(x\right)$ is a constant for $r = |x| \leq h$ we obtain

$$\mu(\Delta^2 F_0, 0; h) = \Delta^2 F_0 = -32d\left(d+2\right)$$

and finally

$$F_0(h) = F_0(0) + a_{d1}h^2 \Delta F_0(0) + \Delta^2 F_0\left(0\right) J^2(1;h) \cdot c_{2,d}$$

It is important to note that we have alternatingly in sign

$$\begin{aligned} F_0(0) &= -1 \\ F_0(h) &= 1 \\ \Delta^2 F_0\left(0\right) &= -32d\left(d+2\right) \end{aligned}$$

which is decisive to obtain the following key identity:

$$\begin{aligned} a_{d1}h^2 \Delta F_0(0) &= -F_0(0) + F_0(h) - \Delta^2 F_0\left(0\right) J^2(1;h) \cdot c_{2,d} = \\ &= 1 + 1 + \left\|\Delta^2 F_0\right\|_{\infty} J^2(1;h) \cdot c_{2,d} \end{aligned}$$

Let now f be an arbitrary function for which $\|f\|_{\infty} \leq 1$, $\left\|\Delta^2 f\right\|_{\infty} \leq \left\|\Delta^2 F_0\right\|_{\infty} = 32d\left(d+2\right)$. From formula (4.3) we express $\Delta f(0)$:

$$a_{d1}h^2 \Delta f(0) = -f(0) + \mu(f, 0; h) - J^2(\mu(\Delta^2 f, 0; \cdot); h) \cdot c_{2,d}$$

Since $J^2(1;h) > 0$, and the kernel of the operator $J^2\left(., r\right)$ is evidently positive we obtain the following inequality:

$$\begin{aligned} a_{d1}h^2 \left|\Delta f(0)\right| &\leq \left\|-f(0) + \mu(f, 0; h) - J^2(\mu(\Delta^2 f, 0; \cdot); h) \cdot c_{2,d}\right\|_{\infty} \leq \\ &\leq \|f\|_{\infty} + \|\mu(f, 0; h)\|_{\infty} + \left\|J^2(\mu(\Delta^2 f, 0; \cdot); h) \cdot c_{2,d}\right\|_{\infty} \\ &\leq \|f\|_{\infty} + \|f\|_{\infty} + \left\|\mu(\Delta^2 f, 0; \cdot)\right\|_{\infty} J^2(1;h) \cdot c_{2,d} \\ &\leq \|f\|_{\infty} + \|f\|_{\infty} + \left\|\Delta^2 f\right\|_{\infty} J^2(1;h) \cdot c_{2,d} \\ &\leq \|f\|_{\infty} + \|f\|_{\infty} + \left\|\Delta^2 F_0\right\|_{\infty} J^2(1;h) \cdot c_{2,d} \\ &\leq 1 + 1 + \left\|\Delta^2 F_0\right\|_{\infty} J^2(1;h) \cdot c_{2,d} \end{aligned}$$

By the proved above we obtain

$$\begin{aligned} a_{d1}h^2 \left|\Delta f(0)\right| &\leq a_{d1}h^2 \Delta F_0(0) \\ &= a_{d1}h^2 \left\|\Delta F_0\right\|_{\infty}. \end{aligned}$$

It follows that

$$|\Delta f(0)| \leq \|\Delta F_0\|_\infty .$$

We may apply the above arguments to the function $f(x + x_0)$, and prove the above inequality for every point $x_0 \in \mathbf{R}^d$, i.e.

$$|\Delta f(x_0)| \leq \|\Delta F_0\|_\infty .$$

Hence it follows that

$$\|\Delta f\|_\infty \leq \|\Delta F_0\|_\infty .$$

This finishes the proof of the Theorem in the formulation of Landau.

If we put

$$\begin{aligned} K_d &= \frac{\|\Delta F_0\|_\infty}{\sqrt{\|\Delta^2 F_0\|_\infty}} \\ &= \frac{8\sqrt{2}d}{\sqrt{32d\,(d+2)}} \\ &= 2\sqrt{\frac{d}{d+2}} \end{aligned}$$

then we obtain equality in (4.1) for the function $f = F_0$. For arbitrary function f inequality (4.1) follows by the above arguments. That finishes the proof of the Theorem also in the formulation of Hadamard.

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