

Extremizers for Multivariate Landau-Kolmogorov Inequality

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Abstract

In the present paper we prove the following sharp inequality:

$$\|\Delta f\|_\infty \leq 2\sqrt{\frac{d}{d+2}}\sqrt{\|f\|_\infty \|\Delta^2 f\|_\infty}$$

in \mathbf{R}^d and we write the explicit solution (the extremizer) for the equality case. This result is an analogue to the famous inequality of Landau in the one-dimensional case: $\|f'\|_\infty \leq \sqrt{2}\sqrt{\|f\|_\infty \|f''\|_\infty}$. In both cases $\|\cdot\|_\infty$ means the L_∞ norm.

1 The inequality of Landau-Kolmogorov

We will give a general setting for inequalities of the type of Landau-Kolmogorov. For that purpose we will first recall the complete result of A.N. Kolmogorov, see [K], which includes the inequality of Landau as a special case. For Lipschitz functions $f \in C^{n-1}(\mathbf{R})$ we use the following notation:

$$M_n(f) = \sup_{x \in \mathbf{R}} |f^{(n)}|.$$

Theorem 1 For every $f \in C^n(\mathbf{R})$ we have

$$M_k \leq C_{nk} M_0^{1-\frac{k}{n}} M_n^{\frac{k}{n}} \quad (1)$$

where $C_{nk} = K_{n-k}/K_n^{1-\frac{k}{n}}$ for $0 < k < n$;

$$K_i = \begin{cases} \frac{4}{\pi} \left(1 - \frac{1}{3^{i+1}} + \frac{1}{5^{i+1}} - \frac{1}{7^{i+1}} + \dots\right) & \text{for even } i \\ \frac{4}{\pi} \left(1 + \frac{1}{3^{i+1}} + \frac{1}{5^{i+1}} - \frac{1}{7^{i+1}} + \dots\right) & \text{for odd } i \end{cases}$$

The extremizing function making (1) equality for all k is given by:

$$f_n(x) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin\left((2m+1)x - \frac{\pi}{2}n\right)}{(2m+1)^{n+1}}.$$

Remark 2 *It is well known that the function f_n is a cardinal spline of degree n , i.e. $f_n(x)$ is a polynomial of degree n on the intervals $(k\pi, (k+1)\pi)$, $k \in \mathbf{Z}$, and $f_n(x) \in C^{(n-1)}(\mathbf{R})$. It is a perfect spline in the sense that $f_n^{(n)}(x) = \pm \text{Const}$, cf. [S]. Another equivalent formulation of Kolmogorov's inequality using the extremizing function f_n is: Let for some function f we have the following inequalities:*

$$\|f\|_\infty \leq 1, \left\| f^{(n)} \right\|_\infty \leq \left\| f_n^{(n)} \right\|_\infty.$$

Then it follows that

$$\left\| f^{(k)} \right\|_\infty \leq \left\| f_n^{(k)} \right\|_\infty, \text{ for } 1 \leq k \leq n.$$

That is the way of formulating the inequalities which appeared in the original paper of E. Landau, [L]. J. Hadamard was the first who formulated in [H] the inequality of Landau in the more compact form (1). Let us point to the paper of de Boor and Schoenberg [B-S] (see also [S]), where an original proof of (1) and the uniqueness of the extremizers is given.

2 General framework for inequalities of Landau-Kolmogorov type

Here we provide a general setting for inequalities of the Landau-Kolmogorov type which will be very useful for generalizations.

Let D_1, D_2 be homogeneous differential operators (of constant or variable coefficients), of orders $d_1 = \deg(D_1) < d_2 = \deg(D_2)$. Now the generalized inequality of Landau-Kolmogorov type may be formulated in the following way:

$$\|D_1 f\| \leq K \|f\|^{1-\frac{d_1}{d_2}} \|D_2 f\|^{\frac{d_1}{d_2}}. \quad (2)$$

It is clear that in the case of inequality (1) we have $D_1 = \frac{d^k}{dx^k}$, $D_2 = \frac{d^n}{dx^n}$.

The sharp equality is obtained for a function f_0 which is a piecewise solution to $D_2 f_0(x) = \pm \text{Const}$, $x \in \mathbf{R}$. It is natural to call f_0 *perfect L-spline*, for the operator $L = D_2$, cf. [Sm].

Let us remark that the powers $1 - \frac{d_1}{d_2}$ and $\frac{d_1}{d_2}$ at which $\|D_1 f\|$ and $\|D_2 f\|$ appear, come in a natural way from the requirement that inequality (2) should hold for all functions of the type

$$af(bx) \quad a, b \in \mathbf{R}$$

with the same constant K !

3 The multivariate case

In the space \mathbf{R}^d the operator D_2 has to be elliptic. This follows, roughly speaking, from the general theory of partial differential operators, cf. [Ho].

The simplest elliptic differential operator in \mathbf{R}^d is the Laplace:

$$\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}.$$

The first result in the multivariate case belongs to Timofeev [T]. He proved the following inequality:

$$\left\| \frac{\partial f}{\partial \xi} \right\|_{\infty} \leq \sqrt{2} \sqrt{\|f\|_{\infty} \|\Delta f\|_{\infty}}.$$

Here $\frac{\partial f}{\partial \xi}$ is an arbitrary directional derivative. The expression Δf is taken in distributional (Sobolev) sense, and it is assumed that $\Delta f \in L_{\infty}(\mathbf{R}^d)$. In view of the general framework, we have in the Timofeev's inequality the operator $D_1 = \frac{\partial}{\partial x_j}$, $D_2 = \Delta$.

The result of Timofeev was later generalized by Kh. Boyadziev [B] and finally by Z. Ditzian [Di]. The main result in [D], in Theorem 6.1, is the following inequality:

$$\left\| \frac{\partial^k f}{\partial \xi_1 \dots \partial \xi_k} \right\|_{\infty} \leq C(n, k) \|f\|_{\infty}^{1-\frac{k}{2n}} \|\Delta^n f\|_{\infty}^{\frac{k}{2n}}, \quad 0 < k < 2n \quad (3)$$

which holds for functions $f \in C(\mathbf{R}^d)$, $\Delta^n f \in L_{\infty}(\mathbf{R}^d)$, where $\Delta^n f$ is taken in distributional (Sobolev) sense. It is also proved that if the right-hand side is bounded then $\frac{\partial^k f}{\partial \xi_1 \dots \partial \xi_k} \in C(\mathbf{R}^d)$.

Clearly, from the point of view of the general framework in the above inequality we have $D_1 = \frac{\partial^k}{\partial \xi_1 \dots \partial \xi_k}$, $D_2 = \Delta^n$.

Let us remark that no question of finding extremizers or proving sharpness of the constants $C(n, k)$ has been solved in the above works.

4 Extremizers in the multivariate case

The main point in the present paper is to find extremizers and to find the sharp constant K_d for the following inequality in \mathbf{R}^d :

$$\|\Delta f\|_{\infty} \leq K_d \sqrt{\|f\|_{\infty} \|\Delta^2 f\|_{\infty}}.$$

From the point of view of the general framework in that case we have $D_1 = \Delta$, $D_2 = \Delta^2$. Let us remark that the existence of a constant $K_d > 0$ follows from (3).

We will need some preliminary definitions in order to formulate and prove the main Theorem of the paper.

Let us introduce the following 2–periodic function:

$$\begin{aligned} E_2(x) &= 1 - 4x^2 \quad \text{in } \left[-\frac{1}{2}, \frac{1}{2}\right] \\ E_2(x) &= -E_2(x-1) \quad \text{in } \left[\frac{1}{2}, \frac{3}{2}\right] \\ E_2(x) &\text{ for } x \notin \left[-\frac{1}{2}, \frac{3}{2}\right], \text{ defined by periodicity} \end{aligned}$$

Following [S] we introduce the following function:

$$g(x) = \begin{cases} 1 - 4x^2 & \text{if } -1/\sqrt{2} \leq x \leq 0 \\ E_2(x) & \text{if } 0 \leq x < \infty \end{cases}$$

and we define

$$f_0(x) = g\left(x - 1/\sqrt{2}\right) \quad \text{for } x \geq 0.$$

We will prove the following

Theorem 3 *For every function the following sharp inequality holds:*

$$\|\Delta f\|_\infty \leq 2\sqrt{\frac{d}{d+2}}\sqrt{\|f\|_\infty \|\Delta^2 f\|_\infty}. \quad (4)$$

The function defined by

$$F_0(x) = f_0(|x|^2).$$

is extremizer

We will need some preparatory work for the proof.

We will introduce the mean value operator: For a domain $D \subset \mathbf{R}^d$, $d \geq 2$, for $f \in C(D)$, $x \in D$ and $h > 0$ such that the open ball $B_h(x) := \{y \in \mathbf{R}^d : |x - y| < h\}$ is strictly contained in D , i. e. $\overline{B_h(x)} \subset D$, we denote by

$$\mu(f, x; h) := \frac{1}{\sigma_d} \int_{S_{d-1}} f(x + h\xi) d\sigma_\xi = \frac{1}{\sigma_d h^{d-1}} \int_{S_{d-1}(h)} f(x + y) d\sigma(y)$$

the surface mean of f over $\partial B_h(x)$. Here $d\sigma_\xi$ denotes the area element of the $(d-1)$ -sphere S_{d-1} and $d\sigma(y)$ is the area element of the sphere $S_{d-1}(h)$. For $h = 0$ we have $\mu(f, x; 0) = f(x)$.

We introduce the linear linear integral operator J defined for a given $h > 0$ by

$$J(\phi; t) := \int_0^t \left(r - \frac{r^{d-1}}{t^{d-2}} \right) \phi(r) dr \quad \text{for } \phi \in C[0, h], \quad 0 \leq t \leq h$$

if $d \geq 3$. In the case $d = 2$, J is defined by

$$J(\phi; t) := \int_0^t r \log \frac{t}{r} \phi(r) dr \quad \text{for } \phi \in C[0, h], \quad 0 \leq t \leq h.$$

The powers of J are defined as usual by $J^1 := J$, and $J^{k+1} := J(J^k)$ for $k \geq 1$. When $\phi = 1$, then

$$J^p(1; t) := \begin{cases} a_{dp} (d-2)^p t^{2p} & \text{for } d \geq 3 \\ \frac{1}{4^p (p!)^2} t^{2p} & \text{for } d = 2 \end{cases}$$

where $p \geq 1$ and

$$a_{dp} := \frac{1}{2^p p! d(d+2) \dots (d+2p-2)} \quad (p \geq 1, d \geq 2).$$

Note that $a_{2p} = \frac{1}{4^p (p!)^2}$. We put $a_{d0} = 1$.

Now we may state the formula of Pizzetti-Nicolescu:

Proposition 4 *i) Suppose $f \in H^{2p}(D)$. Then for any ball $B_h(x)$ strictly contained in D , the following equation holds:*

$$\mu(f, x; h) = f(x) + \sum_{j=1}^{p-1} a_{dj} h^{2j} \Delta^j f(x) + J^p(\mu(\Delta^p f, x; \cdot); h) \cdot \begin{cases} \frac{1}{(d-2)^p} & \text{for } d \geq 3 \\ 1 & \text{for } d = 2 \end{cases}$$

(ii) The remainder in the Pizzetti-Nicolescu formula can be written as:

$$\begin{aligned} J^p(\mu(\Delta^p f, x; \cdot); h) &= \mu(\Delta^p f, x; \vartheta_p h) \cdot J^p(1; h) = \\ &= \Delta^p f(\xi_{x,p}) \begin{cases} a_{dp} h^{2p} (d-2)^p & \text{for } d \geq 3 \\ a_{dp} h^{2p} & \text{for } d = 2 \end{cases} \end{aligned}$$

for suitable $\xi_{x,p} \in B_h(x)$.

For a concise proof of the above results see [H-K], cf. also the original paper of Nicolescu [Ni].

Proof of the Theorem.

We will prove that the function

$$F_0(x) = f_0(r^2), \quad r = |x| \quad (5)$$

is an extremizer.

Let us put $h = (\frac{1}{2})^{\frac{1}{2}}$. Remark that $F_0(0) = -1$, $F_0(h) = 1$.

Applying the formula of Pizzetti-Nicolescu for $p = 2$ and $x = 0$ we obtain:

$$\mu(f, 0; h) = f(0) + a_{d1} h^2 \Delta f(0) + J^2(\mu(\Delta^2 f, 0; \cdot); h) \cdot \begin{cases} \frac{1}{(d-2)^2} & \text{for } d \geq 3 \\ 1 & \text{for } d = 2 \end{cases} \quad (6)$$

On the other hand we have

$$a_{d1} = \frac{1}{2d}, \quad a_{d2} = \frac{1}{8d(d+2)} \quad \text{for } d \geq 2$$

and

$$J^2(1; t) = \begin{cases} a_{d2} (d-2)^2 t^4 & \text{for } d \geq 3 \\ \frac{1}{64} t^4 & \text{for } d = 2 \end{cases}$$

Taking into account that the radial part of the Laplace operator is

$$\Delta_r = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r}$$

and that by definition we have

$$F_0(x) = f_0(|x|^2) = 1 - 4 \left(|x|^2 - \frac{1}{\sqrt{2}} \right)^2 \quad |x| = r \leq h$$

we obtain the following values:

$$\begin{aligned} \Delta F_0(x) &= \Delta_r F_0(r) \\ &= \Delta_r f_0(r^2) \\ &= (f_0(r^2))'' + \frac{d-1}{r} (f_0(r^2))' \\ &= (2r f_0'(r^2))' + 2(d-1) f_0'(r^2) \\ &= 2f_0''(r^2) + 4r^2 f_0'''(r^2) + 2(d-1) f_0'(r^2) \\ &= 4r^2 f_0''(r^2) + 2d f_0'(r^2) \\ &= 4r^2(-8) + 2d \left(-8 \left(r^2 - \frac{1}{\sqrt{2}} \right) \right) \\ &= 16 \left(-2r^2 - dr^2 + \frac{d}{\sqrt{2}} \right). \end{aligned}$$

It follows that the maximum of $|\Delta_r F_0(r)|$ in the interval $r \leq h$ is obtained for $r = 0$, i.e. it is equal to

$$\Delta_r F_0(0) = 8\sqrt{2d}.$$

Due to the symmetry of the function E_2 , it follows that this is also the absolute maximum for every $r \geq 0$, i.e.

$$\|\Delta_r F_0\|_\infty = 8\sqrt{2d}.$$

Also taking into account that for $r \leq h$ the function $F_0(x)$ is a polynomial of degree 4 in r , we obtain

$$\begin{aligned} \Delta^2 F_0(x) &= -4\Delta_r^2(r^4) \\ &= -32d(d+2) \quad \text{for } r = |x| \leq h. \end{aligned}$$

and due to the definition of the function f_0 we have

$$|\Delta^2 F_0(x)| = 32d(d+2) \quad \text{for every } x \in \mathbf{R}^d,$$

i.e.

$$\|\Delta^2 F_0(x)\|_\infty = 32d(d+2).$$

In formula (4.3) we put $f = F_0$. Using the above formulas and the fact that the spherical mean of a radially symmetric function is the value of the function at that radius, we obtain:

$$\mu(F_0, 0; h) = F_0(h)$$

In a similar way since $\Delta^2 F_0(x)$ is a constant for $r = |x| \leq h$ we obtain

$$\mu(\Delta^2 F_0, 0; h) = \Delta^2 F_0 = -32d(d+2)$$

and finally

$$F_0(h) = F_0(0) + a_{d1}h^2\Delta F_0(0) + \Delta^2 F_0(0) J^2(1; h) \cdot c_{2,d}$$

It is important to note that we have alternatingly in sign

$$\begin{aligned} F_0(0) &= -1 \\ F_0(h) &= 1 \\ \Delta^2 F_0(0) &= -32d(d+2) \end{aligned}$$

which is decisive to obtain the following key identity:

$$\begin{aligned} a_{d1}h^2\Delta F_0(0) &= -F_0(0) + F_0(h) - \Delta^2 F_0(0) J^2(1; h) \cdot c_{2,d} = \\ &= 1 + 1 + \|\Delta^2 F_0\|_\infty J^2(1; h) \cdot c_{2,d} \end{aligned}$$

Let now f be an arbitrary function for which $\|f\|_\infty \leq 1$, $\|\Delta^2 f\|_\infty \leq \|\Delta^2 F_0\|_\infty = 32d(d+2)$. From formula (4.3) we express $\Delta f(0)$:

$$a_{d1}h^2\Delta f(0) = -f(0) + \mu(f, 0; h) - J^2(\mu(\Delta^2 f, 0; \cdot); h) \cdot c_{2,d}$$

Since $J^2(1; h) > 0$, and the kernel of the operator $J^2(\cdot, r)$ is evidently positive we obtain the following inequality:

$$\begin{aligned} a_{d1}h^2|\Delta f(0)| &\leq \|-f(0) + \mu(f, 0; h) - J^2(\mu(\Delta^2 f, 0; \cdot); h) \cdot c_{2,d}\|_\infty \leq \\ &\leq \|f\|_\infty + \|\mu(f, 0; h)\|_\infty + \|J^2(\mu(\Delta^2 f, 0; \cdot); h) \cdot c_{2,d}\|_\infty \\ &\leq \|f\|_\infty + \|f\|_\infty + \|\mu(\Delta^2 f, 0; \cdot)\|_\infty J^2(1; h) \cdot c_{2,d} \\ &\leq \|f\|_\infty + \|f\|_\infty + \|\Delta^2 f\|_\infty J^2(1; h) \cdot c_{2,d} \\ &\leq \|f\|_\infty + \|f\|_\infty + \|\Delta^2 F_0\|_\infty J^2(1; h) \cdot c_{2,d} \\ &\leq 1 + 1 + \|\Delta^2 F_0\|_\infty J^2(1; h) \cdot c_{2,d} \end{aligned}$$

By the proved above we obtain

$$\begin{aligned} a_{d1}h^2|\Delta f(0)| &\leq a_{d1}h^2\Delta F_0(0) \\ &= a_{d1}h^2\|\Delta F_0\|_\infty. \end{aligned}$$

It follows that

$$|\Delta f(0)| \leq \|\Delta F_0\|_\infty.$$

We may apply the above arguments to the function $f(x+x_0)$, and prove the above inequality for every point $x_0 \in \mathbf{R}^d$, i.e.

$$|\Delta f(x_0)| \leq \|\Delta F_0\|_\infty.$$

Hence it follows that

$$\|\Delta f\|_\infty \leq \|\Delta F_0\|_\infty.$$

This finishes the proof of the Theorem in the formulation of Landau.

If we put

$$\begin{aligned} K_d &= \frac{\|\Delta F_0\|_\infty}{\sqrt{\|\Delta^2 F_0\|_\infty}} \\ &= \frac{8\sqrt{2}d}{\sqrt{32d(d+2)}} \\ &= 2\sqrt{\frac{d}{d+2}} \end{aligned}$$

then we obtain equality in (4.1) for the function $f = F_0$. For arbitrary function f inequality (4.1) follows by the above arguments. That finishes the proof of the Theorem also in the formulation of Hadamard.

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