Multivariate Bernoulli functions and polyharmonically exact cubature of Euler-Maclaurin

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Abstract

In the present paper we provide a multivariate generalization of the Euler-Maclaurin formula which is based on an appropriate multivariate extension of the Bernoulli functions.

1 Introduction

One of the most beautiful devices of classical analysis which has found numerous applications in number theory and divergent series, see Ramanujan [2, p. 7 in Vol. 1, Chapters 7,8], Hardy [11, Chapter 13], and which plays profound role in numerical integration Davis-Rabinowitz [4, Chapter 2.9], is the Euler-Maclaurin formula.

In the present paper we provide a multivariate generalization of the Euler-Maclaurin formula which is based on an appropriate multivariate extension of the Bernoulli functions. As a cornerstone of our generalization we take the property of the Euler-Maclaurin formula, which might be called *reproduction with respect to neighboring domains* the explanation of which follows.

1. The celebrated *summation formula of Euler-Maclaurin* is the following (cf. [11, p. 323], [4, p. 109], Chakravarti [3], Gel'fond [9], Luogeng-Yuan [12]):

$$\begin{aligned} &\frac{1}{2}f(0)+\sum_{i=1}^{n-1}f(i)+\frac{1}{2}f(n) \\ &=\int_0^n f(t)\,dt+\sum_{j=1}^{k}\frac{B_{2j}}{(2j)!}\left[f^{(2j-1)}(n)-f^{(2j-1)}(0)\right]+R^s(f) \end{aligned} \tag{1}$$

with remainder term $R^s(f)$ of *even order* or of *odd order* as follows

$$R^s(f)=\begin{cases} -\int_0^n P_{2k}(t)\,f^{(2k)}(t)\,dt & \text{for } f \text{ in } C^{2k}[0,n] \\ \int_0^n P_{2k+1}(t)\,f^{(2k+1)}(t)\,dt & \text{for } f \text{ in } C^{2k+1}[0,n]. \end{cases} \tag{2}$$

The Bernoulli functions $P_k(t)$ are defined in Section 2. Here appear the so-called Bernoulli numbers (the even ones), given by $B_{2j}=[(2j)!]\,P_{2j}(0)=[(2j)!]\,P_{2j}(1)$. By making a change of the variables in (1), namely $t\longmapsto a+ht$, where $h=(b-a)/n$, we obtain the *quadrature form* of the Euler-Maclaurin formula (cf. [31, Ch. 7.21], [4, p. 109])

$$\begin{aligned} \int_a^b f(t)\,dt &= h\left[\frac{1}{2}f(a)+\sum_{i=1}^{n-1}f(a+ih)+\frac{1}{2}f(b)\right] \\ &\quad -\sum_{j=1}^{k}\frac{B_{2j}}{(2j)!}h^{2j}\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right]+R^c(f), \end{aligned} \tag{3}$$

where the remainder term $R^c(f)$ has the *even order* or *odd order* form

$$R^c(f) = \begin{cases} h^{2k} \int\limits_a^b P_{2k}\left(n\frac{t-a}{b-a}\right) f^{(2k)}(t)\,dt & \text{for } f \text{ in } C^{2k}[a,b] \\ -h^{2k+1} \int\limits_a^b P_{2k+1}\left(n\frac{t-a}{b-a}\right) f^{(2k+1)}(t)\,dt & \text{for } f \text{ in } C^{2k+1}[a,b]. \end{cases}$$

This formula is called *quadrature* since it gives an approximation to the integral on the left-hand side through the so-called *compound trapezoidal rule*

$$TR(f) = h\left[\frac{1}{2}f(a) + \sum_{i=1}^{n-1} f(a+ih) + \frac{1}{2}f(b)\right]$$

minus linear combinations of the odd order derivatives, which we will call *boundary terms*,

$$BT(f) = \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!} h^{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right].$$

The remainder term $R^c(f)$ on the right-hand side of (3) is thought of as the error of the approximation. This terminology applies clearly also to the summation Euler-Maclaurin formula.

Now we may formulate the meaning of the *reproduction* property. Let us write formula (3) for two neighboring intervals having the same length, then the terms containing derivatives at the common midpoint cancel each other. Or, more generally, imagine that we write formula (3) (putting $n = 1$) for every subinterval $[a+jh, a+(j+1)h]$, $j = 0, ..., n-1$. After summing we see that the terms with interior derivatives cancel and we obtain formula (3) itself, i.e. it has reproduced itself! We see that the periodicity of the Bernoulli polynomials plays an essential role.

In a similar way the *reproduction* property is seen by the summation formula (1).

2. Having in mind the *reproduction property* as a main motivation we will produce a quite different type of multivariate generalization of the Euler-Maclaurin formula compared to the usual.

The usual way in which the Euler-Maclaurin formula is generalized, see Sobolev [26], Mikhlin [20, Chapter 10], Stroud [29, and refs. there], [23], has for an object to obtain an approximation of the integral in terms of values of the function at the integer points (or the points of some lattice), or, from the point of view of summation formulas, the sum over integer points is expressed through the integral over the domain solely, Shaneson [25, and refs. tere]. In general, this is the *standard polynomial paradigm* on which the one-dimensional quadrature and the *multivariate cubature formulas* are based, Engels [6], Lyness-Cools [18], [22], [23], namely, the formulas are *exact for polynomials* up to a certain degree. So far, (to the authors' knowledge) none of the results obtained in these references generalizes the *reproduction with respect to neighboring domains* property.

On the other hand, our multivariate approach gives approximation of the integral of the $d-$variable function f over a $d-$dimensional domain by integrals of the same function f over surfaces of lower dimension, $d-1$, $d-2$, etc., in such a way that the formula obtained would possess the property of *reproduction with respect to neighboring domains* (at least for some special domains). An important feature of this approach is that we obtain formulas which are *exact for functions which are polyharmonic* of a certain degree or which are solutions to similar higher order partial differential equations. The last fact is a manifestation of what we call *polyharmonicity paradigm.* It has proved to be very successful in treating problems in multivariate approximation through polyharmonic functions and in Spline Theory and Wavelet Analysis based on piecewise polyharmonic functions (polysplines), see [13], [14], [16], as well as in applications of multivariate polysplines to optimal recovery and cubature formulas, see [15]. It may be taken as an alternative starting point of our multivariate approach to the Euler-Maclaurin formulas as well.

3. One of the discoveries of the present research was the observation that the theory of the one-dimensional Bernoulli polynomials may be viewed in the framework of the *Hodge theory.* Namely, due to the basic property of the classical Bernoulli functions $\left[B_{k+1}^{*}\left(t\right)\right]' = (k+1)B_{k}^{*}\left(t\right)$ and by their $1-$periodicity the following equality is obvious $\int_0^1 B_k^*\left(t\right)dt = 0$. The last is necessary for the solubility of the problem $u''\left(t\right) = B_k^*\left(t\right)$, for $0 \leq t \leq 1$, in terms of 1-periodic functions u, i.e. of the corresponding Poisson problem on the one-dimensional torus. Further, the solution u of the Poisson problem on the one-dimensional torus is uniquely determined by the orthogonality condition $\int_0^1 u\left(t\right)dt = 0$.

Thus having found the first two $1-$periodic Bernoulli functions $B_1^*(t) = t - 1/2,$ for $0 < t < 1$, $B_1^*(0) = 0$, and $B_0^*(t) = [B_1^*(t)]' = 1 - \delta(t)$, for $0 \leq t < 1$, the series of Bernoulli functions can be constructed by splitting them into two independent series. The first one B_0^*, B_2^*, B_4^*, $\cdots$ consists of all even order Bernoulli functions. By using B_{2k}^* we construct B_{2k+2}^*, $k = 0, 1, 2, \ldots$ as the unique 1-periodic solution of the Poisson problem $u''(t) = B_{2k}^*(t)$, satisfying $\int_0^1 u(t)dt = 0$ on the one-dimensional torus. The second series B_1^*, B_3^*, B_5^*, $\cdots$ consists of all odd order Bernoulli functions. Analgously, B_{2k+3}^* is the unique $1-$periodic solution of the Poisson problem $v''(t) = B_{2k+1}^*(t),$ satisfying $\int_0^1 v(t)dt = 0$ on the one-dimensional torus. This new treatment of splitting the series of the one-variate Bernoulli functions into two independent series on the basis of the Poisson periodic problem gives rise to their multivariate extension that we are going to present in this paper. This general framework motivates our definition of *multivariate Bernoulli functions* and delineates the perspective of even more considerable generalizations.

Let us note that $1-$periodic functions can be considered as functions, defined on the unit circle. As usually, the unit circle $\mathbb{S}^1$ is obtained through factorization of the interval $[0, 1]$ by identifying the points 0 and 1 and having measure 1. The generalized function B_0^* is orthogonal to the constants also, but in generalized sense.

It is of basic importance to choose proper initial functions $P_{0,d}\left(x\right)$ *and*

$P_{1,d}(x)$, the analogs to B_0^* and B_1^* in order to get Euler-Maclaurin cubature possessing desired properties as reproduction with respect to neighboring domains and being exact, say for the infinite dimensional space of all polyharmonic functions of given order. For the proper choice of such we are lead by the idea explained above to approximate the integral $\int_D f(x)dx$ by a linear combination of integrals over manifolds of lower dimension, not only zero-dimensional.

We obtain two series of multivariate Euler-Maclaurin formulas in the unit cube $D_{\text{uc}} = [0,1]^d \subset \mathbb{R}^n$. The first series $P_{2k,d}$ is based on a proper generalization of the *even order Bernoulli functions* P_{2k} which appear in formula (1). We choose as initial function, see below formula (11), the following one:

$$P_{0,d}(x) = d - \sum_{k=1}^{d} \sum_{j=-\infty}^{\infty} \delta(x_k - j).$$

The second series $P_{2k+1,d}$ corresponds to the *odd order Bernoulli functions* P_{2k+1}. Here we choose as initial function the following one:

$$P_{1,d}(x) = \prod_{j=1}^{d} B_1(x_j).$$

The main purpose of the present paper is to study the even order case which is related to the function $P_{0,d}(x)$.

As said above, the one-dimensional Bernoulli functions satisfy $[B_k^*(t)]' = k\,B_{k-1}^*(t)$ or that is equivalent $[P_k^*(t)]' = P_{k-1}^*(t)$. In a similar way our multivariate Bernoulli functions are 1−periodic in every variable and satisfy the equation

$$\Delta P_{2k+1,d} = P_{2k-1,d} \quad \text{and} \quad \Delta P_{2k,d} = P_{2k-2,d} \qquad \text{for } k = 1, 2, \ldots,$$

as well as the orthogonality condition $\int_{D_{\text{uc}}} P_{k,d}(x)\,dx = 0$. Let us remark that the last condition is evidently true for $P_{0,d}$ and $P_{1,d}$. Within the Hodge theory it is a necessary condition for the solubility of the Poisson problem

$$\Delta u(x) = P_{k,d}(x)$$

on the torus $\mathbb{S}^d$, cf. Griffiths-Harris [10, p. 89], and represents the orthogonality to the harmonic space on the torus which consists of the constants only [10, p. 89]. Now the solution u is determined up to a constant on the torus since by the same argument the weak harmonic space consists only of the constant functions. Thus, u is uniquely determined by putting

$$\int_{D_{\text{uc}}} u(x)\,dx = 0.$$

Following this way, having the functions $P_{0,d}$ and $P_{1,d}$ appropriately chosen we produce two independent series of multivariate Bernoulli functions $P_{2k,d}$

(even order multivariate Bernoulli functions) and $P_{2k+1,d}$ (odd order multivariate Bernoulli functions).

We prove that (see (19) below)

$$P_{2k,d}(x) = \sum_{j=1}^{d} P_{2k}(x_j)$$

where P_{2k} are the one-dimensional normalized Bernoulli functions. So far, for the odd order we have multiple Fourier series expression, namely,

$$P_{2k+1,d}(x) = (-1)^{d+k} \frac{1}{(2\pi)^{2k}} \sum_{l_1=1}^{\infty} \dots \sum_{l_d=1}^{\infty} \frac{1}{|l|^{2k}} \left\{ \prod_{j=1}^{d} \frac{2\sin(2\pi l_j x_j)}{2\pi l_j} \right\} ,$$

where $l = (l_1, l_2, \dots, l_d)$ and $|l| = \left(l_1^2 + l_2^2 \cdots + l_d^2\right)^{1/2}$.

4. In this paper we give in details a multivariate extension of the Euler-Maclaurin formula that corresponds to the even order multivariate Bernoulli functions, i.e. with even order remainder term. The *formula of Euler-Maclaurin type* which we obtain for the unit cube $D_{\mathrm{uc}} = [0,1]^d \subset \mathbb{R}^d$ may be written as (see Theorem 3, page 19, below)

$$\begin{aligned} \int_{D_{\mathrm{uc}}} f(x)\,dx \ &= \ \frac{1}{n} \sum_{\substack{0 \le l \le n \\ 1 \le j \le d}} \gamma_l \int_{S_{j,l}} f(x)\,d\sigma_{l,x} \qquad (4) \\ &+ \sum_{j=1}^{d} \sum_{k=1}^{m} \frac{1}{n^{2k}} \overbrace{\int_0^1 \dots \int_0^1}^{d-1} \left\{ \frac{\partial}{\partial x_j} \Delta^{k-1} f(x)_{|x_j=1} - \frac{\partial}{\partial x_j} \Delta^{k-1} f(x)_{|x_j=0} \right\} \\ &\times P_{2k,d}(nx)_{|x_j=0}\, dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_d + \frac{1}{n^{2m}} R_{2m}^{c}(f), \end{aligned}$$

where for the integers l, j, with $0 \le l \le n,\ 1 \le j \le d$, we have denoted the hyperplanes

$$S_{j,l} = \{x = (x_1 \dots, x_d) \in D_{\mathrm{uc}} \colon x_j = l/n\}$$

and the constants

$$\gamma_l = \begin{cases} 1/(2d) & \text{for } l = 0, n, \\ 1/d & \text{for } 1 \le l \le n-1. \end{cases}$$

Here, $d\sigma_{l,x}$ denotes the $d-1$-dimensional Lebesgue measure on every surface $S_{j,l}$.

The term

$$\frac{1}{n} \sum_{\substack{0 \le l \le n \\ 1 \le j \le d}} \gamma_l \int_{S_{j,l}} f(x)\,d\sigma_{l,x}$$

is a generalization of the composed trapezoidal rule.

Now we see that the next terms containing the difference of derivatives

$$\frac{\partial}{\partial x_j}\Delta^{k-1}f\left(x\right)_{|x_j=1}-\frac{\partial}{\partial x_j}\Delta^{k-1}f\left(x\right)_{|x_j=0}$$

are analog to the boundary terms with the difference of derivatives

$$f^{(2j-1)}\left(1\right)-f^{(2j-1)}\left(0\right)$$

in (3), where we have taken for clearness the case of the unit interval, $a=0,\ b=1$. Also the functions $P_{2k,d}$ are $1-$periodic in every variable.

Hence, it follows immediately that formula (4) possesses the above proclaimed property of *reproduction with respect to neighboring domains.* Indeed, if we add more neighboring cubes, the interior terms containing $\frac{\partial}{\partial x_j}\Delta^{k-1}f\left(x\right)$ will cancel due to the periodicity of $P_{2k,d}$. Consequently, we will have also the *stability with respect to subdivisions property.*

Let us point out that the cubature formula (4) possesses an acceleration effect with respect to the order of approximation, and the reproduction with respect to the neighboring domains property *implies a minimal computational complexity of the cubature formula (4) which makes it very efficient from algorithmic point of view.*

The even order remainder term (that is called an approximation error also) is of the form

$$R_{2m}^{c}\left(f\right)\ =\ \frac{1}{d}\int\limits_{D_{\mathrm{uc}}}P_{2m,d}\left(nx\right)\Delta^{m}f\left(x\right)dx$$

which will be considered in general to be bounded.

As seen from the representation of the remainder term $R_{2m}\left(f\right)$, our cubature formula (4) is exact for the class of functions which are polyharmonic of order m in the cube D_{uc}, i.e. satisfy $\Delta^{m}f\left(x\right)=0$ for $x\in D_{\mathrm{uc}}$ (for basic facts on polyharmonic functions see Aronszajn-Creese-Lipkin [1]).

The meaning of formula (4) is that we subdivide the unit cube D_{uc} into n^{d} smaller cubes of sides $\frac{1}{n}$, and on the right-hand side of the cubature we have integrals over the boundaries $S_{j,l}$ of the small cubes. So, the formula (4) is *a compound cubature formula of Euler-Maclaurin type.*

Let us remark that for $d=1$ the cubature formula (4) is the well known univariate Euler–Maclaurin quadrature (3) with even order remainder term.

5. The proofs in general rely upon the invertability of the Laplace operator on the $d-$dimensional torus (the Hodge theory for $0-$forms).

6. After the main results of the present work were obtained the authors made a search through MathSciNet under keyword "Euler-Maclaurin" by title and review text. The number of references on May 16, 2000 was 388. There are quite few dealing with multivariate generalizations of the Euler-Maclaurin formula and we hope that we have mentioned above some of the main references (which also contain further references).

We have to point out to the wide circle of papers of S.L. Sobolev and his school in the area of cubature formulas, [26, and refs. therein], and Müller-Freeden [19], where the integral functional $\int_D f(x)dx$ is approximated through linear combinations of function values, i.e. through Dirac delta functions. The formulas are exact for a given power of a fixed elliptic operator of second order (Δ in [26] and $\Delta+\lambda$ in [19]). The main feature is that they choose as initial function

$$P_{0,d}\left(x\right)=1-\sum_{\alpha\in\mathbb{Z}^{d}}\delta_{d}\left(x-\alpha\right)$$

(considered as error functional in [26]) where δ_d is the $d-$dimensional Dirac delta function. So far such a choice does not produce multivariate Bernoulli functions and respectively does not generalize the reproduction property.

The results of the present paper were announced in [5]. The results concerning the odd order case will be presented in detail in a forthcoming paper.

2 Univariate Bernoulli functions

To make the things clear we remind the basic properties of the one-dimensional Bernoulli polynomials. We shall denote by $B_k\left(t\right)$ the one-dimensional *Bernoulli polynomials* in the interval $[0,1]$ and through $B_k^*\left(t\right)$ their periodic continuation which are called *Bernoulli functions*. We shall give a multivariate extension of a normed version of the Bernoulli functions, the functions $P_k\left(t\right)=B_k^*\left(t\right)/k!$. They seem to be somewhat more convenient to deal with. For simplicity we will call them also Bernoulli functions. We will list some of the properties of the one-dimensional Bernoulli polynomials and functions. We display the $1-$periodic on the whole real line Bernoulli functions $P_k\left(t\right)$ (cf. [11, p. 320-322], [4, p. 107], [17, Chapter 1]) as follows: $P_0\left(t\right)=1-\delta(t)$, for $0\leq t<1$, and $P_1\left(t\right)=t-1/2$, for $0<t<1$, $P_1\left(0\right)=0$. Moreover we have:

$$P_{2k}\left(t\right)=\left(-1\right)^{k-1}\sum_{l=1}^{\infty}\frac{2\cos\left(2\pi lt\right)}{\left(2\pi l\right)^{2k}}\qquad\text{for } k\geq 0,\tag{5}$$

where for $k=0$ the series is considered in generalized sense and

$$P_{2k+1}\left(t\right)=\left(-1\right)^{k-1}\sum_{l=1}^{\infty}\frac{2\sin\left(2\pi lt\right)}{\left(2\pi l\right)^{2k+1}}\qquad\text{for } k\geq 0.$$

The Bernoulli functions have the following properties: $P'_{k+1}\left(t\right)=P_k\left(t\right)$, for $k\geq 0$ and $P_{2k+1}\left(0\right)=P_{2k+1}\left(1\right)=0$, for $k\geq 0$. In addition

$$P_{2k}\left(0\right)=P_{2k}\left(1\right)=\frac{B_{2k}}{(2k)!}=\left(-1\right)^{k-1}\sum_{l=1}^{\infty}\frac{2}{\left(2\pi l\right)^{2k}}\qquad\text{for } k\geq 1.$$

The classical polynomials of Bernoulli are defined like (cf. [17, Chapter 1]) $B_k\left(t\right)=\left(k!\right)P_k\left(t\right)$, for $k\geq 0$ and $0<t<1$, so we may write $B_0(t)=1$, $B_1(t)=t-1/2$ and $B_2(t)=t^2-t+1/6$.

The usual definition of the Bernoulli polynomials is often through generating polynomials, or using the Bernoulli numbers, cf. [11, p. 320-322]. As it is explain in the Introduction, point 3, they can be defined by splitting into two series on the basis of the Poisson problem on the torus $\mathbb{T}^1 = \mathbb{S}^1$ which will be a starting point for the multivariate generalization.

3 Even order Euler-Maclaurin multivariate formula

Our consideration in the multivariate case will parallel the results in the one-dimensional case which may be found in, e.g., [11, Chapter XIII],[4, p. 107]. Here we assume that the appropriate derivatives of the function f exist and are continuous. In the multidimensional setting we shall consider the cube $D_n = [0, n]^d$ and we shall work with integer multi-indexes $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{Z}^d$. We introduce the subdivision of D_n into smaller unit cubes

$$\begin{aligned} D_\alpha &= \prod_{i=1}^{d} [\alpha_i, \alpha_i + 1] \\ &= \alpha + D_{\text{uc}} \qquad \text{for } 0 \le \alpha_i \le n-1, \end{aligned} \tag{6}$$

where $D_{\text{uc}} = [0, 1]^d$. Now the $d-$dimensional torus

$$\mathbb{T}^d = (\mathbb{S})^d$$

is also obtained through factorization of D_{uc} or which is the same

$$\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d.$$

Let us remark that we will use this representation in order to take into account the property of the Bernoulli polynomials being $1-$periodic.

In order to make a reasonable multivariate generalization of the Euler-Maclaurin formula we remark that the *zero order Bernoulli* function is in fact given by

$$P_0(t) = 1 - \sum_{j=-\infty}^{\infty} \delta(t - j) \tag{7}$$

but not simply by 1 as is usually taken in [21, p. 19], [27, p. 120]. This follows through

$$P_1'(t) = P_0(t) = 1 - \delta(t) \qquad \text{for } 0 \le t < 1$$

in distributional sense. Like a distribution, the function P_0 satisfies property

$$\langle P_0, 1 \rangle = \int_0^1 P_0(t)\, dt = 0. \tag{8}$$

Then the problem

$$u''(t) = P_0(t) \qquad (0 \le t \le 1) \tag{9}$$

has a periodic solution, or which is the same, the problem considered on the one-dimensional torus $\mathbb{T}^1$ has a solution determined up to a constant. This is a classical fact from the theory of ordinary differential equations. On the other hand it may be considered as a result about the Laplace operator on the one-dimensional torus $\mathbb{T}^1 = \mathbb{S}^1$. In order to emphasize on the analogy between the one-dimensional and the multidimensional case we will refer to the Hodge theory for $0-$forms on the torus, see ([10, p. 91],[30, Ch. 6]), in which the orthogonality condition (8) is necessary for solubility of problem (9) on the one-dimensional torus $\mathbb{T}^1$.

Now the function $P_2(t)$ is a *continuous* periodic solution to the equation (9) and is uniquely determined like such through the additional condition

$$\int_0^1 P_2(t)\, dt = 0.$$

Hence, we obtain inductively the periodic functions $P_4, P_{6,} \ldots$, using the Green operator, (cf. [10, p. 91]), which also satisfy

$$\int_0^1 P_{2k}(t)\, dt = 0 \tag{10}$$

etc.

Let us go back to the multivariate setting. We introduce the following notations:

$$\widehat{x}_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) \text{ and } \widehat{dx}_j = dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_d.$$

We will define in the $d-$dimensional setting the *zero order* $d-$*dimensional Bernoulli function* by putting:

$$P_{0,d}(x) = d - \sum_{k=1}^{d} \sum_{j=-\infty}^{\infty} \delta(x_k - j) \qquad \text{for } x = (x_1, \ldots, x_d) \in \mathbb{R}^d. \tag{11}$$

Here we have denoted the delta functions of the hyperplanes by

$$\delta(x_k - j) = \delta(x_k - j)\, 1_{\widehat{x}_k}, \tag{12}$$

where $1_{\widehat{x}_k}$ denotes the unity function on the variables $\widehat{x}_k$ and δ denotes the one-dimensional *delta-function.*

Notice that the function $P_{0,d}$ in (11) satisfies the orthogonality condition

$$\langle P_{0,d}\, , \, 1 \rangle = \overbrace{\int_0^1 \ldots \int_0^1}^{d} P_{0,d}(x)\, dx = 0. \tag{13}$$

Then we will look for the $1-$*periodic* solution in all variables to the *Poisson problem*

$$\Delta P_{2,d}(x) = P_{0,d}(x) \tag{14}$$

which satisfies the orthogonality condition

$$\langle P_{2,d}\,,\,1\rangle = 0. \tag{15}$$

Similar to above, condition (13) is a necessary and sufficient one for the solubility of problem (14), and condition (15) provides uniqueness, cf. [10, p. 91]. The uniqueness follows from the fact that the only periodic harmonic functions are the constants, or which is the same, the only harmonic functions on the torus are the constants, see [10, p. 89].

The unique solution is easy to check to be given by *the second Bernoulli function in dimension* d :

$$P_{2,d}\left(x\right) = \sum_{j=1}^{d} P_{2}\left(x_{j}\right).$$

The last function is periodic and satisfies both (14), (15), the last due to (10).

3.1 First Order Formula of Euler-Maclaurin for the Unit Cube $D_{\mathrm{uc}} = [0,1]^d$

The idea is to replace the ordinary differential operator d^2/dx^2 through the Laplace operator Δ. Assuming that $f \in C^2\left(\overline{D_{\mathrm{uc}}}\right)$ we obtain

$$\begin{aligned}\int_{D_{\mathrm{uc}}} \Delta f\left(x\right) P_{2,d} dx &= \int_0^1 \ldots \int_0^1 \Delta f\left(x\right) P_{2,d}\left(x\right) dx \\ &= \sum_{j=1}^{d} \int_0^1 \ldots \int_0^1 \frac{\partial^2}{dx_j^2} f\left(x\right) P_{2,d}\left(x\right) dx.\end{aligned}$$

We apply integration by parts twice (or use the Green formula, which is the same) in the domain D_{uc} and obtain the following:

$$\begin{aligned}
&\int_{D_{\mathrm{uc}}} \frac{\partial^2}{dx_j^2} f(x)\, P_{2,d}(x)\, dx \\
&= \int_0^1 \dots \int_0^1 \frac{\partial^2}{dx_j^2} f(x)\, P_{2,d}(x)\, dx \\
&= \int_0^1 \dots \int_0^1 \sum_{k\neq j} \frac{\partial^2}{dx_j^2} f(x)\, P_2(x_k)\, dx + \int_0^1 \dots \int_0^1 \frac{\partial^2}{dx_j^2} f(x)\, P_2(x_j)\, dx \\
&= \overbrace{\int_0^1 \dots \int_0^1}^{d-1} \left(\frac{\partial}{dx_j} f(x)_{|x_j=1} - \frac{\partial}{dx_j} f(x)_{|x_j=0} \right) \left(\sum_{\substack{k=1\\k\neq j}}^{d} P_2(x_k) + P_2(1) \right) \widehat{dx_j} \\
&- \overbrace{\int_0^1 \dots \int_0^1}^{d} \frac{\partial}{dx_j} f(x)\, P_1(x_j)\, dx.
\end{aligned}$$

Let us denote

$$I_j = \overbrace{\int_0^1 \dots \int_0^1}^{d-1} \left(\frac{\partial}{dx_j} f(x)_{|x_j=1} - \frac{\partial}{dx_j} f(x)_{|x_j=0} \right) \left(\sum_{\substack{k=1\\k\neq j}}^{d} P_2(x_k) + P_2(1) \right) \widehat{dx_j}.$$

Since

$$\begin{aligned}
&\int_0^1 \dots \int_0^1 \frac{\partial}{dx_j} f(x)\, P_1(x_j)\, dx \\
&= -\int_0^1 \dots \int_0^1 f(x)\, dx + \frac{1}{2} \overbrace{\int_0^1 \dots \int_0^1}^{d-1} \left(f(x)_{|x_j=1} + f(x)_{|x_j=0} \right) \widehat{dx_j}
\end{aligned}$$

we obtain

$$\overbrace{\int_0^1 \cdots \int_0^1}^{d} \Delta f(x)\, P_{2,d}(x)\, dx$$

$$= \sum_{j=1}^{d} I_j + d \int_0^1 \cdots \int_0^1 f(x)\, dx - \frac{1}{2} \sum_{j=1}^{d} \overbrace{\int_0^1 \cdots \int_0^1}^{d-1} \left(f(x)_{|x_j=1} + f(x)_{|x_j=0} \right) \widehat{dx}_j .$$

Hence, the simplest *Euler-Maclaurin formula* becomes

$$\overbrace{\int_0^1 \cdots \int_0^1}^{d} f(x)\, dx = TR(f) - BT(f) + \frac{1}{d} \overbrace{\int_0^1 \cdots \int_0^1}^{d} \Delta f(x)\, P_{2,d}(x)\, dx, \tag{16}$$

where we have put for the boundary terms

$$BT(f) = \frac{1}{d} \sum_{j=1}^{d} I_j$$

and for the so-called "trapezoidal rule":

$$\begin{aligned} TR(f) &= \frac{1}{2d} \sum_{j=1}^{d} \overbrace{\int_0^1 \cdots \int_0^1}^{d-1} \left(f(x)_{|x_j=1} + f(x)_{|x_j=0} \right) \widehat{dx}_j \\ &= \frac{1}{2d} \int_{\partial D_{\mathrm{uc}}} f(x)\, d\sigma_x , \end{aligned}$$

where $d\sigma_x$ is the surface element on the piecewise smooth surface ∂D_{uc}. Let us remark that the function $P_2(\cdot)$ is periodic and continuous with period 1 on the whole line, and consequently $P_{2,d}(\cdot)$ is periodic in every variable with period 1.

The above formula expresses the multiple $d-$dimensional integral $\int_0^1 \cdots \int_0^1 f(x)\, dx$ through integrals of dimension $d-1$.

3.2 The Two-dimensional Case

In the two-dimensional case, $d = 2$ we obtain

$$\begin{aligned}
&\int_0^1 \int_0^1 f(x,y)\,dxdy \\
&= \frac{1}{4}\left\{ \int_0^1 [f(1,y) + f(0,y)]\,dy + \int_0^1 [f(x,1) + f(x,0)]\,dx \right\} \\
&- \frac{1}{2}\left\{ \int_0^1 (P_2(1) + P_2(y)) \left[\frac{\partial}{dx} f(1,y) - \frac{\partial}{dx} f(0,y)\right] dy \right. \\
&\left. + \int_0^1 (P_2(1) + P_2(x)) \left[\frac{\partial}{dy} f(x,1) - \frac{\partial}{dy} f(x,0)\right] dx \right\} \\
&+ \frac{1}{2} \int_0^1 \int_0^1 \Delta f(x,y)\, P_{2,d}(x,y)\,dxdy.
\end{aligned}$$

Hence, after putting for the "trapezoidal rule"

$$TR(f) = \frac{1}{4}\left\{ \int_0^1 [f(1,y) + f(0,y)]\,dy + \int_0^1 [f(x,1) + f(x,0)]\,dx \right\}$$

and for the boundary terms

$$\begin{aligned}
BT(f) = \frac{1}{2}&\left\{ \int_0^1 (P_2(1) + P_2(y)) \left[\frac{\partial}{dx} f(1,y) - \frac{\partial}{dx} f(0,y)\right] dy \right. \\
&\left. + \int_0^1 (P_2(1) + P_2(x)) \left[\frac{\partial}{dy} f(x,1) - \frac{\partial}{dy} f(x,0)\right] dx \right\},
\end{aligned}$$

we obtain an expression for the integral

$$\int_0^1 \int_0^1 f(x,y)\,dxdy = TR(f) - BT(f) + \frac{1}{2} \int_0^1 \int_0^1 \Delta f(x,y)\, P_{2,d}(x,y)\,dxdy$$

which is a sum of a cubature formula and a remainder containing $\Delta f(x,y)$.

The above formula evidently expresses the multiple integral $\int_0^1 \int_0^1 f(x,y)\,dxdy$ through integrals of dimension 1 and is *exact for functions harmonic in the cube.* Let us remark the analogy with the one-dimensional case where the simplest Euler-Maclaurin formula is exact for linear polynomials.

3.3 The Higher Order Formula of Euler-Maclaurin for the Unit Cube $[0,1]^d$

Further we will proceed inductively by defining the *multivariate Bernoulli functions* through the recurrent system of Poisson equations

$$\Delta P_{2k,d}(x) = P_{2k-2,d}(x) \qquad \text{for } k \geq 2, \tag{17}$$

where P_{2k} is continuous and periodic and satisfies the orthogonality condition

$$\langle P_{2k,d}(x), 1\rangle = 0. \tag{18}$$

The *unique solution* to (17) is easy to find, it is given by

$$P_{2k,d}(x) = \sum_{j=1}^{d} P_{2k}(x_j). \tag{19}$$

Now we are prepared to carry out the next step. Assuming that $f \in C^4\left(\overline{D_{\mathrm{uc}}}\right)$ we apply the formula of Green ([1, p. 9]) (or make twice integration by parts, which is the same) and due to $\Delta P_{4,d}(x) = P_{2,d}(x)$ we obtain the following:

$$\begin{aligned}
&\int_{D_{\mathrm{uc}}} \Delta f(x)\, P_{2,d}(x)\, dx \\
&= \int_{D_{\mathrm{uc}}} \Delta f(x)\, \Delta P_{4,d}(x)\, dx \\
&= \int_{D_{\mathrm{uc}}} \Delta^2 f(x)\, P_{4,d}(x)\, dx \\
&+ \int\limits_{\partial D_{\mathrm{uc}}} \left\{ \Delta f(x) \frac{\partial}{\partial \nu_x} P_{4,d}(x) - \frac{\partial}{\partial \nu_x} \Delta f(x)\, P_{4,d}(x) \right\} d\sigma_x.
\end{aligned}$$

Using the properties of the one-dimensional *Bernoulli polynomials* we may simplify the boundary integral in above expression and obtain

$$\int\limits_{\partial D_{\mathrm{uc}}} \Delta f(x) \frac{\partial}{\partial \nu_x} P_{4,d}(x)\, d\sigma_x = 0.$$

Indeed, on every flat piece $x_j = 0$ or $x_j = 1$ of the boundary ∂D_{uc} we have the following:

$$\frac{\partial}{\partial \nu_x} P_{4,d}(x) = \frac{\partial}{\partial x_j} P_{4,d}(x)_{|x_j=0 \text{ or } x_j=1} = P_3(x_j)_{|x_j=0 \text{ or } x_j=1} = 0.$$

Hence, the above equality becomes

$$\begin{aligned}
&\int_{D_{\mathrm{uc}}} \Delta f(x)\, P_{2,d}(x)\, dx \\
&= \int_{D_{\mathrm{uc}}} \Delta^2 f(x)\, P_{4,d}(x)\, dx - \int\limits_{\partial D_{\mathrm{uc}}} \frac{\partial}{\partial \nu_x} \Delta f(x)\, P_{4,d}(x)\, d\sigma_x\ .
\end{aligned}$$

Similarly, we make use of

$$\frac{\partial}{\partial \nu_x} P_{2k,d}(x) = \frac{\partial}{\partial x_j} P_{2k,d}(x)_{|x_j=0 \text{ or } x_j=1} = P_{2k-1}(x_j)_{|x_j=0 \text{ or } x_j=1} = 0 \quad (20)$$

for $k \geq 2$.

We will now prove *a formula of Euler-Maclaurin type.*

Theorem 1 *Let the even order Bernoulli function $P_{2k,d}$ be given by (19). Let the function $f \in C^{2m}\left(\overline{D_{\mathrm{uc}}}\right)$. Then*

$$\begin{aligned} &\int_{D_{\mathrm{uc}}} f(x)\,dx \\ &= TR_{D_{\mathrm{uc}}}(f) - \frac{1}{d}\sum_{k=1}^{m} \int\limits_{\partial D_{\mathrm{uc}}} \frac{\partial}{\partial \nu_x}\Delta^{k-1} f(x)\,P_{2k,d}(x)\,d\sigma_x + \\ &\qquad + \frac{1}{d}\int\limits_{D_{\mathrm{uc}}} \Delta^m f(x)\,P_{2m,d}(x)\,dx, \end{aligned} \quad (21)$$

where the trapezoidal sum is

$$TR_{D_{\mathrm{uc}}}(f) = \frac{1}{2d}\int\limits_{\partial D_{\mathrm{uc}}} f(x)\,d\sigma_x.$$

Proof. We apply induction argument and obtain by using (20)

$$\begin{aligned} &\int_{D_{\mathrm{uc}}} \Delta^j f(x)\,P_{2,d}(x)\,dx \\ &= \int_{D_{\mathrm{uc}}} \Delta^{j+1} f(x)\,P_{4,d}(x)\,dx - \int\limits_{\partial D_{\mathrm{uc}}} \frac{\partial}{\partial \nu_x}\Delta^j f(x)\,P_{4,d}(x)\,d\sigma_x \end{aligned}$$

for $j \geq 1$. Hence,

$$\begin{aligned} &\int_{D_{\mathrm{uc}}} f(x)\,dx \\ &= TR_{D_{\mathrm{uc}}}(f) - BT_{D_{\mathrm{uc}}}(f) + \frac{1}{d}\int\limits_{D_{\mathrm{uc}}} \Delta^1 f(x)\,P_{2,d}(x)\,dx \\ &= TR_{D_{\mathrm{uc}}}(f) - \frac{1}{d}\sum_{k=1}^{m}\int\limits_{\partial D_{\mathrm{uc}}} \frac{\partial}{\partial \nu_x}\Delta^{k-1} f(x)\,P_{2k,d}(x)\,d\sigma_x \\ &+ \frac{1}{d}\int\limits_{D_{\mathrm{uc}}} \Delta^m f(x)\,P_{2m,d}(x)\,dx, \end{aligned}$$

which proves the theorem.

■

3.4 The Higher order Summation Formula of Euler-Maclaurin Type for the Cube $[0,n]^d$

Here we do the analog to the one-dimensional subdivision of the interval into n equal subintervals.

Evidently, we have from (6) the following equality

$$D_n = \bigcup_{\substack{i=1,\dots,d \\ \alpha\in\mathbb{Z}^d \\ 0\leq\alpha_i\leq n-1}} D_\alpha \ .$$

We choose a natural number $n \geq 1$. Due to the periodicity of the functions $P_{2k,d}$ we may write formula (21) for every domain D_α, $0 \leq \alpha_i \leq n-1$, namely,

$$\begin{aligned}
&\int_{D_\alpha} f(x)\,dx \\
&= TR_{D_\alpha}(f) - \frac{1}{d}\sum_{k=1}^{m}\int_{\partial D_\alpha}\frac{\partial}{\partial\nu_x}\Delta^{k-1}f(x)\,P_{2k,d}(x)\,d\sigma_x \\
&+ \frac{1}{d}\int_{D_\alpha}\Delta^m f(x)\,P_{2m,d}(x)\,dx \ .
\end{aligned}$$

Finally, we obtain the *summation formula of Euler-Maclaurin type* for the cube $D_n = [0,n]^d$:

Theorem 2 *For every function $f \in C^{2m}\left(\overline{D_n}\right)$ the following formula holds:*

$$\begin{aligned}
&\int_{D_n} f(x)\,dx && (22)\\
&= TR_{D_n}(f) - BT_{D_n}(f) + \frac{1}{d}\int_{D_n}\Delta^m f(x)\,P_{2m,d}(x)\,dx,
\end{aligned}$$

where the trapezoidal sum is

$$TR_{D_n}(f) = \frac{1}{d}\sum_{k=1}^{n-1}\sum_{j=1}^{d}\overbrace{\int_0^n \dots \int_0^n}^{d-1} f(x)_{|x_j=k}\,\widehat{dx}_j + \frac{1}{2d}\int_{\partial D_n} f(x)\,d\sigma_x,$$

and the boundary terms are

$$\begin{aligned}
&BT_{D_n}(f) \\
&= \frac{1}{d}\sum_{k=1}^{m}\sum_{j=1}^{d}\overbrace{\int_0^n \dots \int_0^n}^{d-1}\left\{\frac{\partial}{\partial x_j}\Delta^{k-1}f(x)_{|x_j=n} - \frac{\partial}{\partial x_j}\Delta^{k-1}f(x)_{|x_j=0}\right\}P_{2k,d}(x)_{|x_j=0}\,\widehat{dx}_j.
\end{aligned}$$

Proof. We sum up over $\alpha \in \mathbb{Z}^d$ with $0 \leq \alpha_i \leq n-1$ and obtain

$$\int_{D_n} f(x)\,dx = \sum_{\substack{i=1,\dots,d \\ \alpha\in\mathbb{Z}^d \\ 0\leq\alpha_i\leq n-1}} \int_{D_\alpha} f(x)\,dx$$

$$= TR_{D_n}(f) - \frac{1}{d}\sum_{k=1}^{m}\int_{\partial D_n} \frac{\partial}{\partial\nu}\Delta^{k-1} f(x)\, P_{2k,d}(x)\,d\sigma$$

$$+ \frac{1}{d}\int_{D_n} \Delta^m f(x)\, P_{2m,d}(x)\,dx,$$

where we have put

$$TR_{D_n}(f) = \sum_{\substack{i=1,\dots,d \\ 0\leq\alpha_i\leq n-1}} TR_{D_\alpha}(f)$$

$$= \frac{1}{d}\sum_{k=1}^{n-1}\sum_{j=1}^{d} \overbrace{\int_0^n \dots \int_0^n}^{d-1} f(x)_{|x_j=k}\, \widehat{dx}_j + \frac{1}{2d}\sum_{\substack{k=0\\k=n}}\sum_{j=1}^{d} \overbrace{\int_0^n \dots \int_0^n}^{d-1} f(x)_{|x_j=k}\, \widehat{dx}_j$$

$$= \frac{1}{d}\sum_{k=1}^{n-1}\sum_{j=1}^{d} \overbrace{\int_0^n \dots \int_0^n}^{d-1} f(x)_{|x_j=k}\, \widehat{dx}_j + \frac{1}{2d}\int_{\partial D_n} f(x)\,d\sigma_{d-1,x}.$$

Let us remark that the integrals on the interior boundaries ∂D_α , namely

$$\int_{\partial D_\alpha} \frac{\partial}{\partial\nu_x}\Delta^{k-1} f(x)\, P_{2k,d}(x)\,d\sigma_x,$$

have cancelled due to continuity of $P_{2k,d}(x)$, $k \geq 1$ and the fact that $\frac{\partial}{\partial\nu_x}\Delta^{k-1} f(x)$ is continuous and with different signs when taken for neighboring cubes.

Due to the periodicity of $P_{2k,d}(x)$ the rest of the terms on the boundary ∂D_n group and give the following:

$$BT_{D_n}(f) = \frac{1}{d}\sum_{k=1}^{m}\int_{\partial D_n} \frac{\partial}{\partial\nu_x}\Delta^{k-1} f(x)\, P_{2k,d}(x)\,d\sigma$$

$$= \frac{1}{d}\sum_{k=1}^{m}\sum_{j=1}^{d} \overbrace{\int_0^n \dots \int_0^n}^{d-1} \left\{ \frac{\partial}{\partial x_j}\Delta^{k-1} f(x)_{|x_j=n} - \frac{\partial}{\partial x_j}\Delta^{k-1} f(x)_{|x_j=0} \right\} P_{2k,d}(x)_{|x_j=0}\, \widehat{dx}_j.$$

This finishes the proof.

■

3.5 The cubature Formula of Euler-Maclaurin Type

We apply formula (22) to the function $g(x) = f\left(\frac{x}{n}\right)$ and after making the change of variables $x = ny$, we obtain *a cubature formula of Euler-Maclaurin type:*

Theorem 3 *For every function $f \in C^{2m}\left(\overline{D_{\text{uc}}}\right)$ the following formula holds*

$$\int_{D_{\text{uc}}} f(y)\,dy = TR(f) - BT(f) + R(f), \tag{23}$$

where the trapezoidal sum is given by

$$TR(f) = \frac{1}{nd}\sum_{k=1}^{n-1}\sum_{j=1}^{d}\overbrace{\int_0^1 \dots \int_0^1}^{d-1} f(y)_{|y_j=k/n}\,\widehat{dy}_j + \frac{1}{2nd}\sum_{\substack{k=0\\k=n}}\sum_{j=1}^{d}\overbrace{\int_0^1 \dots \int_0^1}^{d-1} f(y)_{|y_j=k/n}\,\widehat{dy}_j$$

the boundary terms by

$$BT(f) = \frac{1}{d}\sum_{j=1}^{d}\sum_{k=1}^{m}\frac{1}{n^{2k}}\overbrace{\int_0^1 \dots \int_0^1}^{d-1}\left\{\frac{\partial}{\partial y_j}\Delta_y^{k-1} f(y)_{|y_j=1} - \frac{\partial}{\partial y_j}\Delta_y^{k-1} f(y)_{|y_j=0}\right\}$$
$$\times P_{2k,d}(ny)_{|y_j=0}\,\widehat{dy}_j$$

and the remainder by

$$R(f) = \frac{1}{dn^{2m}}\int_{D_{\text{uc}}} \Delta_y^m f(y)\, P_{2m,d}(ny)\,dy.$$

Remark 4 *Let us remark that the above formula possesses the reproduction with respect to neighboring domains property. This is evident due to the specific form of the boundary terms in $BT(f)$ which cancel at the interior points if a neighboring cube is added. Let us remark that for $d = 1$ the above cubature formula coincides with the one - dimensional Euler-Maclaurin quadrature formula (3) with even order remainder term.*

Another immediate result is the following:

Corollary 5 *The summation formula (22) is exact for polyharmonic functions of order m in D_n. The cubature formula (23) is exact for functions polyharmonic of order m in D_{uc}.*

Theorem 3 implies that the cubature formula (23) possesses an acceleration effect concerning the order of approximation. The next corollary shows the approximate effectiveness of the cubature formula (23).

Corollary 6 *Let the function $f \in C^{2m}$ be 1−periodic in every variable. Let $|\Delta^m f| \le M$. Then in the notations of Theorem 3 the following estimate holds*

$$\left| \int_{D_{\mathrm{uc}}} f(y)\, dy - TR(f) \right| \le MC \frac{\zeta(2m)}{n^{2m}}$$

where $C = 2/[(2\pi)^{2m}]$ and $\zeta(\cdot)$ denotes the Riemann zeta function.

Proof. The proof follows directly from Theorem 3 and the representation of the Bernoulli functions P_{2k} in (5),(19). ■.

3.6 The Euler–Maclaurin formula for arbitrary domains

A natural effect of the periodicity of the multivariate Bernoulli functions $P_{2k,d}$ is that all results above hold for the case of an arbitrary domain with smooth boundary. One has to add some more terms on the boundary. We confine ourselves to only stating the cubature formula.

Theorem 7 *Let the bounded domain G have a smooth boundary which does not intersect itself. Then for every function $f \in C^{2m}\left(\overline{G}\right)$ the following cubature formula of Euler–Maclaurin type holds*

$$\begin{aligned} &\int_G f(y)\, dy & (24)\\ &= TR(f) - \sum_{k=1}^{m} \frac{1}{n^{2k}} \int\limits_{\partial G} \frac{\partial}{\partial \nu_y} \Delta^{k-1} f(y)\, P_{2k,d}(ny)\, d\sigma_y \\ &\quad + \sum_{k=1}^{m} \frac{1}{n^{2k}} \int\limits_{\partial G} \Delta^{k-1} f(y) \frac{\partial}{\partial \nu_y} P_{2k,d}(ny)\, d\sigma_y \\ &\quad + \frac{1}{dn^{2m}} \int\limits_{G} \Delta^m f(y)\, P_{2m,d}(ny)\, dy \end{aligned}$$

where the trapezoidal sum is given by

$$TR(f) = \frac{1}{2dn} \sum_{k=-\infty}^{\infty} \sum_{j=1}^{d} \int\limits_{\{G, y_j = k/n\}} ind_G(y)\, f(y)_{|y_j = k/n}\, \widehat{dy}_j$$

and the index $ind_G(y)$ is defined for every point $y \in G$ as 1 if $y \in \partial G$ and as 2 if $y \in G$; for y out of the closure of G we have $ind_G(y) = 0$.

The proof follows from the cubature formula (23) and Green's formula [1].

Let us remark that the additional boundary terms appear since the functions $\frac{\partial}{\partial \nu_y} P_{2k,d}(ny)$ may be non-zero on the boundary ∂G. The above formula may be written also if the boundary is only piecewise smooth. Then the index $ind_G(y)$ has to be changed appropriately.

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