Optimal Recovery of Linear Functionals of Peano Type Through Data on Manifolds

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0. Introduction.

In the present paper we consider a multivariate problem of optimal recovery of a linear functional using data on $(n-1)$ – dimensional manifolds, cf. problem (2.8) below. For its solution we apply the notion of polyspline which was introduced and studied by us in [9,10,11].

Our main purpose is to show that, once given the existence results, the properties of the polysplines are very similar to those of the univariate splines, at least what concerns questions of optimal recovery.

The class of functionals which we recover is a subset of the class of so–called functionals of *Peano type* which was specified in Haußmann–Kounchev [7], cf. (2.4)–(2.5) and Remark 2.3.1 below.

Schoenberg was the first who realized the extraordinary importance of the splines in the optimal recovery problems in one dimension. To explain the analogy of our result with the classical theory, we first recall the optimal recovery theorem in the univariate case which was proved by Schoenberg [18]. We shall follow Laurent [13, Theorem 4.1.4] and Bojanov–Hakopian–Sahakian [4, Theorem 5.10].

Throughout this paper we will make an extensive use of the following classical results from the theory of Sobolev spaces and Elliptic boundary value problems:

1. The *interior regularity theorem*: Lions–Magenes [14, Chapter II, Theorem 3.2].

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2. The *global regularity theorem*, or regularity up to the boundary: Lions–Magenes [Chapter II, Theorem 5.3].

3. The *trace theorem*: Lions–Magenes [14, Chapter I, Theorem 9.4], Adams [1, Chapter VII, Section 7.35, Theorem 7.57, Theorem 7.58].

4. The *Sobolev embedding theorem*: Adams [1, Theorem 7.57].

5. The *density theorem* for Sobolev spaces, e.g. H^k is dense in H^{k-1} : Lions–Magenes [Chapter I, Theorem 8.2], Adams [1, Theorem 3.18].

In fact, we will work only in domains and subdomains where the above results are applicable.

1. Univariate Optimal Recovery of a Linear Functional of Peano Type.

Let J be a linear functional which is defined in the Sobolev space $H^r([a,b])$ on the interval $[a,b]$, where

$$H^r[a,b] = \{f \in C^{r-1}[a,b] : f^{(r-1)}\text{is absolutely continuous and } \| f^{(r)} \|_{L_2[a,b]} < \infty\}.$$

We assume that the functional J is of *Peano type*, i.e. is given by (see Davis [6, p. 70, formula (3.7.1)]):

$$J(f) = \int_a^b (\sum_{j=0}^{r-1} a_j(x) f^{(j)}(x))dx + \sum_{k=1}^{m} b_k f^{(\lambda_k)}(t_k), \tag{1.1}$$

where the functions $\{a_j(x)\}_0^{r-1}$ are integrable in $[a,b]$, $\{b_k\}_1^m$ are real numbers, and the integers λ_k satisfy $0 \le \lambda_k \le r-1, k = 1, \ldots, m$, and the points $t_1, \ldots, t_m \in [a,b]$.

Let $x_1 < \ldots < x_N$ be points in $[a,b]$, and S be an arbitrary linear functional which uses only point evaluation at $x_1, \ldots, x_N$, i.e. information out of the set $T(f) = \{f(x_1), \ldots, f(x_N)\}$. Thus:

$$S(f) = \sum_{j=1}^{N} \lambda_j f(x_j), \tag{1.2}$$

Let us denote by $S_f(x)$ the natural spline of order $2r-1$ such that

$$S_f(x_j) = f(x_j), \qquad j = 1, \ldots, \text{N}.$$

Recall that the natural splines satisfy by definition the boundary conditions

$$S^{(j)}(x_1) = S^{(j)}(x_N) = 0, \qquad j = r, \ldots, 2r-1. \tag{1.3}$$

For an arbitrary linear functional J we may generate immediately the linear functional $\overline{L}(f) = J(S_f)$.

The distance between two functionals L_1 and L_2 will be measured in the class

$$B_{D^r}H^r[a,b] := \{f \in H^r[a,b] : \| D^r f \|_{L_2[a,b]} \leq 1\},$$

where $D^r := f^{(r)}$. Let us put

$$\| L_1 - L_2 \|_{B_{D^r}H^r[a,b]} := \sup_{f \in B_{D^r}H^r[a,b]} | L_1(f) - L_2(f) | .$$

The problem of *optimal recovery of a linear functional* (cf. Sard [17]) can be stated as follows : Find a functional L of type (1.2) solving the extremal problem:

$$\| J - L \|_{B_{D^r}H^r[a,b]} = \inf_S (\| J - S \|_{B_{D^r}H^r[a,b]}), \tag{1.4}$$

where S runs over the class of functionals given by (1.2).

The following theorem of Schoenberg (cf. [18], Micchelli–Rivlin [15], Bojanov [4, Theorem 5.10] shows the fundamental role played by the natural splines:

THEOREM 1.1. Let J be an arbitrary linear functional of Peano type, defined in $H^r[a,b]$, and $\{x_k\}_1^N$ be fixed distinct points in the interval $[a,b]$. Then the functional $\overline{L}(f) = J(S_f)$ is the best method for recovery of the functional J with respect to the class $B_{D^r}H^r[a,b]$ by the information $T(f) = \{f(x_1), \ldots, f(x_N)\}$, i.e. $\overline{L}$ solves problem (1.4).

REMARK 1.2. In fact, Theorem 1.1 holds in a more general class of recovering functionals S than those given by (1.2) above. Namely, we may consider the class of functionals

$$S(f) = S_*(f(x_1), \ldots, f(x_N)). \tag{1.2'}$$

defined by an arbitrary function S_* of N variables rather than the linear combinations used in (1.2). The proof is obtained by using the so–called Smolyak's lemma, cf. Micchelli–Rivlin [15], Bojanov–Hakopian–Sahakian [4].

2. The Multivariate Problem of Optimal Recovery of a Linear Functional.

The very essence of our concept of optimal recovery in the multivariate case is that we assume the information set $T(f)$ to be defined on a union of $(n-1)$ – dimensional manifolds. This reflects the situation in the 1–dimensional case, where one uses information on points $x_1, \ldots, x_N$, i.e. on manifolds of dimension $n-1=0$, as done in Section 1.

Let now $n \geq 2$ and the bounded domain $D \subset \mathbf{R}^n$ have a connected complement, and assume that $N \geq 1$ manifolds, $\Gamma_1, \ldots, \Gamma_N$, of dimension $n-1$ be given, such that $\Gamma_j = \partial D_j, j = 1, \ldots, N-1$, where D_j is a family of nested domains having connected complement and satisfying $D_j \subset \overline{D}_j \subset D_{j+1}$, for $j = 0, \ldots, N-1$. and $D_N = D$. For convenience we have put $D_0 = \emptyset$. We assume that the domains D_j are such that $\Gamma_j = \partial D_j$ are C^∞ manifolds, D_j being locally on one side of Γ_j, cf. Lions–Magenes [14, p. 111], and we can apply without restrictions the theory of elliptic boundary value problems in every of the domains D_j , cf. Lions–Magenes [14, Chapter II]. For every integer $q \geq 1$, we will work in the Sobolev space $H^{2q}(D)$ of functions supplied with the usual norm

$$\| f \|^2_{H^{2q}(D)} = \sum_{|\alpha| \leq 2q} \int_D | D^\alpha f(x) |^2 \, dx \qquad (< \infty),$$

where D^α denotes the partial derivative of order $|\alpha| = \sum_{i=1}^{d} \alpha_i \leq 2p$ with $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$.

For $q \geq 1$ the trace of the function $f \in H^{2q}(D)$ on the manifold Γ_j (that is the "restriction to Γ_j" in the sense explained in Lions–Magenes [14, Chapter I, Section 9.2],[1, p.113]) will be denoted by $\pi_j(f)$. Due to the *trace theorem* the last belongs to the Sobolev space of a fractional order, namely:

$$\pi_j(f) \in H^{2q-\frac{1}{2}}(\Gamma_j). \tag{2.1}$$

The core of our concept is to use only the information

$$T(f) = (\pi_1(f), \ldots, \pi_N(f)) \tag{2.2}$$

for recovering a linear functional J defined on $H^{2q}(D)$.

For the recovery we will use a class $\mathcal{L}$ of linear functionals L using only the information $T(f)$ in the sense that they are given by

$$L(f) = \sum_{j=1}^{N} l_j(\pi_j(f)) \tag{2.3}$$

where l_j are linear functionals of the type

$$l_j(g) = \int_{\Gamma_j} g(x) \lambda_j(x) d\sigma_j(x).$$

More precisely, we say that $L \in \mathcal{L}$ if and only if

$$(2.4) \qquad L(f) = \sum_{j=1}^{N} \int_{\Gamma_j} \pi_j(f)(x)\lambda_j(x)d\sigma_j(x),$$

where $\lambda_j \in L_2(\Gamma_j)$ and $d\sigma_j$ is the area element on Γ_j.

REMARK 2.1. Due to the *Sobolev embedding theorem* we have that $H^s \subset L_2,\ s > 0$, on every compact C^∞ manifold. It follows that $\pi_j(f) \in L_2(\Gamma_j)$ and the existence of the integrals in (2.4).

The functionals J on $H^{2q}(D)$ which we will recover are of *simple Peano type*: They consist of terms

$$L_{S,r,m}(f) := \int_S f(x)m(x)d\sigma(x),$$

where $S \subset D$ is an r–dimensional compact smooth manifold ($0 \le r \le n$), $m \in C(S)$, and σ is the area element on S. Let us remark that in the case $r = 0$ we have that S is a point (or union of such), say x_1, and then $L_{S,r,m}(f) = Cf(x_1)$, for some constant C. We assume that S is such that one may apply the *trace theorem* on manifolds of lower dimension.

DEFINITION 2.2. The functional J is of *simple Peano type* if it may be represented like

$$(2.5) \qquad J(f) \ := \ \sum_{k=1}^{K} L_{S_k,r_k,m_k}(f),$$

where for $r_k = n$ we have $S = D$, and $2q - \frac{(n-r_k)}{2} > 0,\ k = 1, \dots, K$.

PROPOSITION 2.3. The functional J is continuous in $H^{2q}(D)$.

Proof : Applying the *trace theorem* to a manifold S of dimension r, see [1, Theorem 7.58], we obtain that $f_{\big|_S} \in H^{2q-\frac{(n-r_k)}{2}}$. According to the Sobolev embedding theorem it follows that $f_{\big|_S} \in L_2(S)$. The proof is finished. Q.E.D.

REMARK 2.4. 1. The most popular example of a functional of a simple Peano type is

$$J(f) \ := \ \int_D f(x)dx.$$

2. The functionals of *Peano type* were defined in Haußmann–Kounchev [7]. They consist of terms which contain some differentiation of order α, and can be easily written by putting

$$L_{S,r,\alpha,m}(f) = L_{S,r,m}(D^\alpha f).$$

In the present paper we consider only *simple Peano type* functionals to avoid the technical difficulties arising from considering Sobolev spaces of negative order.

3. Note that the functionals in $\mathcal{L}$ are of Peano type, and they are defined on every Sobolev space $H^s(D)$, $s \geq 1$, since due to the trace theorem, if $f \in H^s(D)$ then $\pi_j(f) \in L_2(\Gamma_j)$.

Basic for establishing our main result, Theorem 5.4, is that the functionals J of the type (2.5) satisfy a Peano kernel representation theorem (see Theorem 5.2 below and Haußmann–Kounchev [7]) in case they vanish on a space of polyharmonic functions of some degree.

We will measure the distance between the functionals with respect to the following class:

$$B_{\Delta^q} H^{4q}(D) := \{f \in H^{4q}(D) : \| \Delta^q f \|_{L_2[a,b]} \leq 1\} \tag{2.6}$$

by putting

$$\| L_1 - L_2 \|_{B_{\Delta^q} H^{4q}(D)} := \sup_{f \in B_{\Delta^q} H^{4q}(D)} | L_1(f) - L_2(f) | \tag{2.7}$$

for every two functionals L_1 and L_2 defined on $H^{4q}(D)$.

Now we state the problem of optimal recovery of a linear functional of Peano type (in analogy with problem (1.4)):

Find a functional L solving the extremal problem:

$$\| J - L \|_{B_{\Delta^q} H^{4q}(D)} = \inf_S (\| J - S \|_{B_{\Delta^q} H^{4q}(D)}), \tag{2.8}$$

where S runs the class $\mathcal{L}$ of functionals given by (2.4).

In the previous paragraph we saw that in the univariate case one needs the notion of natural spline in order to solve the problem of optimal recovery. In the next paragraph we will introduce the notion of polyspline needed to solve problem (2.8).

3. Polysplines.

The notion of polyspline of order $2q$ was introduced and studied in [9,10,11]. Using the D_j and the Γ_j as in Section 2, let us put

$$T = \bigcup_{j=1}^{N} \Gamma_j$$

and

$$\widetilde{D}_j = D_j \setminus \overline{D_{j-1}} \qquad \text{for} \quad j = 1, \ldots, N;$$

recall that $\widetilde{D}_0 = \emptyset$.

For a function u defined in the domain D we shall denote by u_j its restriction to the subdomains $\widetilde{D}_j, j = 1, \ldots, N$.

DEFINITION 3.1. Let $q > 0$. We will say that the function $u \in H^{4q}(D \setminus T)$ is a polyspline of order $2q$ for the given data set T if and only if the following conditions hold:

$$(3.1_a) \qquad \Delta^{2q} u_j(x) = 0, \qquad \text{for } x \in \widetilde{D}_j, \quad j = 1, 2, \ldots, N;$$

$$(3.1_b) \qquad \Delta^k u_N(x) = 0, \qquad \text{for } x \in \Gamma_N = \partial D, \quad k = q, \ldots, 2q-2;$$

$$(3.1_c) \qquad \frac{\partial}{\partial n_N} \Delta^k u_N(x) = 0, \qquad \text{for } x \in \Gamma_N = \partial D, \quad k = q, \ldots, 2q-1;$$

$$(3.1_d) \qquad \Delta^p u_j(x) = \Delta^p u_{j+1}(x), \qquad \text{for } x \in \Gamma_j$$

for $\quad j = 1, \ldots, N-1, \quad p = 0, \ldots, 2q-1;$

$$(3.1_e) \qquad \frac{\partial \Delta^p u_j(x)}{\partial n_j} = \frac{\partial \Delta^p u_{j+1}(x)}{\partial n_j}, \quad \text{for } x \in \Gamma_j,$$

for $j = 1, 2, \ldots, N-1$, and $p = 0, 1, \ldots, 2q-2$.

Equalities $(3.1_a)-(3.1_e)$ *should be considered like equalities between traces of functions taken from the appropriate subdomains.*

The space of all polysplines will be denoted by $PS_{2q}(D)$ or simply, by PS_{2q}.

DEFINITION 3.2. By PS'_{2q} we denote the space of all functions $u \in H^{4q-1}(D \setminus T) \cap H^{4q}_{loc}(D \setminus T)$ such that conditions $(3.1_a) - (3.1_e)$ hold.

The following statement can be found in Kounchev [9],[11,Theorem 6.2.4]:

THEOREM 3.3. Let $f_j \in H^{2q-\frac{1}{2}}(\Gamma_j), j = 1, \ldots, N$. Then the following interpolation problem has a unique solution $u \in PS_{2q}(D)$:

$$(3.2) \qquad u = f_j \quad \text{on} \quad \Gamma_j, \quad j = 1, \ldots, N,$$

where the equality is understood in the sense of traces.

THEOREM 3.4. The space PS'_{2q} is dense in PS_{2q} in the topology of $H^{4q-1}(D \setminus T)$.

Proof : The proof follows immediately from the *density theorem for Sobolev spaces* applied to the spaces $H^{4q}(D \setminus T)$ and $H^{4q-1}(D \setminus T)$ in the topology of the last.Q.E.D.

The operator which maps the data $\{f_j\}$ onto the polyspline $u \in PS_{2q}(D)$ will be denoted by S_f, i.e. $u(x) = S_f(x), x \in \overline{D}$.

REMARK 3.5. 1. For proving the uniqueness of problem (3.2) one needs a special identity available in [9,11].

2. For proving the existence in Theorem 3.3, let us remark that problem $(3.1_a), (3.1_d), (3.1_e)$ is in fact an interface problem where the interface surfaces are Γ_j. As such it may be treated through the general approach described e.g. in Chazarain–Piriou [5, Chapter 6.9, p. 277].

3. Let us note that the conditions on the "external boundary" $\Gamma_N = \partial D$ given by $(3.1_b), (3.1_c)$ are a subject of rather free choice; we may take arbitrary set of boundary operators $\{id, B_2(f), \ldots, B_{2q}(f)\}$ which covers the operator Δ^{2q} in the sense explained in Lions–Magenes [14, p.128]. Our choice here is dictated by the analogy with conditions (1.3) satisfied by the natural splines. Such a choice guarantees that every function $f \in H^{4q}(D \backslash T)$ satisfying $\Delta^q f = 0$ in D, is in fact in $PS_{2q}(D)$.

4. Polyspline Green Function.

The main result of the present paragraph is to show that the linear functional $\overline{L}$ in $H^{4q}(D)$ given by

$$\overline{L}(f) = J(S_f) \tag{4.1}$$

is a functional of the type (2.3),(2.4).

Later we will prove that in fact $\overline{L}$ solves problem (2.8). Thus we will obtain a full analogy with the univariate case using the same construct just by substituting the natural splines through the polysplines.

THEOREM 4.1. The functional $\overline{L}$ defined by $\overline{L}(f) = J(S_f)$ has the form (2.4).

REMARK 4.2. Actually, by Theorem 4.1, formula (2.4) shows that although $\overline{L}$ is initially defined only on $H^{4q}(D)$, we may extend it at least to $H^{2q}(D)$.

For proving Theorem 4.1 we will introduce the *polyspline Green function*.
We will need some definitions.

As in the previous sections we put $T = \bigcup_{j=1}^{N} \Gamma_j$.

We introduce the space of polyharmonic functions of order p in $H^{2p}(D)$ by putting:

$$PH_p(D) := \{h \in H^{2p}(D) : \Delta^p h = 0 \text{ in } D\}.$$

Here Δ^p is the p–th power of the Laplacian operator Δ with $\Delta^0 := id$.

Denote by R_p the fundamental solution for the operator Δ^p in $\mathbb{R}^n$, $p \in \mathbb{N}$, which is given by (cf. Aronszajn–Creese–Lipkin [3])

$$R_p(x) = R_p(|x|) = r^{2p-n}(A_{p,n} \log(r) + B_{p,n})$$

where $r := |x|$, and $A_{p,n}$ and $B_{p,n}$ are appropriate constants with $A_{p,n} \neq 0$ only for even n and $p \geq \frac{n}{2}$. Note that throughout this paper we use the notation $f(x) = f(|x|)$ to indicate the radial symmetry of a function f.

The function R_p has the property that

$$\Delta^k R_p = R_{p-k} \quad \text{for } 0 \leq k < p. \tag{4.2}$$

For details we refer again to [3].

We will need also the constant Ω_n by

$$\Omega_n = \begin{cases} \dfrac{1}{(n-2)\sigma_n} & \text{for } n \geq 3, \\ \dfrac{1}{\sigma_n} & \text{for } n = 2, \end{cases}$$

where $\sigma_n := \dfrac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ denotes the area of the $(n-1)$–sphere $S_{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$.

For every $y \in D \setminus T$ consider the function $x \to R_{2q}(x-y)$. For every $j = 1, \ldots, N$ we have that $\pi_j(R_{2q}(.-y)) \in C^\infty(\Gamma_j)$. Hence, for every $y \in \bigcup D \setminus T$ the interpolation problem

$$v(x) + R_{2q}(x-y) = 0, \qquad x \in \bigcup \Gamma_j \tag{4.3$_a$}$$

$$v \in PS_{2q}(D) \tag{4.3$_b$}$$

has a unique solution $v(x) = v_y(x)$, as explained in Section 3.

For $y \in T = \bigcup \Gamma_j$ we will put

$$v_y(x) = -R_{2q}(x-y). \tag{4.3$_c$}$$

Note that such v_y satisfies conditions $(3.1_a), (3.1_d), (3.1_e)$ but $v_y(x) \notin H^{4q}(D \setminus T)$. Let us define the *polyspline Green function* by

$$G(x,y) = v_y(x) + R_{2q}(x-y). \tag{4.4}$$

LEMMA 4.3. For every $s \in PS_{2q}(D)$ and every $y \in D \setminus T$ the following equality holds:

$$s(y) = \sum_{j=1}^{N} \int_{\Gamma_j} s(x) \{ \frac{\partial \Delta^{2q-1} G(x,y)}{\partial \nu_x} \}_j d\sigma_j(x), \tag{4.5}$$

where ν_x is the inner unit normal to Γ_j at the point x and $d\sigma_j$ denotes the area element of Γ_j, and $\{\partial \Delta^{2q-1}/\partial \nu_x G(x,y)\}_j$ denotes the jump of the normal derivative on Γ_j which is equal to

$$\frac{\partial \Delta^{2q-1} G_{j-1}(x,y)}{\partial \nu_x} - \frac{\partial \Delta^{2q-1} G_j(x,y)}{\partial \nu_x} =$$
$$\frac{\partial \Delta^{2q-1} v_{y,j-1}(x)}{\partial \nu_x} - \frac{\partial \Delta^{2q-1} v_{y,j}(x)}{\partial \nu_x}, \qquad j = 1, \ldots, N-2.$$

Here G_j denotes the restriction of G to the domain $\widetilde{D}_j$. For $j = N$ it is equal to

$$-\frac{\partial \Delta^{2q-1} G_N(x,y)}{\partial \nu_x}.$$

Proof : . Let $s \in PS_{2q}(D)$. According to the first Green formula, see Aronszajn–Creese–Lipkin [3, p.10, formula (2.9)], Lions–Magenes [14, Remark 2.2 on p. 120] we have that

$$\sum_{l=0}^{2q-1} \int_{\partial \widetilde{D}_j} \left(\Delta^l s(x) \frac{\partial}{\partial n} \Delta^{2q-1-l} v(x) - \Delta^{2q-1-l} v(x) \frac{\partial}{\partial n} \Delta^l s(x) \right) d\sigma(x) = 0 \tag{4.6}$$

for every two functions $s, v \in H^{4q}(D \setminus \bigcup \Gamma_j)$ such that $\Delta^{2q} s = \Delta^{2q} u = 0, x \in D \setminus \bigcup \Gamma_j$. Recall that $\partial \widetilde{D}_j = \Gamma_j \cup \Gamma_{j-1}$. Here n and later n_x denote the inner unit normal vector to $\partial \widetilde{D}_j$ at the point x. Note that on $\partial \widetilde{D}_j$ we have $n = \nu_j$ for $x \in \Gamma_j$ and $n = -\nu_j$ for $x \in \Gamma_{j-1}$.

On the other hand, by the second Green formula, cf. Aronszajn–Creese–Lipkin [3, p.10, formula (2.11)] we have for $s \in PH_{2q}(\widetilde{D}_j)$:

$$\Omega_n \cdot \sum_{l=0}^{2q-1} \int_{\partial \widetilde{D}_j} \left(\Delta^l s(x) \frac{\partial}{\partial n_x} R_{l+1}(x-y) - R_{l+1}(x-y) \frac{\partial}{\partial n_x} \Delta^l s(x) \right) d\sigma(x)$$

$$
= \begin{cases} s(y) & \text{if } y \in \widetilde{D}_j, \\ 0 & \text{if } y \notin \overline{\widetilde{D}_j} \ . \end{cases} \tag{4.7}
$$

for every function $s \in H^{4q}(D \setminus \bigcup \Gamma_j)$, and every $j = 1, \ldots, N$.

Let us sum up (4.6) and (4.7). We obtain the following identity:

$$
\Omega_n \sum_{l=0}^{2q-1} \int_{\partial \widetilde{D}_j} \left(\Delta^l s(x) \frac{\partial}{\partial n} \Delta^{2q-1-l} G(x,y) - \Delta^{2q-1-l} G(x,y) \frac{\partial}{\partial n} \Delta^l s(x) \right) d\sigma(x)
$$

$$
= \begin{cases} s(y) & \text{if } y \in \widetilde{D}_j, \\ 0 & \text{if } y \notin \overline{\widetilde{D}_j}. \end{cases} \tag{4.8}
$$

Finally, let us sum up over $j = 1, \ldots, N$, recalling that $\partial \widetilde{D}_j \cap \partial \widetilde{D}_{j+1} = \Gamma_j$. Due to the fact that the unit normal n_x has opposite directions on Γ_j when taken from $\widetilde{D}_j$ and from $\widetilde{D}_{j+1}$, and also due to the properties $(3.1_b), (3.1_c), (3.1_d), (3.1_e)$, we obtain a simplified identity:

$$
s(y) = \sum_{j=1}^{N} \int_{\Gamma_j} s(x) \{ \partial \Delta^{2q-1} / \partial \nu_x G(x,y) \}_j d\sigma_j(x),
$$

for every $y \in D \setminus \bigcup \Gamma_j$. Q.E.D.

Let us note that the polyspline Green function enjoys all properties typical for the usual Green function for elliptic boundary value problems, e.g. it is also symmetric in x and y. The technics of proving such properties is essentially the same as the classical. In particular, we need the following Proposition for proving smoothness properties in the variable y:

PROPOSITION 4.4. The polyspline Green function satisfies the following inequality:

$$
\mid \frac{\partial \Delta^{2q-1} G(x,y)}{\partial \nu_x} \mid \leq M \{ \mid x - y \mid^{-n+1} + M_0 \}
$$

with M and M_0 positive constants, $x \in T = \bigcup \Gamma_j, y \in \overline{D}$.

The **proof** of this inequality does not differ essentially from that of Krasovskii's inequality for the usual Green function and its derivatives, see Krasovskii [12], Aronszajn–Creese–Lipkin [3, p. 187].

PROPOSITION 4.5. Let the linear functional J be of the type (2.5). Assume that for $r_k < n$ the manifolds $S_k \bigcap \Gamma_j = \emptyset, k = 1, \ldots, K, \ j = 1, \ldots, N$. Then for every $s \in PS_{2q}(D)$ we have that

$$J(s) = \sum_{j=1}^{N} \int_{\Gamma_j} s(x)\varphi_j(x)d\sigma_j(x), \tag{4.9}$$

where $\varphi_j \in C(\Gamma_j)$.

Proof : 1. By Lemma 4.3 we have

$$s(y) = \sum_{j=1}^{N} \int_{\Gamma_j} s(x)\{\frac{\partial\Delta^{2q-1}G(x,y)}{\partial\nu_x}\}_j d\sigma_j(x).$$

2. First consider the case of S_k with $r_k =$ n. Then by Definition 2.2 of *simple Peano class* $S_k =$ D. By Proposition 4.4 it follows that the integral

$$I_{k,j}(x) := \int_D \frac{\partial\Delta^{2q-1}G_j(x,y)}{\partial\nu_x} m_k(y)dy$$

satisfies the sufficient condition for *uniform convergence*, see [Sobolev, pp. 126–129]. Hence, $I_{k,j}(x)$ is a continuous function for $x \in T$, see the same reference, Lemma 1.

3. On a manifold S_k with $r_k < n$ the integral

$$I_{k,j}(x) := \int_{S_k} \frac{\partial\Delta^{2q-1}G_j(x,y)}{/\partial\nu_x} m_k(y)d\sigma_j(x)$$

is uniformly convergent due to Proposition 4.4, and the assumption that $S_k \bigcap T = \emptyset$. Hence, as in 2.) we have that $I_{k,j}(x)$ is continuous.

4. Finally we sum up in $k = 1, \ldots, K$, and obtain that the functions

$$\varphi_j(x) = J_y(\{\frac{\partial\Delta^{2q-1}G(x,y)}{\partial\nu_x}\}_j) \in C(\Gamma_j) \qquad j = 1, \ldots, N.$$

This proves also the convergence of the corresponding integrals in (4.9). Q.E.D.

Proof of Theorem 4.1. Since by definition $S_f(x) = f(x)$ for $x \in \bigcup \Gamma_j$, the proof follows immediately by Proposition 4.5. Q.E.D.

5. Multivariate Optimal Recovery of Linear Functionals.

In order to pass to the solution of problem (2.8), we will need also the following technical Lemma:

LEMMA 5.1. For $p = 2q$ the single layer potential

$$F(x) = \sum_{j=1}^{N} \int_{\Gamma_j} R_p(x-y)\varphi_j(y)d\sigma_j(y), \tag{5.1}$$

defined on the set $T = \bigcup \Gamma_j$, with densities $\varphi_j \in L_2(\Gamma_j)$, satisfies conditions (3.1_a), (3.1_d), (3.1_e) in the definition of polyspline. Also for every integer $p \geq 1$ we have that $F \in H^p_{loc}(D \setminus T) \bigcap H^{p-1}(D \setminus T)$.

The proof of the Lemma does not seem to be readily available in the literature. For that reason we will provide a detailed proof in the Appendix.

In [7] we proved the following theorem concerning the Peano kernel of a linear functional vanishing on polyharmonic functions:

THEOREM 5.2. Let L be a linear functional of the Peano type (2.5), such that $L(g) = 0$ for every $g \in PH_p(D)$. Then for all $f \in H^{2p}(D)$ we have the representation

$$L(f) = \int P(x) \Delta^p f(x) dx,$$

where $P(x) := -\Omega_n K(x) := -\Omega_d L_y(R_p(x-y))$. Also, $\Delta^j K \in L_2(D),\ j = 0, \ldots, p-1$. Here the notation L_y means that the functional L is applied with respect to the y–variable.

REMARK 5.3. In fact in [7] we stated Theorem 5.2. only for functions $f \in C^{2p}$ on a neighbourhood of D. Due to the *density theorem for Sobolev spaces* we may extend the representation to $f \in H^{2p}(D)$.

Now we may prove the main result of the paper:

THEOREM 5.4. Let J be an arbitrary linear functional of Peano type (2.5), where the compact manifolds $S_k \bigcap \Gamma_j = \emptyset, k = 1, \ldots, K,$ and $j = 1, \ldots, N$. Let us define the linear functional $\overline{L}$ on $H^{4q}(D)$ by putting $\overline{L}(f) = J(S_f)$. Consider the class $\mathcal{L}$ of all linear functionals using only the information $T(f) = (\pi_1(f), \ldots, \pi_N(f))$, (where $\pi_j(f)$ is the trace of $f \in H^{4q}(D)$ on the manifold Γ_j). We assume that they are given by (2.4), i.e.

$$L(f) = \sum_{j=1}^{N} \int_{\Gamma_j} \pi_j(f)(x) \lambda_j(x) d\sigma_j(x),$$

where the densities $\lambda_j \in L_2(\Gamma_j)$.

Then $\overline{L} \in \mathcal{L}$, and $\overline{L}$ solves the problem of optimal recovery (2.8) in the class $BH^{4q}(D)$, i.e.

$$\| J - \overline{L} \|_{BH^{4q}(D)} = \inf_{L} (\| J - L \|_{BH^{4q}(D)}),$$

where $L \in \mathcal{L}$.

Proof : We will follow the idea of the proof in the univariate case provided in Bojanov–Hakopian–Sahakian [4, Theorem 5.10]:

1.) Let us first remark that if $\| J - L \|_{BH^{4q}(D)} < \infty$ for some $L \in \mathcal{L}$ then $L(f) - J(f) = 0$ for every $f \in PH_q(D) \cap H^{4q}(D)$. Indeed, if $L(f_1) - J(f_1) = \epsilon$

for some $f_1 \in PH_q(D) \cap H^{4q}(D)$, then $cf_1 \in BH^{4q}(D)$ for every c and consequently,

$$\| J - L \|_{BH^{4q}(D)} = \infty,$$

which is a contradiction. Clearly, it makes sense to consider only L satisfying

$$\| J - L \|_{BH^{4q}(D)} < \infty.$$

2.) The functional $\overline{L}$ satisfies the property $\overline{L} = J$ on $PH_q(D)$ as well. Indeed, by Theorem 4.1 the functional $\overline{L}(f) = J(S_f)$ is of a *simple Peano type* (2.5). Since for $\phi \in PS_{2q}(D)$ we have $S_\phi = \phi$, it follows that $\overline{L}(\phi) = J(\phi)$. But $PH_q(D) \bigcap H^{4q}(D) \subset PS_{2q}(D)$, hence $\overline{L}(\phi) = J(\phi)$ for $\phi \in PH_q(D) \bigcap H^{4q}(D)$.

3.) Since the boundary ∂D is C^∞ we may apply the *density theorem* to obtain that $H^{4q}(D)$ is dense in $H^{2q}(D)$. Hence $PH_q(D) \cap H^{4q}(D)$ is dense in $PH_q(D) \cap H^{2q}(D) = PH_q(D)$. By Proposition 2.3 the functional J is continuous on $H^{2q}(D)$. Evidently, the functionals $\overline{L}, L$ are continuous on $H^{2q}(D)$ as well. This implies that $\overline{L}(f) = L(f) = J(f)$ for every $f \in PH_q(D)$. Let us remark that $\overline{L}(\phi) = J(\phi)$ also for every $f \in PS_{2q}'$ due to the density Theorem 3.4, and the continuity of the functionals J and $\overline{L}$ in $H^{4q-1}(D \setminus T)$.

4.) From above we see that the functionals $R = J - \overline{L}, R_1 = J - L, R_2 = \overline{L} - L$ are of Peano type and vanish on $PH_q(D)$. We may apply Theorem 5.2 to obtain the following representations:

$$R(f) = \int_D K(x) \Delta^q f(x) dx$$

and

$$R_i(f) = \int_D K_i(x) \Delta^q f(x) dx, \qquad i = 1, 2,$$

for every $f \in H^{4q}(D)$.

By the Cauchy-Kovalevski's inequality we have

$$R(f) \leq \| K \|_{L_2(D)} . \| \Delta^q f \|_{L_2(D)} .$$

Consequently,

$$\| R \| := \sup\{R(f) : f \in BH^{4q}(D)\} =$$

$$\sup\{R(f) : f \in BH^{2q}(D)\} = \| K \|_{L_2(D)} .$$

The equality of the two *suprema* follows from the density of $H^{4q}(D)$ in $H^{2q}(D)$ stated in point 3.). We have $\| R \| = R(f)$ only for functions f such that $\Delta^q f = K(x) / \| K \|_{L_2(D)}$. Such function f is obtained by putting,

$$f(x) = \frac{1}{\| K \|_{L_2(D)}} \int_D K(y) R_q(x - y) dy.$$

This is a classical fact in potential theory and follows e.g. by the second Green formula, cf. Aronszajn–Creese–Lipkin [3, p.10] and Lions–Magenes [14, Chapter II, Remark 2.2, p.120].

In particular, we get that $\| J - \overline{L} \|_{BH^{4q}(D)} < \infty$.

Similarly, we obtain

$$\| R_i \| = \| K_i \|_{L_2(D)}, \quad i = 1, 2.$$

5.) Let us see that $\| K \|_{L_2(D)} \leq \| K_1 \|_{L_2(D)}$ and equality is attained only for $K = K_1$, i.e. for $L = \bar{\mathrm{L}}$.

For that purpose, consider the Peano kernel

$$K_2 = -\Omega_n(\overline{L}_y - L_y)(R_q(x-y)).$$

Let us put

$$u(x) = -\Omega_n(\overline{L}_y - L_y)(R_{2q}(x-y)).$$

Due to the dominated convergence theorem, we may differentiate under the sign of the integrals and we obtain $\Delta^q u = K_2$.

We will prove that $u \in PS'_{2q}(D)$.

Since $\overline{L}, L \in \mathcal{L}$, we have the representation

$$u(x) = -\Omega_n(\sum_{j=1}^{N} \int_{\Gamma_j} (\overline{\lambda}_j(y) - \lambda_j(y)) R_{2q}(x-y) d\sigma_j(y),$$

where the densities $\overline{\lambda}_j(y), \lambda_j \in L_2(\Gamma_j)$. By Lemma 5.1 $u(x)$ satisfies conditions $(3.1_a), (3.1_d), (3.1_e)$, and

$$u \in H^{4q-1}(D \setminus T).$$

For proving (3.1_b) and (3.1_c) let us remark that

$$\Delta^{q+k} u(x) = \Delta^k K_2(x) =$$
$$\Delta^k(\overline{L}_y - L_y)(R_q(x-y)) = (\overline{L}_y - L_y)(R_{q-k}(x-y)),$$

for $k = 0, \ldots, q-2$, and

$$\partial/\partial\nu_x \Delta^{q+k} u(x) = \partial/\partial\nu_x \Delta^k K_2(x) =$$
$$\partial/\partial\nu_x \Delta^k(\overline{L}_y - L_y)(R_q(x-y)) = (\overline{L}_y - L_y)(\partial/\partial\nu_x R_{q-k}(x-y)),$$

for $k = 0, \ldots, q-1$. On the other hand we have the inclusion

$$R_{q-k}(x - .), \partial/\partial\nu_x R_{q-k}(x - .) \in PH_q(D),$$

for every $x \notin$ D.

Since we assume that $\parallel L - J \parallel_{BH^{4q}(D)} < \infty$, by points 2.) and 3.) we have that $\overline{L} = L$ on $PH_q(D)$. Hence,

$$\overline{L_y}(R_{q-k}(x-y)) = L_y(R_{q-k}(x-y)) \qquad \text{for every } x \notin D,$$

and

$$\overline{L}(\partial/\partial\nu_x R_{q-k}(x-.)) = L(\partial/\partial\nu_x R_{q-k}(x-.)) \qquad \text{for every } x \notin \text{ D.}$$

Passing to the traces on ∂D, it follows by the definition of u that

$$\Delta^{q+k}u(x) = 0, \qquad x \in \partial D, \quad k = 0, \dots, q-2,$$

and

$$\partial/\partial\nu_x \Delta^{q+k}u(x) = 0, \qquad x \in \partial D, \quad k = 0, \dots, q-1.$$

The above shows that $u \in PS'_{2q}(D)$.

6.) Since by 3.) $u \in PS'_{2q}(D)$, we obtain that $R(u) = J(u) - \overline{L}(u) = 0$ which implies $J(u) - \overline{L}(u) = \int_D K(x)\Delta^q u(x)dx = \int_D K(x)K_2(x)dx = 0$. Then we have evidently the following inequality:

$$\int_D K(x)^2 dx = \int_D [(K(x) + K_2(x))^2 - K_2^2]dx \leq$$

$$\int_D (K(x) + K_2(x))^2 dx,$$

and equality is attained only for $K_2 \equiv 0$.

On the other hand, we have $R_2 = R_1 - R$. Consequently, $K_2 = K_1 - K$. It follows that

$$\int_D K(x)^2 dx \leq \int_D (K(x) + K_2(x))^2 dx = \int_D K_1(x)^2 dx,$$

and equality is attained only for $K_2 \equiv 0$, i.e. $K \equiv K_1$. Q.E.D.

REMARKS 5.5. 1. The above Theorem has to be compared with the univariate Theorem 1.1. The major difference is in the class of functionals J.

2. Let us remark that we are working in the class $B_{\Delta^q}H^{4q}(D)$ instead of the more natural one $B_{\Delta^q}H^{2q}(D)$ which would be the direct analog to the univariate case, since we have to use the existence result Theorem 3.3. which holds in the space H^{4q}. This creates technical problems typical only for the multivariate case. For example, we have to work with smoother solutions in $PH_q \bigcap H^{4q}$, instead of simply PH_q.

3. The strong restriction that $S_k \bigcap \bigcup \Gamma_j = \emptyset$ is difficult to relax without losing the relative simplicity of the proofs. It requires working in Sobolev spaces of negative order. That is also the major difference with the univariate case, it reflects the specifics of the multivariate Sobolev spaces.

6. Conclusion.

Following rather natural intuitive reasoning in the Euclidean space $\mathbb{R}^n$ we have taken the data set T(f) supported on a union of $(n-1)$–dimensional manifolds, and thus T(f) being functions on these manifolds. The linear recovering functionals L are also taken through very natural considerations. The class of the linear functionals J to be recovered contains a large class of functionals of interest including point evaluations as well as integral operators.

It is thus remarkable that in order to solve the very simply formulated problem of optimal recovery (2.8) one needs the concept of polyspline.

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Appendix
On Polyharmonic Single Layer Potentials

We refer to Kellogg [8], Agmon–Douglis-Nirenberg [2], Michlin–Prössdorf [16, Chapter XI, Section 5, Chapter XIII], Sobolev [19], where fundamental properties of integrals of the type (5.1) are considered.

Proof of Lemma 5.1:

1. We have to check that $\Delta^k F(x), k = 0, \ldots, 2q-1$, and $\partial \Delta^k / \partial \nu_x F(x), k = 0, \ldots, 2q-2$, have the same values on Γ_j no matter from D_j or D_{j+1} we approach it.

Clearly, it is enough to check only the "worst case" with highest derivatives, namely $\Delta^{2q-1} F(x)$. We have

$$\Delta^{2q-1} F(x) = \sum_{j=1}^{N} \int_{\Gamma_j} \Delta^{2q-1} R_{2q}(x-y) \varphi_j(y) d\sigma_j(y) = \tag{5.2}$$

$$\sum_{j=1}^{N} \int_{\Gamma_j} R_1(x-y) \varphi_j(y) d\sigma_j(y).$$

The last integral is in fact a simple layer for the Newtonian potential. It is a classical fact that it is a continuous function of x, cf. Sobolev [19, p. 210, Lecture 15, Section 3].

2. It is a simple fact from the classical potential theory that for $x \in D \setminus T$ we have $\Delta^{2q} F(x) = 0$, cf. [2], Sobolev [19, Lecture 7, Section 2]. Hence, by the *interior regularity theorem* we obtain immediately that $F \in H^{4q}_{loc}(D \setminus T)$.

3. More subtle is to prove that $F \in H^{4q-1}(D \setminus \bigcup \Gamma_j)$. It follows by the properties of a simple layer potential. Indeed, assume that the point $0 \in \Gamma_j$. We may change the variables and reduce the above problem locally to $\Gamma_j = \mathbb{R}^{n-1}$. We obtain a function (cf. Michlin–Prössdorf [16, Chapter XIII]):

$$\Phi(x) = \int\limits_{\mathbb{R}^{n-1}} R_{2q}(x-y)\varphi(y)dy,$$

where the function φ has a compact support.

Let us remark that we need the regularity of the function Φ near the surface $\mathbb{R}^{n-1} = \{t = 0\}$. For that reason we will consider for every $\epsilon > 0$ the bounded region given by:

$$\mathbf{R}^n_\epsilon = \{(z,t) : z \in \mathbb{R}^{n-1}, \mid z \mid < \epsilon, 0 < t < \epsilon\},$$

and also we put

$$B^{n-1}_\epsilon = \{z \in \mathbb{R}^{n-1}, \mid z \mid < \epsilon\}.$$

We will consider a more general problem than needed. We put

$$\Phi_p(z,t) = \int\limits_{\mathbb{R}^{n-1}} R_p(z-y,t)\varphi(y)dy.$$

Let us remark that for every $z \in \mathbb{R}^{n-1}$, $t > 0$ and $p \geq 1$ the above integral has a weak singularity, and exists for functions $\varphi \in L_2(\mathbb{R}^{n-1})$, cf. Michlin–Prössdorf [16, Chapter VIII, p. 209] or Sobolev [19, loc.cit.]. This fact will be used several times below without mentioning.

Proceeding by induction we will prove that $\Phi_p \in H^{2p-1}(\mathbb{R}^n_\epsilon)$ for every $\epsilon > 0$.

4.) We will first prove that $\Phi_1 \in H^1(\mathbb{R}^n_\epsilon)$.

We have to see that $\Phi_1, \partial\Phi_1/\partial t \in L_2(\mathbb{R}^n_\epsilon)$, and $\partial\Phi_1/\partial z_j \in L_2(\mathbb{R}^n_\epsilon)$, for every $j = 1, \ldots, n-1$.

For $n \geq 3$, *a* simple differentiation gives for every $t > 0$:

$$\frac{\partial \Phi_1(z,t)}{\partial t} = (2-n) \int\limits_{\mathbb{R}^{n-1}} t((z-y)^2 + t^2)^{n/2}\varphi(y)dy =$$

$$\frac{(2-n)}{c_{n-1}} \int\limits_{\mathbb{R}^{n-1}} P(z-y,t)\varphi(y)dy,$$

where $P(z,t)$ is the so-called Poisson kernel, and the constant $c_{n-1} = \Gamma(\frac{n}{2})/\pi^{\frac{n}{2}}$, cf. Stein–Weiss [21, p. 6].

According to Stein–Weiss [21, Chapter II, Theorem 2.1], we have the following inequality:

$$\int\limits_{\mathbb{R}^{n-1}} \mid \frac{\partial \Phi_1(z,t)}{\partial t} \mid^2 dz \leq \parallel \varphi(y) \parallel^2_{L_2(\mathbb{R}^{n-1})}$$

for all $t > 0$. Hence, we obtain

$$\int_0^\epsilon \int\limits_{\mathbb{R}^{n-1}} \mid \frac{\partial \Phi_1(z,t)}{\partial t} \mid^2 dzdt \leq \epsilon \parallel \varphi(y) \parallel^2_{L_2(\mathbb{R}^{n-1})},$$

which proves that $\partial\Phi_1/\partial t \in L_2(\mathbb{R}^n_\epsilon)$.

According to the same theorem in Stein–Weiss [21] cited above, the function $\partial\Phi_1/\partial t$ is harmonic in

$$\mathbb{R}^n_+ = \{(z,t) : z \in \mathbb{R}^{n-1}, t > 0\},$$

and satisfies for almost every $z \in \mathbb{R}^{n-1}$ the following relation:

$$\lim_{t\to 0} \frac{\partial \Phi_1(z,t)}{\partial t} = \varphi(z).$$

Hence, for almost every $z \in \mathbb{R}^{n-1}$ we have the representation

$$\Phi_1(z,t) = \int_0^t \frac{\partial \Phi_1(z,\tau)}{\partial t} d\tau + \varphi(z).$$

By the Cauchy-Kovalevski's inequality, it follows that

$$\int_{B^{n-1}_\epsilon} \mid \Phi_1(z,t) \mid^2 dz \leq \int_0^t \int_{B^{n-1}_\epsilon} \mid \frac{\partial \Phi_1(z,\tau)}{\partial t} \mid^2 dzd\tau$$

which immediately gives

$$\int_0^\epsilon \int_{B^{n-1}_\epsilon} \mid \Phi_1(z,t) \mid^2 dzdt \leq C \parallel \varphi \parallel^2_{L_2(\mathbb{R}^{n-1})} \quad ,$$

for some constant $C > 0$. This proves that $\Phi_1 \in L_2(\mathbb{R}^n_\epsilon)$.

Another differentiation in z_j immediately gives that

$$\frac{\partial \Phi_1(z,t)}{\partial z_j} = (2-n) \int\limits_{\mathbb{R}^{n-1}} (z_j - y_j)((z-y)^2 + t^2)^{n/2} \varphi(y)dy =$$

$$\frac{(2-n)}{c_{n-1}} \int\limits_{\mathbb{R}^{n-1}} K_j(z-y,t)\varphi(y)dy,$$

where the functions $K_j(z,t), j = 1, \ldots, n-1$, are the so-called conjugate Poisson kernels, cf. Stein–Weiss [21, p.224, Chapter VI, Theorem 2.6] or Stein [20, p.126, formula (20)].

According to Stein–Weiss [21, Chapter VI, Theorem 4.17(ii)], we have that

$$\frac{\partial \Phi_1(z,t)}{\partial z_j} = \int\limits_{\mathbb{R}^{n-1}} P(z-y,t)\mathcal{R}_j\varphi(y)dy,$$

where $\mathcal{R}_j\varphi, j = 1, \ldots, n-1$ denote the Riesz transforms in $\mathbb{R}^{n-1}$.

Now by [SW, Chapter 6, Theorem 2.6] it follows that

$$\| \mathcal{R}_j\varphi \|^2_{L_2(\mathbb{R}^{n-1})} \leq \| \varphi \|^2_{L_2(\mathbb{R}^{n-1})} .$$

By the already cited [SW, Chapter 2, Theorem 2.1] it follows that

$$\int\limits_{\mathbb{R}^{n-1}} | \frac{\partial \Phi_1(z,t)}{\partial z_j} |^2 \, dz \ \leq \| \varphi(y) \|^2_{L_2(\mathbb{R}^{n-1})} .$$

Hence

$$\int_0^\epsilon \int_{B_\epsilon^{n-1}} | \frac{\partial \Phi_1(z,t)}{\partial z_j} |^2 \, dzdt \leq C_\epsilon \| \varphi(y) \|^2_{L_2(\mathbb{R}^{n-1})} .$$

The last proves that

$$\frac{\partial \Phi_1(z,t)}{\partial z_j} \in L_2(\mathbb{R}^n_\epsilon).$$

This finishes the proof of $\Phi_1 \in H^1(\mathbb{R}^n_\epsilon)$ for $n \geq 3$. For $n = 2$ the proof is essentially the same.

5.) To proceed by induction, we assume that $\Phi_{p-1} \in H^{2p-3}(\mathbb{R}^n_\epsilon)$. It is clear that we have $\Delta\Phi_p(z,t) = \Phi_{p-1}(z,t)$ in $\mathbb{R}^n_+$ due to property (4.2) of the fundamental solutions R_p. In order to apply the *global regularity theorem* to the Dirichlet problem for the Laplace operator, we have first to prove that

$$\Phi_{p|t=0} \in H^{2p-\frac{3}{2}}_{loc}(\mathbb{R}^{n-1}).$$

Here and further the subscript $_{|t=0}$ will denote the restriction from $\mathbb{R}^n_+$ to $\mathbb{R}^{n-1} = \{t = 0\}$.

6.) Let us first consider the case of odd dimension n. Then the fundamental solution has the simpler form

$$R_p(z,t) = A_{p,n}(z^2+t^2)^{\frac{2p-n}{2}}.$$

By the property of the fundamental solutions R_p, we have

$$R_{p-1}(z,t) = \Delta R_p(z,t) = \Delta_y R_p(z,t) + \frac{\partial^2}{\partial t^2} R_p(z,t).$$

Hence,

$$R_{p-1}(z,t)_{|t=0} = \Delta R_p(z,t)_{|t=0} = \Delta_z R_p(z,t)_{|t=0} + \frac{\partial^2}{\partial t^2} R_p(z,t)_{|t=0}.$$

A direct computation shows that

$$\frac{\partial^2}{\partial t^2} R_p(z,t)_{|t=0} = A_{p,n} \frac{\partial^2}{\partial t^2} \left((z^2+t^2)^{\frac{2p-n}{2}} \right)_{|t=0} =$$

$$A_{p,n} \frac{\partial}{\partial t} \left((2p-n) t (z^2+t^2)^{\frac{2p-n-2}{2}} \right)_{|t=0} = A_{p,n}(2p-n) \left((z^2+t^2)^{\frac{2p-n-2}{2}} \right)_{|t=0} =$$

$$\frac{A_{p,n}}{A_{p-1,n}} (2p-n) R_{p-1}(z,0).$$

The last implies

$$\Delta R_p(z,t)_{|t=0} = \left(1 - \frac{A_{p,n}}{A_{p-1,n}} (2p-n) \right) R_{p-1}(z,0).$$

Let us put $\psi_p(z) = \Phi_p(z,t)_{|t=0}$ for $p \geq 1$.

By the above we see that

$$\Delta_z \psi_p(z) = \int\limits_{\mathbb{R}^{n-1}} \Delta_y R_p(z-y,0) \varphi(y) dy =$$

$$\left(1 - \frac{A_{p,n}}{A_{p-1,n}} (2p-n) \right) \int\limits_{\mathbb{R}^{n-1}} R_{p-1}(z-y,0) \varphi(y) dy =$$

$$\left(1 - \frac{A_{p,n}}{A_{p-1,n}} (2p-n) \right) \psi_{p-1}(z).$$

On the other hand according to our inductive hypothesis, for every $\epsilon > 0$, $\Phi_{p-1} \in H^{2p-3}(\mathbb{R}^n_\epsilon)$. By the trace theorem it follows that

$$\psi_{p-1}(z) = \Phi_{p-1|t=0} \in H_{loc}^{2p-3\frac{1}{2}}(\mathbb{R}^n_\epsilon).$$

Now only the *interior regularity theorem* is enough to obtain

$$\psi_p(z) \in H_{loc}^{2p-1\frac{1}{2}}(\mathbb{R}^{n-1}).$$

By the *global regularity theorem* we obtain that

$$\Phi_p \in H^{2p-1}(\mathbb{R}^n_\epsilon).$$

7.) In the case of even dimension $n = 2m, m \geq 1$, we will consider another inductive hypothesis.

We consider all integrals of the type

$$\Phi_p(z,t) = \int\limits_{\mathbb{R}^{n-1}} \left(R_p(z-y,t) + \alpha((z-y)^2 + t^2)^{(2p-n)/2} + \beta \right) \varphi(y)dy,$$

where α and β are arbitrary constants.

For $p = 1$ the statement $\Phi_1 \in H^1(\mathbb{R}^n_\epsilon)$ follows immediately by points 3.) and 4.) since then we have

$$\alpha((z-y)^2 + t^2)^{(2p-n)/2} = \begin{cases} \alpha & \text{for } n = 2, \\ \\ \alpha R_1(z-y,t) & \text{for } n > 2, \end{cases}$$

The inductive hypothesis now states that $\Phi_{p-1} \in H^{2p-3}(\mathbb{R}^n_\epsilon)$ for arbitrary α and β.

8.) By a direct computation as in 6.) we see that

$$I = \frac{\partial^2}{\partial t^2} R_p(z,t)_{|t=0} =$$

$$\frac{\partial^2}{\partial t^2} \left((z^2 + t^2)^{\frac{2p-n}{2}} (A_{p,n} \log(z^2 + t^2)^{\frac{1}{2}} + B_{p,n}) \right)_{|t=0} =$$

$$(2p-n) \mid z \mid^{2p-n-2} (A_{p,n} \log \mid z \mid + B_{p,n}) + A_{p,n} \mid z \mid^{2p-n-2} .$$

Since $\Delta R_{p|t=0} = R_{p-1|t=0}$, we obtain from above that

$$\Delta R_p(z,t)_{|t=0} = R_{p-1}(z,0) - I = C_1 R_{p-1}(z,0) + C_2 \mid z \mid^{2p-n-2} .$$

9.) According to our inductive hypothesis the function

$$\Phi_{p-1}(z,t) = \int\limits_{\mathbb{R}^{n-1}} \left(C_1 R_{p-1}(z-y,t) + C_2 \mid (z-y)^2 + t^2 \mid^{\frac{2p-n-2}{2}} \right) \varphi(y)dy$$

belongs to $H^{2p-3}(\mathbb{R}^n_\epsilon)$. Hence by the trace theorem the function

$$\psi_{p-1}(z) = \Phi_{p-1}(z,t)_{|t=0} =$$

$$\int_{\mathbb{R}^{n-1}} \left(C_1 R_{p-1}(z-y,0) + C_2 \mid z-y \mid^{\frac{2p-n-2}{2}} \right) \varphi(y) dy$$

is in $H^{2p-3\frac{1}{2}}_{loc}(\mathbb{R}^n_\epsilon)$. Thus we see that if we put $\psi_p(z) = \Phi_p(z,t)_{|t=0}$, by 8.) we have

$$\Delta_z \psi_p(z) = \Delta_z \Phi_p(z,t)_{|t=0} = \psi_{p-1}.$$

We proceed further like in point 7.).

The proof of the Lemma is finished. Q.E.D.

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