Symmetry properties of cardinal interpolation L-splines and polysplines

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February 23, 2004

Abstract

Recently the authors have proved existence and uniqueness of cardinal polysplines of order p. In this paper it is shown that a cardinal fundamental polyspline $L_f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ of order p possess the following symmetry property:

$$L_f \left(r^{-1}\theta\right) = r^{n-2p} L_f \left(r\theta\right) \qquad \text{for all } r > 0, \theta \in \mathbb{S}^{n-1}.$$

The proof is based on symmetry properties of univariate cardinal interpolation $L-$splines for a certain class of linear differential operators L.

1 Introduction

A main result in the area of cardinal polynomial splines is the existence of cardinal interpolation splines for certain classes of data, proved by *Schoenberg* [9]: suppose that a bi–infinite sequence of real numbers $y_j, j \in \mathbb{Z}$ of *power growth* with exponent $\gamma \geq 0$ be given, i.e. that there exists $C > 0$ such that $|y_j| \leq C \cdot |j|^{\gamma}$ for all $j \in \mathbb{Z}$, and suppose further that p is an *odd integer.* Then there *exists* a *unique* cardinal polynomial spline S of polynomial degree p such that the following conditions hold:

$$\begin{aligned} S(j) &= y_j && \text{for all } j \in \mathbb{Z}, \quad \text{and} \\ |S(x)| &\leq O\left(|x|^{\gamma}\right) && \text{for all } x \in \mathbb{R}. \end{aligned} \tag{1}$$

Hence S is a cardinal spline of power growth on the whole real line interpolating the data $(y_j)_{j\in\mathbb{Z}}$. Furthermore, the interpolation spline has the following *symmetry property*: The spline S is *even* (which means that $S(-x) = S(x)$ for all $x \in \mathbb{R}$) provided $y_{-j} = y_j$ for all $j \in \mathbb{Z}$. In particular, the so–called fundamental interpolation spline (the one satisfying (1) with data $y_j = 0$ for $j \neq 0$ and $y_0 = 1$) is even.

The existence and uniqueness results of Schoenberg were generalized to cardinal $L-$splines by Ch. Micchelli in [5], [6], where L is a differential operator with constant real coefficients. In this paper we are interested in symmetry properties of the interpolation cardinal $L-$splines and the fundamental cardinal $L-$spline. Intuitively, it is clear that the interpolation $L-$splines will have symmetry properties only if the differential operator L has certain kind of symmetry properties which we formulate below.

Let us assume that the operator L is defined by the expression

$$L := \prod_{j=1}^{N+1} \left(\frac{d}{dx} - \lambda_j \right).$$

Hence L is characterized by the (unordered) vector $\Lambda := (\lambda_1, \lambda_2, ..., \lambda_{N+1})$ and *throughout the paper we will assume that all* λ_j *are real.* We call Λ to be *nearly symmetric with respect to* $c \in \mathbb{R}$ if there exists a permutation π of the set $\{1, ..., N+1\}$ such that $-\lambda_j = c + \lambda_{\pi(j)}$ for $j = 1, ..., N+1$, or, symbolically, $-\Lambda = c + \Lambda$. In the case $c = 0$ we call Λ *symmetric*; indeed then Λ is a symmetric set with respect to 0.

Let us denote by L_0 the fundamental $L-$spline[1], i.e. the interpolation $L-$spline for the data $y_0 = 1$ and $y_j = 0$ for all $j \in \mathbb{Z}$, $j \neq 0$. Then our main result states that if Λ is *nearly symmetric with respect to* c, then L_0 satisfies the following symmetry property:

$$L_0(-x) = e^{cx} L_0(x) \qquad \text{for all } x \in \mathbb{R}.$$

This leads to a symmetry property of the cardinal interpolation $L-$splines which is analogous to the polynomial case. Furthermore we prove that the *basis* $L-$spline (or the $TB-$spline) which we denote by Q_Λ (recall that it has support equal to the interval $[0, N+1]$) satisfies the following symmetry property

$$Q_\Lambda(N+1-x) = e^{cx} Q_\Lambda(x) \qquad \text{for all } x \in \mathbb{R}.$$

These results are finally used to prove the symmetry properties of the fundamental polysplines.

2 Symmetry of the Euler–Frobenius polynomial

Let us recall briefly the definition of a cardinal $L-$spline: Let L be a linear differential operator L (of order $N+1$) with constant coefficients. The set of all homogeneous solutions is defined by

$$U_L := U_\Lambda := \{ f \in C^\infty(\mathbb{R}) : Lf(x) = 0 \text{ in } \mathbb{R} \}.$$

It is well known from Ordinary Differential Equations that U_Λ is the linear space generated by all functions of the type $x \mapsto e^{\lambda_j x} x^i$ where $j \in \{1, ..., N+1\}$ and the integer $i \geq 0$ is smaller than the multiplicity of λ_j. Then a function $u \colon \mathbb{R} \to \mathbb{R}$ is called a *cardinal* $L-$*spline* if

[1] We hope that the reader will not mix the operator L and the fundamental spline L_0; the notation for the last has been systematically used by Schoenberg, and the authors have decided to follow this tradition.

- for each $j \in \mathbb{Z}$ there exists $f_j \in U_L$ such that $u(t) = f_j(t)$ for all $t \in (j, j+1)$ and
- u is $N-1$ times continuously differentiable.

Define $e^{\Lambda} = \left\{ e^{\lambda_j} : j = 1, ..., N+1 \right\}$. In the theory of cardinal $L-$splines the function $A_{\Lambda} : \mathbb{R} \times \left(\mathbb{C} \setminus e^{\Lambda} \right) \to \mathbb{C}$ (cf. [6], p. 223) defined by

$$A_{\Lambda}(x, \lambda) = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\prod_{j=1}^{N+1} (z - \lambda_j)} \frac{e^{xz}}{e^z - \lambda} dz \tag{2}$$

is of fundamental importance. Here Γ is a closed simple curve in the complex plane surrounding all $\lambda_j, j = 1, ..., N+1$ whereby the zeros of the function $e^z - \lambda$ lie in the exterior of Γ. We will call $A_{\Lambda}(x, \lambda)$ the *Euler-Schoenberg function.* The *Euler-Frobenius function* is defined by

$$\Pi_{\Lambda}(x, \lambda) := A_{\Lambda}(x, \lambda) \cdot \prod_{j=1}^{N+1} \left(e^{\lambda_j} - \lambda \right). \tag{3}$$

For $x = 0$ it is a polynomial of degree at most N in the variable λ (Corollary 2.1 in [6]) and $\Pi_{\Lambda}(0, \lambda)$ is called the *Euler-Frobenius polynomial.*

Proposition 1 *Let Λ be nearly symmetric with respect to $c \in \mathbb{R}$. For all $\lambda \notin e^{\Lambda} \cup e^{-\Lambda} \cup \{0\}$ and all $x \in \mathbb{R}$ the following equality holds for*

$$A_{\Lambda}\left(1 - x, \frac{1}{\lambda}\right) = (-1)^{N+1} \lambda \, e^{(x-1)c} \, A_{\Lambda}\left(x, \lambda e^{-c}\right). \tag{4}$$

Proof. Let $\Lambda = (\lambda_1, ..., \lambda_{N+1})$ and define $-\Lambda = (-\lambda_1, ..., -\lambda_{N+1})$. Then for all $x \in \mathbb{R}$ and $\lambda \notin e^{\Lambda} \cup e^{-\Lambda} \cup \{0\}$ we have (see [6, p. 213])

$$A_{\Lambda}\left(1 - x, \frac{1}{\lambda}\right) = (-1)^{N+1} \lambda \cdot A_{-\Lambda}(x, \lambda). \tag{5}$$

Assume now that Λ is nearly symmetric with respect to $c \in \mathbb{R}$. A straightforward computation shows that for all $x \in \mathbb{R}$ and $\lambda \notin e^{\Lambda} \cup e^{-\Lambda} \cup \{0\}$

$$A_{-\Lambda}(x, \lambda) = e^{(x-1)c} A_{\Lambda}\left(x, \lambda e^{-c}\right). \tag{6}$$

Combining equation (5) and (6) the proof is accomplished. ■

By taking $x = 0$ in (4) using $A_{\Lambda}(1, \lambda) = \lambda A_{\Lambda}(0, \lambda)$ (see [6]) we obtain the equality

$$(-1)^{N+1} \lambda \, e^{-c} \, A_{\Lambda}\left(0, \lambda e^{-c}\right) = A_{\Lambda}\left(1, \frac{1}{\lambda}\right) = \frac{1}{\lambda} \cdot A_{\Lambda}\left(0, \frac{1}{\lambda}\right).$$

It follows that

$$(-1)^{N+1} \; e^{-c} \lambda^2 \, A_{\Lambda}\left(0, \lambda e^{-c}\right) = A_{\Lambda}\left(0, \frac{1}{\lambda}\right) \tag{7}$$

Suppose now that $\lambda = \mu_i$ is a zero of the function $\lambda \mapsto A_\Lambda(0,\lambda)$. Put $\rho_i = e^c \mu_i$. Then the identity (7) tells us that $v_i := 1/\rho_i$ is again a zero. It follows that $\mu_i \cdot v_i = e^{-c}$. Since all zeros are negative and simple it follows that for each μ_i there exists exactly one v_i which is reciprocal. Hence, we have proved the following important result:

Proposition 2 *Assume that Λ is nearly symmetric with respect to $c \in \mathbb{R}$ and let $\mu_{N-1} < \ldots < \mu_1 < 0$ be the zeros of $\lambda \mapsto A_\Lambda(0,\lambda)$ in $(-\infty, 0)$. Then $\mu_j \mu_{N-j} = e^{-c}$ for all $j = 1, \ldots., N-1$.*

Proposition 3 *Let Λ be nearly symmetric with respect to $c \in \mathbb{R}$. Then the Euler–Frobenius polynomial Π_Λ satisfies the equality*

$$\boxed{\Pi_\Lambda(x, \lambda e^{-c}) = \lambda^N e^{-(x-1)c} \cdot e^{-\frac{1}{2}(N+1)c} \Pi_\Lambda\left(1 - x, \tfrac{1}{\lambda}\right).} \tag{8}$$

Proof. It is easy to see that $\lambda_1 + \ldots + \lambda_{N+1} = -\frac{1}{2}(N+1)c$ since $\lambda_j + \lambda_{\pi(j)} = -c$ for $j = 1, \ldots N+1$. Define $r_\Lambda(\lambda) = \prod_{j=1}^{N+1}\left(e^{\lambda_j} - \lambda\right)$. Since $\lambda_j + \lambda_{\pi(j)} = -c$ it follows that $r_\Lambda(\lambda e^{-c})$ is equal to

$$(-\lambda)^{N+1} e^{(\lambda_1 + \ldots + \lambda_{N+1})} \prod_{j=1}^{N+1}\left(e^{\lambda_{\pi(j)}} - \frac{1}{\lambda}\right) = e^{-\frac{1}{2}(N+1)c}(-\lambda)^{N+1} r_\Lambda\left(\frac{1}{\lambda}\right).$$

The last equation with (4) shows that the Euler-Frobenius function $\Pi_\Lambda(x, \lambda e^{-c})$ (which was defined by $A_\Lambda(x, \lambda e^{-c}) \cdot r_\Lambda(\lambda e^{-c})$ in (3)) is equal to

$$A_\Lambda\left(1 - x, \frac{1}{\lambda}\right)(-1)^{N+1}\frac{1}{\lambda} e^{-(x-1)c} \cdot e^{-\frac{1}{2}(N+1)c}(-\lambda)^{N+1} r_\Lambda\left(\frac{1}{\lambda}\right).$$

This is just the statement. ■

3 Symmetry property of the basis $L-$spline Q_Λ

Define the function $s_\Lambda(\lambda) := \prod_{j=1}^{N+1}\left(e^{-\lambda_j} - \lambda\right)$ and let s_j, $j = 0, \ldots, N+1$ be the coefficients of $s_\Lambda(\lambda)$, i.e. $s_\Lambda(\lambda) = \sum_{j=0}^{N+1} s_j \lambda^j$. Due to the choice of the real numbers s_j it is straightforward to prove that the following cardinal $L-$spline has support in the interval $[0, N+1]$,

$$Q_\Lambda(x) := \sum_{j=0}^{N+1} s_j \cdot A_\Lambda(x + 1 - j, 0) \cdot 1_{[0,\infty)}(x). \tag{9}$$

We call Q_Λ the *basis-spline* or *B-spline* or *TB-spline*.[2] It has the property that $Q_\Lambda(x) > 0$ for all $x \in (0, N+1)$ and Q_Λ is unique up to a positive constant.

[2] It is the analog to the $B-$spline in the polynomial case, see e.g. [11].

Proposition 4 *Suppose that Λ is nearly symmetric with respect to the constant $c \in \mathbb{R}$. If $S(x)$ is a cardinal $L-$spline with respect to Λ then $e^{-cx}S(m-x)$ is a cardinal $L-$spline with respect to Λ for every integer m.*

Proof. Define $f(x) := e^{-cx}S(m-x)$ for all $x \in \mathbb{R}$. Clearly f is $(N-1)$-continuously differentiable. We know that S is on each interval $(j, j+1)$, $j \in \mathbb{Z}$, a solution of the given differential operator L, i.e., that the restriction S_j of S to $(j, j+1)$ gives a function in U_Λ. Hence S_j is a linear combination of exponentials $x^s e^{\lambda_j x}$ with $j \in \{1, ..., N+1\}$ and s is an integer smaller than the multiplicity of λ_j. Now consider $f_j(x) = e^{-cx}S_j(m-x)$. Since $-\lambda_j = c + \lambda_{\pi(j)}$ it follows that f_j is in U_Λ and the proof is complete. ■

Theorem 5 *Suppose that Λ is nearly symmetric with respect to the constant $c \in \mathbb{R}$. Then for all $x \in \mathbb{R}$ the following equality holds*

$$\boxed{Q_\Lambda(N+1-x) = e^{cx}e^{-\frac{1}{2}(N+1)c}Q_\Lambda(x).}$$

Proof. Define $f(x) := e^{-cx}Q_\Lambda(N+1-x) \geq 0$ for all $x \in \mathbb{R}$. Clearly f is $(N-1)$-continuously differentiable and it has support in $[0, N+1]$. The last Proposition shows that f is an $L-$spline with respect to Λ. Then the above-mentioned uniqueness property of the basis spline Q_Λ yields that there exists a constant $D > 0$ such that $f = DQ_\Lambda$. By putting $x = \frac{1}{2}(N+1)$ one obtains the constant D and the proof is accomplished. ■

The following fundamental formula relates the Euler-Frobenius function with the basis–spline Q_Λ (cf. [6, p. 221 and p. 222])

$$R_\Lambda^x(\lambda) := \sum_{j=0}^{N} \lambda^{N-j}Q_\Lambda(x+j) = \frac{(-1)^N}{e^{(\lambda_1+...+\lambda_{N+1})}} \cdot \Pi_\Lambda(x, \lambda). \tag{10}$$

The following result will be needed in the next section

Proposition 6 *Suppose that Λ is nearly symmetric with respect to the constant $c \in \mathbb{R}$. Then the polynomial $P_\Lambda^0(\lambda) = R_\Lambda^0\left(\frac{1}{\lambda}\right)\lambda^N$ is proportional to the polynomial $\Pi_\Lambda(0, \lambda e^{-c})$, or more precisely,*

$$P_\Lambda^0(\lambda) = (-1)^N \lambda e^{Nc} \cdot \Pi_\Lambda\left(0, \lambda e^{-c}\right).$$

Proof. Since $\lambda_1 + ... + \lambda_{N+1} = -\frac{1}{2}(N+1)c$ the equation (10) yields

$$P_\Lambda^\alpha(\lambda) := R_\Lambda^\alpha\left(\frac{1}{\lambda}\right)\lambda^N = (-1)^N e^{\frac{1}{2}(N+1)c}\lambda^N \Pi_\Lambda\left(\alpha, \frac{1}{\lambda}\right).$$

Now we use the symmetry property of the Euler-Frobenius function $\Pi_\Lambda(x, \lambda e^{-c})$ in (8) applied to $\alpha = 1-x$ and obtain that

$$P_\Lambda^\alpha(\lambda) = (-1)^N e^{-c\alpha}e^{(N+1)c}\,\Pi_\Lambda\left(1-\alpha, \lambda e^{-c}\right).$$

Now take $\alpha = 0$ and use the fact that $A_\Lambda(1, \mu) = \mu A_\Lambda(0, \mu)$ and the proof is accomplished. ■

4 Symmetry of the fundamental $L-$spline L_0

We come to the main point of our study. Let us now consider interpolation problems for cardinal $L-$splines. Let us fix a number $\alpha \in [0,1)$. A cardinal $L-$spline L_α is called a *fundamental $L-$spline* with respect to α if $L_\alpha(\alpha) = 1$ and $L_\alpha(\alpha + j) = 0$ for all $j \in \mathbb{Z}$, $j \neq 0$ and if it decays exponentially, i.e. if there exist two constants $A, B > 0$ such that

$$|L_\alpha(x)| \leq Ae^{-B|x|} \qquad \text{for all } x \in \mathbb{R}. \tag{11}$$

The existence and uniqueness of fundamental $L-$spline L_α with respect to α was proved by Micchelli in [6, p. 213] under the assumption $A_\Lambda(\alpha, -1) \neq 0$. Let us recall from [10, p. 271] the construction of the fundamental $L-$spline: Define as in the last section the function

$$P_\Lambda^\alpha(\lambda) := R_\Lambda^\alpha\left(\frac{1}{\lambda}\right)\lambda^N = \sum_{j=0}^{N} \lambda^j Q_\Lambda(\alpha + j).$$

Assume now that the function $\lambda \to 1/P_\Lambda^\alpha(\lambda)$ is holomorphic on the annulus $\{R_1 < |\lambda| < R_2\}$ in the complex plane such that $R_1 < 1 < R_2$. We consider the Laurent series

$$\frac{1}{P_\Lambda^\alpha(\lambda)} = \sum_{j=-\infty}^{\infty} \omega_j \lambda^j.$$

Then the fundamental $L-$spline can be defined by

$$L_\alpha(x) := \sum_{j=-\infty}^{\infty} \omega_j Q_\Lambda(x - j). \tag{12}$$

The following Proposition provides in fact an improvement of the constants A and B in the above inequality (11) but only on the half axis $x \geq 0$. It is so far essential for the proof of our main result

Proposition 7 *Let Λ be nearly symmetric with respect to $c > 0$ and suppose that $A_\Lambda(0, -1) \neq 0$. Then there exists $\varepsilon > 0$ and $A > 0$ such that*

$$|L_0(x)| \leq Ae^{-(c+\varepsilon)x} \qquad \text{for all } x \geq 0.$$

Proof. 1. For the proof we need at first a result about the zeros of $\lambda \mapsto A_\Lambda(0, \lambda)$. Let $\mu_{N-1} < \cdots < \mu_k < -1 < \mu_{k-1} < \cdots < \mu_1 < 0$ be the zeros of $\lambda \mapsto A_\Lambda(0, \lambda)$ in $(-\infty, 0)$. By Proposition 2 we know that $\mu_k \mu_{k-1} = e^{-c}$. Clearly $A_\Lambda(0, \mu) \neq 0$ for all μ with $\mu_k < \mu < \mu_{k-1}$. Since $|\mu_k| > 1$ it follows that $|\mu_{k-1}| = \frac{1}{|\mu_k|} e^{-c} < e^{-c}$. We conclude that there exists $\varepsilon > 0$ such that $|\mu_{k-1}| < e^{-c-\varepsilon}$. By taking $\varepsilon > 0$ somewhat smaller we can also assume that $|\mu_k| > e^{\varepsilon}$. Since $A_\Lambda(0, \lambda)$ has zeros only in $(-\infty, 0]$ we conclude that $A_\Lambda(0, \lambda) \neq 0$ for all $e^{-c-\varepsilon} \leq |\lambda| \leq e^{\varepsilon}$. This shows that

$$\Pi_\Lambda(0, \lambda) = r_\Lambda(\lambda) A_\Lambda(0, \lambda) \neq 0 \qquad \text{for all } e^{-c-\varepsilon} \leq |\lambda| \leq e^{\varepsilon}.$$

By Proposition 6 $P_\Lambda^0(\lambda) = (-1)^N \lambda e^{Nc} \cdot \Pi_\Lambda(0, \lambda e^{-c})$. We conclude that $P_\Lambda^0(\lambda) \neq 0$ for all λ satisfying $e^{-\varepsilon} \leq |\lambda| \leq e^{c+\varepsilon}$.

2. Put $M_\Lambda := \max_{y\in(0,N+1)} |Q_\Lambda(y)|$. We want to estimate $L_0(x)$ as defined in (12). Note that only $j \in \mathbb{Z}$ with $x - j \in (0, N+1)$ contributes non-trivially to the sum (12) so we consider only j satisfying $x - N - 1 < j < x$. Denote by $[x]$ the largest integer $k \in \mathbb{Z}$ with $k \leq x$. Note that j can be represented as $j = [x] - N - 1 + k$ with $k \in N_0$. Then the series $L_0(x)$ can be estimated by

$$|L_0(x)| \leq \sum_{j\in\mathbb{Z},[x]-N-1\leq j} |\omega_j| M_\Lambda = M_\Lambda \cdot \sum_{k=0}^{\infty} \left|\omega_{[x]-N-1+k}\right|.$$

Above we have shown that $P_\Lambda^0(\lambda) \neq 0$ is holomorphic on the annulus $A(R_1, R_2)$ where $R_1 = e^{-\varepsilon} < 1 < e^{c+\varepsilon} = R_2$. For any r with $R_1 < r < R_2$ we have the following well known Cauchy estimates (see [1, p. 107]) for every $j \in \mathbb{Z}$,

$$|\omega_j| \leq \max_{|\lambda|=r} \left| \frac{1}{P_\Lambda^\alpha(\lambda)} \right| \cdot r^{-j}. \tag{13}$$

Hence the following inequality holds where we take $r = e^{c+\varepsilon\frac{1}{2}} > 1$

$$\begin{aligned} |L_0(x)| \leq\ & M_\Lambda \max_{|\lambda|=r} \frac{1}{|P_\Lambda^\alpha(\lambda)|} \sum_{k=0}^{\infty} r^{-[x]+N+1-k} \\ &= M_\Lambda \max_{|\lambda|=r} \frac{1}{|P_\Lambda^\alpha(\lambda)|} r^{-[x]} r^{N+1} \frac{1}{1-r^{-1}}. \end{aligned}$$

Since $0 \leq [x] \leq x$ we have $-x \leq -[x] \leq -x+1$. Since $r > 1$ we see that $r^{-[x]} \leq r r^{-x} = r e^{-x \ln r}$ and we obtain the inequality

$$|L_0(x)| \leq\ A e^{-x \ln r} \qquad \text{for all } x \geq 0.$$

Since $r = e^{c+\frac{1}{2}\varepsilon}$ the proof is complete. ■

We will prove now the following Theorem which shows that the symmetry property of the $TB-$spline Q_Λ is shared also by the fundamental $L-$spline L_0.

Theorem 8 *Let Λ be nearly symmetric with respect to $c \in \mathbb{R}$ and suppose that $A_\Lambda(0, -1) \neq 0$. Then the fundamental $L-$spline L_0 has the following symmetry property:*

$$L_0(-x) = e^{cx} L_0(x) \qquad \text{for all } x \in \mathbb{R}.$$

Proof. 1. Define $M(x) = e^{-cx} L(-x)$. By Proposition 4 the function M is a cardinal $L-$spline and clearly we have $M(0) = 1$ and $M(j) = 0$ for all $j \in \mathbb{Z}$, $j \neq 0$. If we know that M is of exponential decay (or of polynomial growth) then the uniqueness property for the interpolation spline implies that $M = L$ and the proof is done.

2. Let us prove that M is of exponential decay: Assume at first that $c \geq 0$. For $y = -x$ with $x \geq 0$ we obtain by the last Proposition

$$|M(y)| = e^{cx} \cdot |L_0(x)| \leq e^{cx} A e^{-(c+\varepsilon)x} = A e^{-\varepsilon|y|}.$$

For $x \geq 0$ we estimate (note that $c \geq 0$)

$$|M(x)| = e^{-cx} |L_0(-x)| \leq |L_0(-x)| \leq Ae^{-\varepsilon|-x|} = Ae^{-\varepsilon|x|}.$$

It follows that M is of exponential decay and the first case is proven.

3. Let us consider the case $c < 0$. Note that T defined by $T(x) = L_0(-x)$ is a fundamental cardinal spline with respect to $-\Lambda$. Further, it is obvious that $-\Lambda$ is nearly symmetric with respect to $-c$. By the first case applied to T we obtain

$$L_0(x) = T(-x) = e^{-cx}T(x) = e^{-cx}L_0(-x)$$

and the proof is complete. ■

For interpolation $L-$splines we have the following symmetry result:

Theorem 9 *Let Λ be nearly symmetric with respect to $c \in \mathbb{R}$ and suppose that $A_\Lambda(0,-1) \neq 0$. Let $(y_j)_{j\in\mathbb{Z}}$ be a bi–infinite sequence of power growth, say $|y_j| \leq A|j|^\gamma$ for all $j \in \mathbb{Z}$. Then there exists a unique interpolation $L-$spline S_0 of polynomial growth, i.e. satisfying*

$$\begin{aligned} S_0(j) &= y_j && \text{for all } j \in \mathbb{Z}, \\ |S_0(x)| &\leq O(|x|^\gamma) && \text{for } x \to \infty, \end{aligned}$$

which has the symmetry property

$$S_0(-x) = e^{cx}S_0(x) \qquad \text{for all } x \in \mathbb{R}$$

if and only if $y_{-j} = e^{cj}y_j$ for all $j \in \mathbb{Z}$.

Proof. We refer to [6] for the existence and uniqueness of S_0. The unique interpolation $L-$spline S_0 is defined by the series (cf. [6, p. 213])

$$S_0(x) = \sum_{k=-\infty}^{\infty} y_k L_0(x-k). \tag{14}$$

The necessity of the condition follows by putting $x = j \in \mathbb{Z}$ and Theorem 8. For the sufficiency note that by Theorem 8 we have

$$\begin{aligned} S_0(-x) &= \sum_{j=-\infty}^{\infty} y_j L_0(-x-j) = \sum_{j=-\infty}^{\infty} y_j e^{c(x+j)} L_0(x+j) \\ &= e^{cx} \sum_{j=-\infty}^{\infty} y_{-j} e^{-cj} L_0(x-j) = e^{cx} S_0(x). \end{aligned}$$

The proof is complete. ■

5 Symmetry of Polysplines

A function[3] $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called a *cardinal polyspline (on spheres) of order* p if S is $(2p-2)$ −times continuously differentiable and the restriction of S to each open annulus $A_j := \left\{ x \in \mathbb{R}^n : e^j < |x| < e^{j+1} \right\}$ is a polyharmonic function of order p for all $j \in \mathbb{Z}$. Recall that a function f defined on an open set U in the euclidean space $\mathbb{R}^n$ is *polyharmonic of order* p if f is $2p$-times continuously differentiable and $\Delta^p f(x) = 0$ for all $x \in U$ where Δ is the Laplace operator and Δ^p its p–th iteration.

Before we can explain the results we have to recall at first some notations: Denote by $Y_{k,l}(\theta)$, $k \in \mathbb{N}_0, l = 1, ..., a_k$ an orthonormal basis of the space $\mathcal{H}_k$ of all spherical harmonics of degree k with respect to the measure $d\theta$ and the L^2-norm

$$\|f\|_{L^2} := \frac{1}{\omega_n} \left(\int_{\mathbb{S}^{n-1}} |f(\theta)|^2 \, d\theta \right)^{\frac{1}{2}}$$

where ω_n denotes the surface area of the sphere $\mathbb{S}^{n-1}$. The *Fourier–Laplace coefficients* of a square integrable function $f : \mathbb{S}^{n-1} \to \mathbb{C}$ are defined by

$$f_{k,l} := \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} f(\theta) Y_{k,l}(\theta) \, d\theta \tag{15}$$

for all $k \in \mathbb{N}_0$, and $l = 1, ..., a_k$. The L_1–*Sobolev norm of exponent* s of a square integrable function $f : \mathbb{S}^{n-1} \to \mathbb{C}$ is defined by[4]

$$\|f\|_{W^{1,s}} := \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} |f_{k,l}| \cdot (1+k)^s . \tag{16}$$

In a recent paper the authors discussed the interpolation results for polysplines and proved the following Theorem: assume that for all $j \in \mathbb{Z}$ the data function d_j be given on the sphere $e^j \mathbb{S}^{n-1}$ of radius e^j, and assume that the functions $f_j : \mathbb{S}^{n-1} \to \mathbb{R}$ defined by $f_j(\theta) := d_j\left(e^j \theta\right)$ have finite L_1−Sobolev norms for $s = 2(p-1) + n/2 - 1$. If the estimate

$$\left\| d_j\left(e^j \theta\right) \right\|_{W^{1,s}} \leq C |j|^\gamma = C \left| \log e^j \right|^\gamma$$

holds for all $\theta \in \mathbb{S}^{n-1}$, and $j \in \mathbb{Z}$, then there exists a polyspline S of order p interpolating the data functions d_j, i.e.

$$S\left(e^j \theta\right) = d_j\left(e^j \theta\right) \qquad \text{for all } \theta \in \mathbb{S}^{n-1}, \ j \in \mathbb{Z}, \tag{17}$$

and obeying the estimate

$$|S(r\theta)| \leq O\left(|\log r|^\gamma\right) \tag{18}$$

[3]The first author introduced in 1991 polysplines in a more general setting, see [2].

[4]Note that this quantity may be infinite for some functions in L_2.

for all $\theta \in \mathbb{S}^{n-1}$ and $r > 0$, i.e. S satisfies the same growth condition as the data functions.

It is a remarkable fact that the polysplines S of order p are of the form

$$S(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} S_{k,\ell}(\log r)\, Y_{k,l}(\theta), \tag{19}$$

where for all k and ℓ the functions $S_{k,\ell} : \mathbb{R} \to \mathbb{R}$ are **cardinal** $L-$splines of order p with respect to a linear differential operator $L = L_{k,p}$ defined by a non–ordered vector $\Lambda(k,p)$. The last is given by

$$\Lambda(k,p) := \Lambda_+(k,p) \cup \Lambda_-(k,p) \tag{20}$$

with

$$\Lambda_+(k,p) := \{k, k+2, ..., k+2p-2\}, \tag{21}$$

$$\Lambda_-(k,p) := \{-k-n+2, -k-n+4, ..., -k-n+2p\}. \tag{22}$$

It is evident that $\Lambda(k,p)$ is nearly symmetric with respect to $c = n - 2p$.

Theorem 10 *Let S be the interpolation polyspline for the data $d_j\left(e^j\theta\right)$ as described above in (17) and (18). Then S satisfies the symmetry property*

$$S\left(r^{-1}\theta\right) = r^{n-2p} S(r\theta) \qquad \text{for all } r > 0,\ \theta \in \mathbb{S}^{n-1}$$

if and only if

$$f_{-j}(\theta) = e^{(n-2p)j} f_j(\theta) \qquad \text{for all } j \in \mathbb{Z},\ \theta \in \mathbb{S}^{n-1}.$$

Before we prove the Theorem we will discuss the special case of "fundamental polysplines".

Let us define a *fundamental polyspline* $L_f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ *for the data function* $f : \mathbb{S}^{n-1} \to \mathbb{C}$ as the polyspline of order p such that for each $j \in \mathbb{Z}$ the interpolation condition

$$L_f\left(e^j\theta\right) = \delta_{j0} f(\theta) \qquad \text{for all } \theta \in \mathbb{S}^{n-1}$$

holds as well as the following growth condition

$$|L_f(r\theta)| \leq M e^{-\varepsilon|\log r|} \|f\|_{W^{1,0}} \qquad \text{for all } r > 0 \text{ and } \theta \in \mathbb{S}^{n-1}. \tag{23}$$

The existence of the fundamental polyspline L_f is proved in the following:

Theorem 11 *Let q be a natural number such that $q \geq p - 1 + \left[\frac{n}{2}\right]$. Then for any $2q$–continuously differentiable function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ there exists a fundamental polyspline L_f of order p such that condition (23) holds with constants $M > 0$ and $\varepsilon > 0$ independent of the choice of the data function f. Furthermore, the fundamental polyspline satisfies the symmetry property*

$$L_f\left(r^{-1}\theta\right) = r^{n-2p} L_f(r\theta) \qquad \text{for all } r > 0 \text{ and } \theta \in \mathbb{S}^{n-1}.$$

Proof. Existence is proved in [4]. For our purposes we need only the defining formula for L_f from [4]. Let $f_{k,l}$ be the Laplace-Fourier coefficient of f defined in (15). The fundamental polyspline $L_f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is defined by the following series (cf. [4]):

$$L_f(r\theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l} \cdot L_{0,k}(\log r) \cdot Y_{k,l}(\theta).$$

Here $L_{0,k}$ denotes the fundamental cardinal $L-$spline with respect to the differential operator L which is defined through the vector $\Lambda(k) = \Lambda(k,p)$. By Theorem 8 we know that $L_{0,k}(-v) = e^{cv} L_{0,k}(v)$ for all $v \in \mathbb{R}$. It follows that $L_{0,k}\left(\log r^{-1}\right) = L_{0,k}(-\log r) = r^c L_{0,k}(\log r)$. Hence we obtain

$$L_f\left(r^{-1}\theta\right) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l} \cdot L_{0,k}\left(\log r^{-1}\right) \cdot Y_{k,l}(\theta) = r^c L_f(r\theta)$$

for all $r > 0$ and $\theta \in \mathbb{S}^{n-1}$. This ends the proof. ∎

Finally we will prove Theorem 10:

Proof. For each f_j we can define a fundamental polyspline L_{f_j} as in Theorem 11. The interpolation polyspline is defined for all $r > 0$ and $\theta \in \mathbb{S}^{n-1}$ by

$$S(r\theta) := \sum_{j=-\infty}^{\infty} L_{f_j}\left(re^{-j}\theta\right).$$

The proof is now straightforward using the last Theorem. ∎

Acknowledgement. *The second author thanks the Alexander von Humboldt-Stiftung for supporting him within the framework of the Feodor-Lynen-program.*

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