Symmetry properties of cardinal interpolation L-splines and polysplines

O. Kounchev and H. Render
Institute of Mathematics, Bulgarian Academy of Sciences

February 23, 2004

Abstract

Recently the authors have proved existence and uniqueness of cardinal polysplines of order p. In this paper it is shown that a cardinal fundamental polyspline $L_f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ of order p possess the following symmetry property:

$$L_f\left(r^{-1}\theta\right) = r^{n-2p} L_f\left(r\theta\right) \qquad \text{for all } r > 0, \theta \in \mathbb{S}^{n-1}.$$

The proof is based on symmetry properties of univariate cardinal interpolation $L-$splines for a certain class of linear differential operators L.

1 Introduction

A main result in the area of cardinal polynomial splines is the existence of cardinal interpolation splines for certain classes of data, proved by *Schoenberg* [9]: suppose that a bi–infinite sequence of real numbers $y_j, j \in \mathbb{Z}$ of *power growth* with exponent $\gamma \geq 0$ be given, i.e. that there exists $C > 0$ such that $|y_j| \leq C \cdot |j|^{\gamma}$ for all $j \in \mathbb{Z}$, and suppose further that p is an *odd integer.* Then there *exists* a *unique* cardinal polynomial spline S of polynomial degree p such that the following conditions hold:

$$\begin{aligned} S\left(j\right) &= y_j && \text{for all } j \in \mathbb{Z}, \quad \text{and} \\ \left|S\left(x\right)\right| &\leq O\left(|x|^{\gamma}\right) && \text{for all } x \in \mathbb{R}. \end{aligned} \tag{1}$$

Hence S is a cardinal spline of power growth on the whole real line interpolating the data $\left(y_j\right)_{j\in\mathbb{Z}}$. Furthermore, the interpolation spline has the following *symmetry property*: The spline S is *even* (which means that $S\left(-x\right) = S\left(x\right)$ for all $x \in \mathbb{R}$) provided $y_{-j} = y_j$ for all $j \in \mathbb{Z}$. In particular, the so–called fundamental interpolation spline (the one satisfying (1) with data $y_j = 0$ for $j \neq 0$ and $y_0 = 1$) is even.

The existence and uniqueness results of Schoenberg were generalized to cardinal $L-$splines by Ch. Micchelli in [5], [6], where L is a differential operator with constant real coefficients. In this paper we are interested in symmetry properties of the interpolation cardinal $L-$splines and the fundamental cardinal $L-$spline. Intuitively, it is clear that the interpolation $L-$splines will have symmetry properties only if the differential operator L has certain kind of symmetry properties which we formulate below.

Let us assume that the operator L is defined by the expression

$$L := \prod_{j=1}^{N+1} \left(\frac{d}{dx} - \lambda_j \right).$$

Hence L is characterized by the (unordered) vector $\Lambda := (\lambda_1, \lambda_2, ..., \lambda_{N+1})$ and *throughout the paper we will assume that all λ_j are real.* We call Λ to be *nearly symmetric with respect to $c \in \mathbb{R}$* if there exists a permutation π of the set $\{1, ..., N+1\}$ such that $-\lambda_j = c + \lambda_{\pi(j)}$ for $j = 1, ..., N+1$, or, symbolically, $-\Lambda = c+\Lambda$. In the case $c = 0$ we call Λ *symmetric*; indeed then Λ is a symmetric set with respect to 0.

Let us denote by L_0 the fundamental $L-$spline[1], i.e. the interpolation $L-$spline for the data $y_0 = 1$ and $y_j = 0$ for all $j \in \mathbb{Z}$, $j \neq 0$. Then our main result states that if Λ is *nearly symmetric with respect to c*, then L_0 satisfies the following symmetry property:

$$L_0(-x) = e^{cx} L_0(x) \qquad \text{for all } x \in \mathbb{R}.$$

This leads to a symmetry property of the cardinal interpolation $L-$splines which is analogous to the polynomial case. Furthermore we prove that the *basis* $L-$spline (or the $TB-$spline) which we denote by Q_Λ (recall that it has support equal to the interval $[0, N+1]$) satisfies the following symmetry property

$$Q_\Lambda(N+1-x) = e^{cx} Q_\Lambda(x) \qquad \text{for all } x \in \mathbb{R}.$$

These results are finally used to prove the symmetry properties of the fundamental polysplines.

2 Symmetry of the Euler–Frobenius polynomial

Let us recall briefly the definition of a cardinal $L-$spline: Let L be a linear differential operator L (of order $N+1$) with constant coefficients. The set of all homogeneous solutions is defined by

$$U_L := U_\Lambda := \{ f \in C^\infty(\mathbb{R}) : Lf(x) = 0 \text{ in } \mathbb{R} \}.$$

It is well known from Ordinary Differential Equations that U_Λ is the linear space generated by all functions of the type $x \mapsto e^{\lambda_j x} x^i$ where $j \in \{1, ..., N+1\}$ and the integer $i \geq 0$ is smaller than the multiplicity of λ_j. Then a function $u \colon \mathbb{R} \to \mathbb{R}$ is called a *cardinal $L-$spline* if

[1] We hope that the reader will not mix the operator L and the fundamental spline L_0; the notation for the last has been systematically used by Schoenberg, and the authors have decided to follow this tradition.

- for each $j \in \mathbb{Z}$ there exists $f_j \in U_L$ such that $u(t) = f_j(t)$ for all $t \in (j, j+1)$ and
- u is $N-1$ times continuously differentiable.

Define $e^\Lambda = \left\{ e^{\lambda_j} : j = 1, ..., N+1 \right\}$. In the theory of cardinal $L-$splines the function $A_\Lambda \colon \mathbb{R} \times \left(\mathbb{C} \setminus e^\Lambda\right) \to \mathbb{C}$ (cf. [6], p. 223) defined by

$$A_\Lambda(x, \lambda) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\prod_{j=1}^{N+1}(z - \lambda_j)} \frac{e^{xz}}{e^z - \lambda} dz \tag{2}$$

is of fundamental importance. Here Γ is a closed simple curve in the complex plane surrounding all $\lambda_j, j = 1, ..., N+1$ whereby the zeros of the function $e^z - \lambda$ lie in the exterior of Γ. We will call $A_\Lambda(x, \lambda)$ the *Euler-Schoenberg function.* The *Euler-Frobenius function* is defined by

$$\Pi_\Lambda(x, \lambda) := A_\Lambda(x, \lambda) \cdot \prod_{j=1}^{N+1} \left(e^{\lambda_j} - \lambda\right). \tag{3}$$

For $x = 0$ it is a polynomial of degree at most N in the variable λ (Corollary 2.1 in [6]) and $\Pi_\Lambda(0, \lambda)$ is called the *Euler-Frobenius polynomial.*

Proposition 1 *Let Λ be nearly symmetric with respect to $c \in \mathbb{R}$. For all $\lambda \notin e^\Lambda \cup e^{-\Lambda} \cup \{0\}$ and all $x \in \mathbb{R}$ the following equality holds for*

$$A_\Lambda\left(1 - x, \frac{1}{\lambda}\right) = (-1)^{N+1} \lambda \, e^{(x-1)c} \, A_\Lambda\left(x, \lambda e^{-c}\right). \tag{4}$$

Proof. Let $\Lambda = (\lambda_1, ..., \lambda_{N+1})$ and define $-\Lambda = (-\lambda_1, ..., -\lambda_{N+1})$. Then for all $x \in \mathbb{R}$ and $\lambda \notin e^\Lambda \cup e^{-\Lambda} \cup \{0\}$ we have (see [6, p. 213])

$$A_\Lambda\left(1 - x, \frac{1}{\lambda}\right) = (-1)^{N+1} \lambda \cdot A_{-\Lambda}(x, \lambda). \tag{5}$$

Assume now that Λ is nearly symmetric with respect to $c \in \mathbb{R}$. A straightforward computation shows that for all $x \in \mathbb{R}$ and $\lambda \notin e^\Lambda \cup e^{-\Lambda} \cup \{0\}$

$$A_{-\Lambda}(x, \lambda) = e^{(x-1)c} A_\Lambda\left(x, \lambda e^{-c}\right). \tag{6}$$

Combining equation (5) and (6) the proof is accomplished. ■

By taking $x = 0$ in (4) using $A_\Lambda(1, \lambda) = \lambda A_\Lambda(0, \lambda)$ (see [6]) we obtain the equality

$$(-1)^{N+1} \lambda \, e^{-c} \, A_\Lambda\left(0, \lambda e^{-c}\right) = A_\Lambda\left(1, \frac{1}{\lambda}\right) = \frac{1}{\lambda} \cdot A_\Lambda\left(0, \frac{1}{\lambda}\right).$$

It follows that

$$(-1)^{N+1} \; e^{-c} \lambda^2 \, A_\Lambda\left(0, \lambda e^{-c}\right) = A_\Lambda\left(0, \frac{1}{\lambda}\right) \tag{7}$$

Suppose now that $\lambda = \mu_i$ is a zero of the function $\lambda \mapsto A_\Lambda(0,\lambda)$. Put $\rho_i = e^c \mu_i$. Then the identity (7) tells us that $v_i := 1/\rho_i$ is again a zero. It follows that $\mu_i \cdot v_i = e^{-c}$. Since all zeros are negative and simple it follows that for each μ_i there exists exactly one v_i which is reciprocal. Hence, we have proved the following important result:

Proposition 2 *Assume that Λ is nearly symmetric with respect to $c \in \mathbb{R}$ and let $\mu_{N-1} < ... < \mu_1 < 0$ be the zeros of $\lambda \mapsto A_\Lambda(0,\lambda)$ in $(-\infty, 0)$. Then $\mu_j \mu_{N-j} = e^{-c}$ for all $j = 1, ..., N-1$.*

Proposition 3 *Let Λ be nearly symmetric with respect to $c \in \mathbb{R}$. Then the Euler–Frobenius polynomial Π_Λ satisfies the equality*

$$\boxed{\Pi_\Lambda\left(x, \lambda e^{-c}\right) = \lambda^N e^{-(x-1)c} \cdot e^{-\frac{1}{2}(N+1)c} \Pi_\Lambda\left(1-x, \tfrac{1}{\lambda}\right).} \tag{8}$$

Proof. It is easy to see that $\lambda_1 + ... + \lambda_{N+1} = -\frac{1}{2}(N+1)c$ since $\lambda_j + \lambda_{\pi(j)} = -c$ for $j = 1, ...N+1$. Define $r_\Lambda(\lambda) = \prod_{j=1}^{N+1}\left(e^{\lambda_j} - \lambda\right)$. Since $\lambda_j + \lambda_{\pi(j)} = -c$ it follows that $r_\Lambda\left(\lambda e^{-c}\right)$ is equal to

$$(-\lambda)^{N+1} e^{(\lambda_1 + ... + \lambda_{N+1})} \prod_{j=1}^{N+1}\left(e^{\lambda_{\pi(j)}} - \frac{1}{\lambda}\right) = e^{-\frac{1}{2}(N+1)c} (-\lambda)^{N+1} r_\Lambda\left(\frac{1}{\lambda}\right).$$

The last equation with (4) shows that the Euler-Frobenius function $\Pi_\Lambda\left(x, \lambda e^{-c}\right)$ (which was defined by $A_\Lambda\left(x, \lambda e^{-c}\right) \cdot r_\Lambda\left(\lambda e^{-c}\right)$ in (3)) is equal to

$$A_\Lambda\left(1-x, \frac{1}{\lambda}\right)(-1)^{N+1} \frac{1}{\lambda} e^{-(x-1)c} \cdot e^{-\frac{1}{2}(N+1)c} (-\lambda)^{N+1} r_\Lambda\left(\frac{1}{\lambda}\right).$$

This is just the statement. ■

3 Symmetry property of the basis $L-$spline Q_Λ

Define the function $s_\Lambda(\lambda) := \prod_{j=1}^{N+1}\left(e^{-\lambda_j} - \lambda\right)$ and let s_j, $j = 0, ..., N+1$ be the coefficients of $s_\Lambda(\lambda)$, i.e. $s_\Lambda(\lambda) = \sum_{j=0}^{N+1} s_j \lambda^j$. Due to the choice of the real numbers s_j it is straightforward to prove that the following cardinal $L-$spline has support in the interval $[0, N+1]$,

$$Q_\Lambda(x) := \sum_{j=0}^{N+1} s_j \cdot A_\Lambda(x+1-j, 0) \cdot 1_{[0,\infty)}(x). \tag{9}$$

We call Q_Λ the *basis-spline* or *B-spline* or *TB-spline*.[2] It has the property that $Q_\Lambda(x) > 0$ for all $x \in (0, N+1)$ and Q_Λ is unique up to a positive constant.

[2]It is the analog to the $B-$spline in the polynomial case, see e.g. [11].

Proposition 4 *Suppose that Λ is nearly symmetric with respect to the constant $c \in \mathbb{R}$. If $S(x)$ is a cardinal $L-$spline with respect to Λ then $e^{-cx}S(m-x)$ is a cardinal $L-$spline with respect to Λ for every integer m.*

Proof. Define $f(x) := e^{-cx}S(m-x)$ for all $x \in \mathbb{R}$. Clearly f is $(N-1)$-continuously differentiable. We know that S is on each interval $(j, j+1), j \in \mathbb{Z}$, a solution of the given differential operator L, i.e., that the restriction S_j of S to $(j, j+1)$ gives a function in U_Λ. Hence S_j is a linear combination of exponentials $x^s e^{\lambda_j x}$ with $j \in \{1, ..., N+1\}$ and s is an integer smaller than the multiplicity of λ_j. Now consider $f_j(x) = e^{-cx}S_j(m-x)$. Since $-\lambda_j = c + \lambda_{\pi(j)}$ it follows that f_j is in U_Λ and the proof is complete. ■

Theorem 5 *Suppose that Λ is nearly symmetric with respect to the constant $c \in \mathbb{R}$. Then for all $x \in \mathbb{R}$ the following equality holds*

$$\boxed{Q_\Lambda(N+1-x) = e^{cx}e^{-\frac{1}{2}(N+1)c}Q_\Lambda(x).}$$

Proof. Define $f(x) := e^{-cx}Q_\Lambda(N+1-x) \geq 0$ for all $x \in \mathbb{R}$. Clearly f is $(N-1)$-continuously differentiable and it has support in $[0, N+1]$. The last Proposition shows that f is an $L-$spline with respect to Λ. Then the above-mentioned uniqueness property of the basis spline Q_Λ yields that there exists a constant $D > 0$ such that $f = DQ_\Lambda$. By putting $x = \frac{1}{2}(N+1)$ one obtains the constant D and the proof is accomplished. ■

The following fundamental formula relates the Euler-Frobenius function with the basis–spline Q_Λ (cf. [6, p. 221 and p. 222])

$$R_\Lambda^x(\lambda) := \sum_{j=0}^{N} \lambda^{N-j}Q_\Lambda(x+j) = \frac{(-1)^N}{e^{(\lambda_1+...+\lambda_{N+1})}} \cdot \Pi_\Lambda(x, \lambda). \tag{10}$$

The following result will be needed in the next section

Proposition 6 *Suppose that Λ is nearly symmetric with respect to the constant $c \in \mathbb{R}$. Then the polynomial $P_\Lambda^0(\lambda) = R_\Lambda^0\left(\frac{1}{\lambda}\right)\lambda^N$ is proportional to the polynomial $\Pi_\Lambda(0, \lambda e^{-c})$, or more precisely,*

$$P_\Lambda^0(\lambda) = (-1)^N \lambda e^{Nc} \cdot \Pi_\Lambda\left(0, \lambda e^{-c}\right).$$

Proof. Since $\lambda_1 + ... + \lambda_{N+1} = -\frac{1}{2}(N+1)c$ the equation (10) yields

$$P_\Lambda^\alpha(\lambda) := R_\Lambda^\alpha\left(\frac{1}{\lambda}\right)\lambda^N = (-1)^N e^{\frac{1}{2}(N+1)c}\lambda^N \Pi_\Lambda\left(\alpha, \frac{1}{\lambda}\right).$$

Now we use the symmetry property of the Euler-Frobenius function $\Pi_\Lambda(x, \lambda e^{-c})$ in (8) applied to $\alpha = 1-x$ and obtain that

$$P_\Lambda^\alpha(\lambda) = (-1)^N e^{-c\alpha}e^{(N+1)c}\,\Pi_\Lambda\left(1-\alpha, \lambda e^{-c}\right).$$

Now take $\alpha = 0$ and use the fact that $A_\Lambda(1, \mu) = \mu A_\Lambda(0, \mu)$ and the proof is accomplished. ■

4 Symmetry of the fundamental $L-$spline L_0

We come to the main point of our study. Let us now consider interpolation problems for cardinal $L-$splines. Let us fix a number $\alpha \in [0,1)$. A cardinal $L-$spline L_α is called a *fundamental $L-$spline* with respect to α if $L_\alpha(\alpha) = 1$ and $L_\alpha(\alpha + j) = 0$ for all $j \in \mathbb{Z}$, $j \neq 0$ and if it decays exponentially, i.e. if there exist two constants $A, B > 0$ such that

$$|L_\alpha(x)| \leq Ae^{-B|x|} \qquad \text{for all } x \in \mathbb{R}. \tag{11}$$

The existence and uniqueness of fundamental $L-$spline L_α with respect to α was proved by Micchelli in [6, p. 213] under the assumption $A_\Lambda(\alpha, -1) \neq 0$. Let us recall from [10, p. 271] the construction of the fundamental $L-$spline: Define as in the last section the function

$$P_\Lambda^\alpha(\lambda) := R_\Lambda^\alpha\left(\frac{1}{\lambda}\right)\lambda^N = \sum_{j=0}^{N} \lambda^j Q_\Lambda(\alpha + j).$$

Assume now that the function $\lambda \to 1/P_\Lambda^\alpha(\lambda)$ is holomorphic on the annulus $\{R_1 < |\lambda| < R_2\}$ in the complex plane such that $R_1 < 1 < R_2$. We consider the Laurent series

$$\frac{1}{P_\Lambda^\alpha(\lambda)} = \sum_{j=-\infty}^{\infty} \omega_j \lambda^j.$$

Then the fundamental $L-$spline can be defined by

$$L_\alpha(x) := \sum_{j=-\infty}^{\infty} \omega_j Q_\Lambda(x - j). \tag{12}$$

The following Proposition provides in fact an improvement of the constants A and B in the above inequality (11) but only on the half axis $x \geq 0$. It is so far essential for the proof of our main result

Proposition 7 *Let Λ be nearly symmetric with respect to $c > 0$ and suppose that $A_\Lambda(0, -1) \neq 0$. Then there exists $\varepsilon > 0$ and $A > 0$ such that*

$$|L_0(x)| \leq Ae^{-(c+\varepsilon)x} \qquad \textit{for all } x \geq 0.$$

Proof. 1. For the proof we need at first a result about the zeros of $\lambda \mapsto A_\Lambda(0, \lambda)$. Let $\mu_{N-1} < \cdots < \mu_k < -1 < \mu_{k-1} < \cdots < \mu_1 < 0$ be the zeros of $\lambda \mapsto A_\Lambda(0, \lambda)$ in $(-\infty, 0)$. By Proposition 2 we know that $\mu_k \mu_{k-1} = e^{-c}$. Clearly $A_\Lambda(0, \mu) \neq 0$ for all μ with $\mu_k < \mu < \mu_{k-1}$. Since $|\mu_k| > 1$ it follows that $|\mu_{k-1}| = \frac{1}{|\mu_k|} e^{-c} < e^{-c}$. We conclude that there exists $\varepsilon > 0$ such that $|\mu_{k-1}| < e^{-c-\varepsilon}$. By taking $\varepsilon > 0$ somewhat smaller we can also assume that $|\mu_k| > e^{\varepsilon}$. Since $A_\Lambda(0, \lambda)$ has zeros only in $(-\infty, 0]$ we conclude that $A_\Lambda(0, \lambda) \neq 0$ for all $e^{-c-\varepsilon} \leq |\lambda| \leq e^{\varepsilon}$. This shows that

$$\Pi_\Lambda(0, \lambda) = r_\Lambda(\lambda) A_\Lambda(0, \lambda) \neq 0 \qquad \text{for all } e^{-c-\varepsilon} \leq |\lambda| \leq e^{\varepsilon}.$$

By Proposition 6 $P_{\Lambda}^{0}(\lambda) = (-1)^{N} \lambda e^{Nc} \cdot \Pi_{\Lambda}(0, \lambda e^{-c})$. We conclude that $P_{\Lambda}^{0}(\lambda) \neq 0$ for all λ satisfying $e^{-\varepsilon} \leq |\lambda| \leq e^{c+\varepsilon}$.

2. Put $M_{\Lambda} := \max_{y \in (0,N+1)} |Q_{\Lambda}(y)|$. We want to estimate $L_0(x)$ as defined in (12). Note that only $j \in \mathbb{Z}$ with $x - j \in (0, N+1)$ contributes non-trivially to the sum (12) so we consider only j satisfying $x - N - 1 < j < x$. Denote by $[x]$ the largest integer $k \in \mathbb{Z}$ with $k \leq x$. Note that j can be represented as $j = [x] - N - 1 + k$ with $k \in N_0$. Then the series $L_0(x)$ can be estimated by

$$|L_0(x)| \leq \sum_{j \in \mathbb{Z}, [x]-N-1 \leq j} |\omega_j| M_{\Lambda} = M_{\Lambda} \cdot \sum_{k=0}^{\infty} \left|\omega_{[x]-N-1+k}\right|.$$

Above we have shown that $P_{\Lambda}^{0}(\lambda) \neq 0$ is holomorphic on the annulus $A(R_1, R_2)$ where $R_1 = e^{-\varepsilon} < 1 < e^{c+\varepsilon} = R_2$. For any r with $R_1 < r < R_2$ we have the following well known Cauchy estimates (see [1, p. 107]) for every $j \in \mathbb{Z}$,

$$|\omega_j| \leq \max_{|\lambda|=r} \left| \frac{1}{P_{\Lambda}^{\alpha}(\lambda)} \right| \cdot r^{-j}. \tag{13}$$

Hence the following inequality holds where we take $r = e^{c+\varepsilon\frac{1}{2}} > 1$

$$\begin{aligned} |L_0(x)| \leq \; & M_{\Lambda} \max_{|\lambda|=r} \frac{1}{|P_{\Lambda}^{\alpha}(\lambda)|} \sum_{k=0}^{\infty} r^{-[x]+N+1-k} \\ = \; & M_{\Lambda} \max_{|\lambda|=r} \frac{1}{|P_{\Lambda}^{\alpha}(\lambda)|} r^{-[x]} r^{N+1} \frac{1}{1-r^{-1}}. \end{aligned}$$

Since $0 \leq [x] \leq x$ we have $-x \leq -[x] \leq -x+1$. Since $r > 1$ we see that $r^{-[x]} \leq r r^{-x} = r e^{-x \ln r}$ and we obtain the inequality

$$|L_0(x)| \leq \; A e^{-x \ln r} \qquad \text{for all } x \geq 0.$$

Since $r = e^{c+\frac{1}{2}\varepsilon}$ the proof is complete. ■

We will prove now the following Theorem which shows that the symmetry property of the $TB-$spline Q_{Λ} is shared also by the fundamental $L-$spline L_0.

Theorem 8 *Let Λ be nearly symmetric with respect to $c \in \mathbb{R}$ and suppose that $A_{\Lambda}(0, -1) \neq 0$. Then the fundamental $L-$spline L_0 has the following symmetry property:*

$$L_0(-x) = e^{cx} L_0(x) \qquad \text{for all } x \in \mathbb{R}.$$

Proof. 1. Define $M(x) = e^{-cx} L(-x)$. By Proposition 4 the function M is a cardinal $L-$spline and clearly we have $M(0) = 1$ and $M(j) = 0$ for all $j \in \mathbb{Z}$, $j \neq 0$. If we know that M is of exponential decay (or of polynomial growth) then the uniqueness property for the interpolation spline implies that $M = L$ and the proof is done.

2. Let us prove that M is of exponential decay: Assume at first that $c \geq 0$. For $y = -x$ with $x \geq 0$ we obtain by the last Proposition

$$|M(y)| = e^{cx} \cdot |L_0(x)| \leq e^{cx} A e^{-(c+\varepsilon)x} = A e^{-\varepsilon|y|}.$$

For $x \geq 0$ we estimate (note that $c \geq 0$)

$$|M(x)| = e^{-cx} |L_0(-x)| \leq |L_0(-x)| \leq Ae^{-\varepsilon|-x|} = Ae^{-\varepsilon|x|}.$$

It follows that M is of exponential decay and the first case is proven.

3. Let us consider the case $c < 0$. Note that T defined by $T(x) = L_0(-x)$ is a fundamental cardinal spline with respect to $-\Lambda$. Further, it is obvious that $-\Lambda$ is nearly symmetric with respect to $-c$. By the first case applied to T we obtain

$$L_0(x) = T(-x) = e^{-cx}T(x) = e^{-cx}L_0(-x)$$

and the proof is complete. ■

For interpolation $L-$splines we have the following symmetry result:

Theorem 9 *Let Λ be nearly symmetric with respect to $c \in \mathbb{R}$ and suppose that $A_\Lambda(0, -1) \neq 0$. Let $(y_j)_{j\in\mathbb{Z}}$ be a bi–infinite sequence of power growth, say $|y_j| \leq A|j|^\gamma$ for all $j \in \mathbb{Z}$. Then there exists a unique interpolation $L-$spline S_0 of polynomial growth, i.e. satisfying*

$$\begin{aligned} S_0(j) &= y_j & &\text{for all } j \in \mathbb{Z}, \\ |S_0(x)| &\leq O(|x|^\gamma) & &\text{for } x \to \infty, \end{aligned}$$

which has the symmetry property

$$S_0(-x) = e^{cx}S_0(x) \qquad \text{for all } x \in \mathbb{R}$$

if and only if $y_{-j} = e^{cj}y_j$ for all $j \in \mathbb{Z}$.

Proof. We refer to [6] for the existence and uniqueness of S_0. The unique interpolation $L-$spline S_0 is defined by the series (cf. [6, p. 213])

$$S_0(x) = \sum_{k=-\infty}^{\infty} y_k L_0(x-k). \tag{14}$$

The necessity of the condition follows by putting $x = j \in \mathbb{Z}$ and Theorem 8. For the sufficiency note that by Theorem 8 we have

$$\begin{aligned} S_0(-x) &= \sum_{j=-\infty}^{\infty} y_j L_0(-x-j) = \sum_{j=-\infty}^{\infty} y_j e^{c(x+j)} L_0(x+j) \\ &= e^{cx} \sum_{j=-\infty}^{\infty} y_{-j} e^{-cj} L_0(x-j) = e^{cx} S_0(x). \end{aligned}$$

The proof is complete. ■

5 Symmetry of Polysplines

A function[3] $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called a *cardinal polyspline (on spheres) of order* p if S is $(2p-2)$ −times continuously differentiable and the restriction of S to each open annulus $A_j := \left\{x \in \mathbb{R}^n : e^j < |x| < e^{j+1}\right\}$ is a polyharmonic function of order p for all $j \in \mathbb{Z}$. Recall that a function f defined on an open set U in the euclidean space $\mathbb{R}^n$ is *polyharmonic of order* p if f is $2p$-times continuously differentiable and $\Delta^p f(x) = 0$ for all $x \in U$ where Δ is the Laplace operator and Δ^p its p–th iteration.

Before we can explain the results we have to recall at first some notations: Denote by $Y_{k,l}(\theta)$, $k \in \mathbb{N}_0, l = 1, ..., a_k$ an orthonormal basis of the space $\mathcal{H}_k$ of all spherical harmonics of degree k with respect to the measure $d\theta$ and the L^2-norm

$$\|f\|_{L^2} := \frac{1}{\omega_n}\left(\int_{\mathbb{S}^{n-1}} |f(\theta)|^2 \, d\theta\right)^{\frac{1}{2}}$$

where ω_n denotes the surface area of the sphere $\mathbb{S}^{n-1}$. The *Fourier–Laplace coefficients* of a square integrable function $f : \mathbb{S}^{n-1} \to \mathbb{C}$ are defined by

$$f_{k,l} := \frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}} f(\theta)\, Y_{k,l}(\theta)\, d\theta \tag{15}$$

for all $k \in \mathbb{N}_0$, and $l = 1, ..., a_k$. The L_1–*Sobolev norm of exponent* s of a square integrable function $f : \mathbb{S}^{n-1} \to \mathbb{C}$ is defined by[4]

$$\|f\|_{W^{1,s}} := \sum_{k=0}^{\infty}\sum_{l=1}^{a_k} |f_{k,l}| \cdot (1+k)^s . \tag{16}$$

In a recent paper the authors discussed the interpolation results for polysplines and proved the following Theorem: assume that for all $j \in \mathbb{Z}$ the data function d_j be given on the sphere $e^j\mathbb{S}^{n-1}$ of radius e^j, and assume that the functions $f_j : \mathbb{S}^{n-1} \to \mathbb{R}$ defined by $f_j(\theta) := d_j\left(e^j\theta\right)$ have finite L_1−Sobolev norms for $s = 2(p-1) + n/2 - 1$. If the estimate

$$\left\|d_j\left(e^j\theta\right)\right\|_{W^{1,s}} \leq C\,|j|^{\gamma} = C\left|\log e^j\right|^{\gamma}$$

holds for all $\theta \in \mathbb{S}^{n-1}$, and $j \in \mathbb{Z}$, then there exists a polyspline S of order p interpolating the data functions d_j, i.e.

$$S\left(e^j\theta\right) = d_j\left(e^j\theta\right) \qquad \text{for all } \theta \in \mathbb{S}^{n-1},\ j \in \mathbb{Z}, \tag{17}$$

and obeying the estimate

$$|S(r\theta)| \leq O\left(|\log r|^{\gamma}\right) \tag{18}$$

[3] The first author introduced in 1991 polysplines in a more general setting, see [2].

[4] Note that this quantity may be infinite for some functions in L_2.

for all $\theta \in \mathbb{S}^{n-1}$ and $r > 0$, i.e. S satisfies the same growth condition as the data functions.

It is a remarkable fact that the polysplines S of order p are of the form

$$S(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} S_{k,\ell}(\log r)\, Y_{k,l}(\theta), \tag{19}$$

where for all k and ℓ the functions $S_{k,\ell} : \mathbb{R} \to \mathbb{R}$ are **cardinal** $L-$splines of order p with respect to a linear differential operator $L = L_{k,p}$ defined by a non–ordered vector $\Lambda(k,p)$. The last is given by

$$\Lambda(k,p) := \Lambda_+(k,p) \cup \Lambda_-(k,p) \tag{20}$$

with

$$\Lambda_+(k,p) := \{k, k+2, ..., k+2p-2\}, \tag{21}$$

$$\Lambda_-(k,p) := \{-k-n+2, -k-n+4, ..., -k-n+2p\}. \tag{22}$$

It is evident that $\Lambda(k,p)$ is nearly symmetric with respect to $c = n-2p$.

Theorem 10 *Let S be the interpolation polyspline for the data $d_j\left(e^j\theta\right)$ as described above in (17) and (18). Then S satisfies the symmetry property*

$$S\left(r^{-1}\theta\right) = r^{n-2p} S(r\theta) \qquad \textit{for all } r > 0,\ \theta \in \mathbb{S}^{n-1}$$

if and only if

$$f_{-j}(\theta) = e^{(n-2p)j} f_j(\theta) \qquad \textit{for all } j \in \mathbb{Z},\ \theta \in \mathbb{S}^{n-1}.$$

Before we prove the Theorem we will discuss the special case of "fundamental polysplines".

Let us define a *fundamental polyspline* $L_f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ *for the data function* $f : \mathbb{S}^{n-1} \to \mathbb{C}$ as the polyspline of order p such that for each $j \in \mathbb{Z}$ the interpolation condition

$$L_f\left(e^j\theta\right) = \delta_{j0} f(\theta) \qquad \text{for all } \theta \in \mathbb{S}^{n-1}$$

holds as well as the following growth condition

$$|L_f(r\theta)| \le M e^{-\varepsilon|\log r|} \|f\|_{W^{1,0}} \qquad \text{for all } r > 0 \text{ and } \theta \in \mathbb{S}^{n-1}. \tag{23}$$

The existence of the fundamental polyspline L_f is proved in the following:

Theorem 11 *Let q be a natural number such that $q \ge p-1+\left[\frac{n}{2}\right]$. Then for any $2q$–continuously differentiable function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ there exists a fundamental polyspline L_f of order p such that condition (23) holds with constants $M > 0$ and $\varepsilon > 0$ independent of the choice of the data function f. Furthermore, the fundamental polyspline satisfies the symmetry property*

$$L_f\left(r^{-1}\theta\right) = r^{n-2p} L_f(r\theta) \qquad \textit{for all } r > 0 \textit{ and } \theta \in \mathbb{S}^{n-1}.$$

Proof. Existence is proved in [4]. For our purposes we need only the defining formula for L_f from [4]. Let $f_{k,l}$ be the Laplace-Fourier coefficient of f defined in (15). The fundamental polyspline $L_f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is defined by the following series (cf. [4]):

$$L_f\left(r\theta\right) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l} \cdot L_{0,k}\left(\log r\right) \cdot Y_{k,l}\left(\theta\right).$$

Here $L_{0,k}$ denotes the fundamental cardinal $L-$spline with respect to the differential operator L which is defined through the vector $\Lambda\left(k\right) = \Lambda\left(k,p\right)$. By Theorem 8 we know that $L_{0,k}\left(-v\right) = e^{cv} L_{0,k}\left(v\right)$ for all $v \in \mathbb{R}$. It follows that $L_{0,k}\left(\log r^{-1}\right) = L_{0,k}\left(-\log r\right) = r^c L_{0,k}\left(\log r\right)$. Hence we obtain

$$L_f\left(r^{-1}\theta\right) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l} \cdot L_{0,k}\left(\log r^{-1}\right) \cdot Y_{k,l}\left(\theta\right) = r^c L_f\left(r\theta\right)$$

for all $r > 0$ and $\theta \in \mathbb{S}^{n-1}$. This ends the proof. ■

Finally we will prove Theorem 10:

Proof. For each f_j we can define a fundamental polyspline L_{f_j} as in Theorem 11. The interpolation polyspline is defined for all $r > 0$ and $\theta \in \mathbb{S}^{n-1}$ by

$$S\left(r\theta\right) := \sum_{j=-\infty}^{\infty} L_{f_j}\left(re^{-j}\theta\right).$$

The proof is now straightforward using the last Theorem. ■

Acknowledgement. *The second author thanks the Alexander von Humboldt-Stiftung for supporting him within the framework of the Feodor-Lynen-program.*

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