Wavelet Analysis of Cardinal L-splines and Construction of Multivariate Prewavelets.

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Abstract. In the fundamental paper [1] non-stationary multiresolution analysis was developed and applied to the case of cardinal $L-$splines. In the first part of this paper we review wavelet analysis of cardinal $L-$splines in more detail by exploiting special properties of the Euler-Frobenius polynomial for cardinal $L-$splines. For example, the Riesz bounds of the mother and father wavelets of cardinal $L-$splines are determined in a constructive way. The main result states that the system of all translations of the wavelets ψ_j over all levels $j \in \mathbb{Z}$ induces a stable basis of $L^2(\mathbb{R})$. In the second part we use these results in order to discuss multivariate prewavelets based on polysplines.

§1. Introduction

Wavelet analysis of polysplines has been recently introduced by the first author in [7]. This approach is based on the definition of a *cardinal polyspline* (for definition see Section 5) — a concept which was introduced by the first author in [6] and which was successfully exploited in the papers [8,9,10]. In this paper it is not assumed that the reader is familiar with the results in the monograph [7]: we will present here parts of the theory in a rather compact form (taking advantage of rather general results in [1]) and we shall bring some of the results in a new perspective. However, our main result in Section 4 is an interesting improvement of a stability result in [7].

The strength of the above-mentioned approach depends on the *computational character of the concept of polysplines. Indeed, the study of cardinal polysplines can be reduced to the study of a sequence of one-dimensional objects.* This fact depends on the orthogonal decomposition of

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a function $f \in L^2(\mathbb{R}^n)$ via spherical harmonics where $L^2(\mathbb{R}^n)$ denotes the set of all square-integrable functions $f : \mathbb{R}^n \to \mathbb{C}$ with respect to Lebesgue measure. Let $Y_{k,l}(\theta)$, $k = 0, 1, 2, ..., l = 1, ..., a_k$, be a standard basis of the set of all spherical harmonics where k denotes the degree of the spherical harmonic (for definition see Section 5) and S^{n-1} denotes the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$. For the reader who is not familiar with spherical harmonics, it might be useful just to consider the two-dimensional case: identify S^1 with $[0, 2\pi]$ and choose $Y_0 = \frac{1}{\sqrt{2\pi}}$, and for $k \in \mathbb{N}$

$$Y_{k,1}(t) = \frac{1}{\sqrt{\pi}} \cos kt \quad \text{and} \quad Y_{k,2}(t) = \frac{1}{\sqrt{\pi}} \sin kt \tag{1}$$

which gives a standard basis of spherical harmonics. Roughly speaking, polysplines S of order p can be characterized as functions of the form (where $x = r\theta, \theta \in S^{n-1}$ and $r > 0$)

$$S(r\theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} S_{k,l}(\log|x|)\, Y_{k,l}(\theta), \tag{2}$$

with the property that for each $k \in \mathbb{N}_0$ the function $v \mapsto S_{k,l}(v)$ is a cardinal $L-$spline with respect to a certain linear differential operator with constant coefficients of order $2p$ which depends on the degree k. In order to describe wavelet analysis of polysplines, it is necessary to have a good picture of the theory of cardinal $L-$splines. For this reason we shall continue a detailed discussion of wavelet analysis of polysplines in Section 5. The reader who is already familiar with wavelet analysis of cardinal $L-$splines therefore may skip the next sections.

We begin with a discussion of wavelet analysis of cardinal $L-$splines. Important results can already be found in [1], and a detailed account is given in [7]. Further in [11] a discussion of $L-$spline wavelets with arbitrary knots is given. Let us recall the definition of a cardinal $L-$spline: let L be a linear differential operator of order $N+1$ with constant coefficients. A function $u : \mathbb{R} \to \mathbb{R}$ is called a cardinal $L-$spline on the mesh $2^{-j}\mathbb{Z}$ if u is $(N-1)-$times continuously differentiable and if for each $k, j \in \mathbb{Z}$ there exists $f_{k,j} \in U_L := \{f \in C^\infty(\mathbb{R}) : Lf(x) = 0 \text{ in } \mathbb{R}\}$ such that $u(t) = f_{k,j}(t)$ for all $t \in \left(2^{-j}k, 2^{-j}(k+1)\right)$. Throughout the paper we assume that the linear differential operator L is given by

$$L = M_\Lambda := \prod_{j=1}^{N+1} \left(\frac{d}{dx} - \lambda_j \right), \tag{3}$$

where the parameter vector $\Lambda = (\lambda_1, ..., \lambda_{N+1})$ consists of *real numbers and at least some λ_j is non-negative*. The space of cardinal $L-$splines on the mesh $2^{-j}\mathbb{Z}$ is denoted by $\mathcal{S}_{2^{-j}\mathbb{Z}}(\Lambda)$.

To a large extent wavelet analysis of cardinal $L-$splines can be developed analogously to the polynomial case which corresponds to $\Lambda = (0, ..., 0)$, or equivalently, $L = \left(\frac{d}{dx}\right)^{N+1}$. Scaling spaces are defined by

$$V_j(\Lambda) = L^2(\mathbb{R})\text{-closure } \left(L^2(\mathbb{R}) \cap \mathcal{S}_{2^{-j}\mathbb{Z}}(\Lambda)\right). \tag{4}$$

It is clear that the spaces $V_j(\Lambda)$ are $2^{-j}\mathbb{Z}$-invariant, and obviously

(i) $V_j(\Lambda) \subset V_{j+1}(\Lambda)$ for all $j \in \mathbb{Z}$.

Theorem 4.3 in [1] in combination with (5) and (6) shows that

(ii) $\bigcup_{j\in\mathbb{Z}} V_j(\Lambda)$ is dense in $L^2(\mathbb{R})$.

Finally it can be proved that

(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ provided that there exists $\lambda_j \geq 0$ in Λ, cf. [1] and [8].

The wavelet spaces are defined by $W_j(\Lambda) := V_{j+1}(\Lambda) \ominus V_j(\Lambda)$. Property (ii) and (iii) show that $L^2(\mathbb{R})$ is equal to the orthogonal sum of the wavelet spaces $W_j(\Lambda)$, $j \in \mathbb{Z}$. Hence, according to [1], the sequence $(V_j(\Lambda))_{j\in\mathbb{Z}}$ is a multiresolution provided some λ_j is non-negative.

An important tool in $L-$spline analysis is the existence of a basic spline which will denoted by Q_Λ: it has the property that $Q_\Lambda(x) > 0$ for all $x \in (0, N+1)$ and $Q_\Lambda(x) = 0$ for all $x \in \mathbb{R} \setminus (0, N+1)$, and it is uniquely determined up to a positive constant through this property, see [13]. The Fourier transform of Q_Λ is given by

$$\widehat{Q_\Lambda}(\xi) = \frac{\prod_{j=1}^{N+1}\left(e^{-\lambda_j} - e^{-i\xi}\right)}{\prod_{j=1}^{N+1}(i\xi - \lambda_j)}. \tag{5}$$

Note that in [1] the basic spline is denoted by N_Λ and it differs by a constant, namely $Q_\Lambda = e^{-c_\Lambda} N_\Lambda$, where $c_\Lambda = \lambda_1 + \cdots + \lambda_{N+1}$.

It is well known that the space $V_0(\Lambda)$ is the $L^2(\mathbb{R})$-closure of the linear hull of the translates $Q_\Lambda(\cdot - l)$, $l \in \mathbb{Z}$. More generally, we have that

$$V_j(\Lambda) = L^2(\mathbb{R})\text{-closure of linear span}\left(Q_{2^{-j}\Lambda}\left(2^j x - k\right), \quad k \in \mathbb{Z}\right). \tag{6}$$

It is important to note that V_j is not the 2^j-dilate of V_0. Indeed, the function $Q_\Lambda(2x)$ is *not* in V_1 since in general it is not a $L-$spline on $\frac{1}{2}\mathbb{Z}$ with respect to Λ. Thus, we are in the case of non-stationary multiresolution analysis where the scaling function φ_j is not just the 2^j-dilate of φ_0. As already noted in [1], the discussion of stability (for definition see Section 4) in the non-stationary case might be much harder.

The paper is organized as follows. In Section 2 we use the fundamental results in [1, p. 152] to describe wavelets and Riesz bounds of cardinal

$L-$splines. It was proved in [1] that the wavelet space $W_j(\Lambda)$ is generated by $2^{-j}\mathbb{Z}$-shifts of a single function $\psi_{j,0}$. Further the system $\psi_{j,l}, l \in \mathbb{Z}$, (cf. formula (17)) is stable: Roughly speaking, the Riesz bounds for the mother and father wavelet can be expressed by the bounds of the function

$$S_{Q_\Lambda}(\xi) := \sum_{j=-\infty}^{\infty} \left| \widehat{Q_\Lambda}(\xi + 2\pi j) \right|^2 . \tag{7}$$

In Section 3 the Riesz bounds are described by the *Euler-Frobenius polynomial* $\Pi_{\widetilde{\Lambda}}\left(0, e^{i\xi}\right)$ (see Section 3) of the symmetrized vector

$$\widetilde{\Lambda} := (\lambda_1, ..., \lambda_{N+1}, -\lambda_1, ..., -\lambda_{N+1}) =: (\Lambda, -\Lambda). \tag{8}$$

As a consequence the following *constructive bounds* for S_{Q_Λ} are exact: using (25) and (21) (the latter applied to the symmetrized vector $\tilde{\Lambda}$) we obtain for all $\xi \in \mathbb{R}$

$$e^{c_\Lambda} \left| \sum_{k=0}^{2N+1} (-1)^{N-k} Q_{\tilde{\Lambda}}(k) \right| \leq \ |S_{Q_\Lambda}(\xi)| \ \leq e^{c_\Lambda} \sum_{k=0}^{2N+1} Q_{\tilde{\Lambda}}(k) , \tag{9}$$

where $c_\lambda := \lambda_1 + \cdots + \lambda_{N+1}$.

It follows from (25) and a well-known criterion for stability (Theorem 3.24 in [3, p. 76]) that the family $Q_\Lambda(\cdot - l), l \in \mathbb{Z}$, is stable, and the Riesz bounds are described in (9). Similarly the stability of the wavelet system $\psi_j(x-l), l \in \mathbb{Z}$, for fixed level $j \in \mathbb{Z}$ can be proved as done in [1]. However, as already pointed out in [1], this does not imply that the full system $\psi_{j,l} := N_j \psi_j\left(2^j \cdot - l\right), \ j, l \in \mathbb{Z}$, provides a stable basis for $L^2(\mathbb{R})$ for suitable constants N_j. Our main result, given in Section 4, says that N_j can be chosen in such a way that $N_j \psi_{j,l}, \ j, l \in \mathbb{Z}$, is indeed a stable basis. The proof requires a deeper analysis of the Euler-Frobenius polynomial for cardinal $L-$splines. In Section 5 a description of wavelet analysis for polysplines is given.

§2. Basic Constructions

Let φ be a function in $L^2(\mathbb{R})$ and let $\widehat{\varphi}(\xi)$ be the Fourier transform of φ. Then the function

$$S_\varphi(\xi) := \sum_{j=-\infty}^{\infty} |\widehat{\varphi}(\xi + 2\pi j)|^2 \tag{10}$$

is of fundamental importance in wavelet analysis. In the notation of [1], we have $S_\varphi = [\widehat{\varphi}, \widehat{\varphi}]$, where the bracket product of two functions $f, g \in L^2(\mathbb{R})$ is defined by $[f, g] := \sum_{j=-\infty}^{\infty} f(\cdot + 2\pi j) \overline{g(\cdot + 2\pi j)}$.

Let us recall the construction of wavelets in [1, Theorem 5.5]: Let φ and η be functions in $L^2(\mathbb{R})$ such that the support of $\widehat{\varphi}$ and $\widehat{\eta}$ is equal to $\mathbb{R}$. Assume that V_0 is generated by integer shifts of φ and V_1 is generated by $\frac{1}{2}\mathbb{Z}$-shifts of the function η. Let A be a 4π-periodic function such that $\widehat{\varphi} = A\widehat{\eta}$. Define a function $\psi \in L^2(\mathbb{R})$ by its Fourier transform

$$\widehat{\psi}(\xi) = 2e^{-\frac{1}{2}i\xi}\overline{A(\xi+2\pi)}\left(\widetilde{\widetilde{\eta}}(\xi+2\pi)\right)^2\widehat{\eta}(\xi), \tag{11}$$

where the function $\widetilde{\widetilde{\eta}}$ is defined by $\widetilde{\widetilde{\eta}}(x) = \left(\sum_{k=-\infty}^{\infty}|\widehat{\eta}(x+4\pi k)|^2\right)^{\frac{1}{2}}$. Then the wavelet space $W_0 = V_1 \ominus V_0$ is the L^2-closure of the linear span of the translates $\psi(\cdot - l), l \in \mathbb{Z}$. The Riesz bounds of this family can be computed by the bounds of the function S_ψ for which the following identity holds:

$$S_\psi(\xi) = 4\left(\widetilde{\widetilde{\eta}}(\xi+2\pi)\right)^2\left(\widetilde{\widetilde{\eta}}(\xi)\right)^2 S_\varphi(\xi). \tag{12}$$

Let us apply this construction to cardinal $L-$splines. Let Q_Λ be the basic spline. Let us define $\varphi(x) = Q_\Lambda(x)$ and $\eta(x) = Q_{\frac{1}{2}\Lambda}(2x)$. As in [1, p. 151] define a 4π-trigonometric polynomial by

$$A_\Lambda(\xi) = 2^{-N}\prod_{j=1}^{N+1}\left(e^{\lambda_j - i\frac{\xi}{2}} + 1\right). \tag{13}$$

A short computation (using (5) and (13) and $\widehat{\eta}(\xi) = \frac{1}{2}\widehat{Q_{\frac{1}{2}\Lambda}}\left(\frac{\xi}{2}\right)$) shows that

$$\widehat{\varphi}(\xi) = \widehat{Q_\Lambda}(\xi) = A_{\frac{1}{2}\Lambda}(\xi)\,e^{-\frac{1}{2}(\lambda_1+\ldots\lambda_{N+1})}\widehat{\eta}(\xi). \tag{14}$$

Therefore we define $A(\xi) := A_{\frac{1}{2}\Lambda}(\xi)\,e^{-\frac{1}{2}(\lambda_1+\ldots\lambda_{N+1})}$. Moreover, we have

$$\left(\widetilde{\widetilde{\eta}}(\xi+2\pi)\right)^2 = \sum_{k=-\infty}^{\infty}\frac{1}{4}\left|\widehat{Q_{\frac{1}{2}\Lambda}}\left(\frac{\xi}{2}+\pi+2\pi k\right)\right|^2 = \frac{1}{4}S_{Q_{\frac{1}{2}\Lambda}}\left(\frac{\xi}{2}+\pi\right).$$

Hence we *define for given* Λ *a wavelet* $\psi_\Lambda \in W_0(\Lambda)$ through the formula

$$\widehat{\psi_\Lambda}(\xi) = \frac{1}{4}e^{-\frac{1}{2}(\lambda_1+\ldots+\lambda_{N+1})}e^{-\frac{1}{2}i\xi}\overline{A_{\frac{1}{2}\Lambda}(\xi+2\pi)}S_{Q_{\frac{1}{2}\Lambda}}\left(\frac{\xi}{2}+\pi\right)\widehat{Q_{\frac{1}{2}\Lambda}}\left(\frac{\xi}{2}\right). \tag{15}$$

Moreover, (12) shows that

$$S_{\psi_\Lambda}(\xi) = \frac{1}{4}S_{Q_{\frac{1}{2}\Lambda}}\left(\frac{\xi}{2}+\pi\right)S_{Q_{\frac{1}{2}\Lambda}}\left(\frac{\xi}{2}\right)S_{Q_\Lambda}(\xi). \tag{16}$$

Let us now consider the case of arbitrary scaling.

Proposition 1. *Let Λ be fixed, $j \in \mathbb{Z}$, and let $\psi_{2^{-j}\Lambda}$ be defined by (15). Then $W_j(\Lambda)$ is the closure of the linear hull of*

$$\psi_{j,l}(x) := \psi_{2^{-j}\Lambda}\left(2^j x - l\right) \quad \text{for } l \in \mathbb{Z}. \tag{17}$$

Proof: Clearly $\psi_{2^{-j}\Lambda}$ is in $W_0\left(2^{-j}\Lambda\right) = V_1\left(2^{-j}\Lambda\right) \ominus V_0\left(2^{-j}\Lambda\right)$: thus it is a cardinal $L-$spline with respect to the differential operator associated to $2^{-j}\Lambda$ defined on the mesh $\mathbb{Z}$, and the orthogonality relations

$$\int \psi_{2^{-j}\Lambda}(y)\overline{r(y)}dy = 0 \tag{18}$$

hold for all $r \in V_0\left(2^{-j}\Lambda\right)$. Define now $\psi_j(x) := \psi_{2^{-j}\Lambda}\left(2^j x\right)$. Then ψ_j is a cardinal $L-$spline with respect to Λ defined on the mesh $2^{-j}\mathbb{Z}$, hence $\psi_j \in V_{j+1}(\Lambda)$. By making the substitution $y = 2^j x$ in (18) we see that $\psi_j \in W_j(\Lambda) = V_{j+1}(\Lambda) \ominus V_j(\Lambda)$. □

§3. Riesz Bounds and the Euler-Frobenius Polynomial

In Section 2 we have seen that the Riesz bounds of the translates of the mother or father wavelet can be characterized by the bounds of the function S_{Q_Λ}. We now want to relate S_{Q_Λ} to the Euler-Frobenius polynomial for cardinal $L-$splines. For this reason let us recall some basic facts and definitions: Define $e^\Lambda = \left\{e^{\lambda_j} : j = 1, ..., N+1\right\}$. The function $A_\Lambda : \mathbb{R} \times \left(\mathbb{C} \setminus e^\Lambda\right) \to \mathbb{C}$ (cf. [13], p. 223) is defined by

$$A_\Lambda(x, \lambda) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{q_\Lambda(z)} \frac{e^{xz}}{e^z - \lambda} dz, \text{ where } q_\Lambda(z) = \prod_{j=1}^{N+1}(z - \lambda_j). \tag{19}$$

Here Γ is a closed simple curve in the complex plane surrounding all λ_j, $j = 1, ..., N+1$, and having the zeros of the function $e^z - \lambda$ in the exterior of Γ. The Euler-Frobenius function is defined by

$$\Pi_\Lambda(x, \lambda) := A_\Lambda(x, \lambda) \cdot r_\Lambda(\lambda), \text{ where } r_\Lambda(\lambda) = \prod_{j=1}^{N+1}\left(e^{\lambda_j} - \lambda\right). \tag{20}$$

The following fundamental formula relates the Euler-Frobenius function with the basic spline (cf. [13, p. 221 and p. 222]):

$$\sum_{j=0}^{N} \lambda^{N-j} Q_\Lambda(x + j) = \frac{(-1)^N}{e^{(\lambda_1 + ... + \lambda_{N+1})}} \cdot \Pi_\Lambda(x, \lambda). \tag{21}$$

It follows that the function $\Pi_\Lambda(x,\lambda)$ is a polynomial of degree $\leq N$ in the variable λ. For $x=0$ it is called the Euler-Frobenius polynomial which is of degree $\leq N-1$ since $Q_\Lambda(0)=0$. Next we consider

$$\Phi_\Lambda(x,\lambda) := \sum_{j\in\mathbb{Z}} \lambda^j Q_\Lambda(x-j). \tag{22}$$

For fixed λ the function $x \to \Phi_\Lambda(x,\lambda)$ is a cardinal $L-$spline and it satisfies the "exponential equation"

$$\Phi_\Lambda(x+1,\lambda) = \lambda\Phi_\Lambda(x,\lambda), \tag{23}$$

which reminds of the exponential equation $\lambda^{x+1}=\lambda\lambda^x$. Since Q_Λ has support in $[0,N+1]$, we have for $0\leq x<1$

$$\sum_{j=0}^{N} \lambda^{N-j} Q_\Lambda(x+j) = \lambda^N \sum_{j=0}^{N} \lambda^{-j} Q_\Lambda(x+j) = \lambda^N \sum_{j=-\infty}^{\infty} \lambda^{-j} Q_\Lambda(x+j).$$

This formula and (21) yields the following identity valid for all $0\leq x<1$ and for all λ:

$$\Phi_\Lambda(x,\lambda) = \frac{(-1)^N}{e^{(\lambda_1+\cdots+\lambda_{N+1})}\lambda^N} \cdot \Pi_\Lambda(x,\lambda). \tag{24}$$

Theorem 1. *Suppose that $\Lambda\in\mathbb{R}^{N+1}$. Then the following identity holds for all $\xi\in\mathbb{R}$:*

$$S_{Q_\Lambda}(\xi) = (-1)\, e^{-(\lambda_1+\ldots+\lambda_{N+1})} e^{-Ni\xi} \cdot \Pi_{\widetilde{\Lambda}}\left(0, e^{i\xi}\right) \neq 0, \tag{25}$$

where $\Pi_{\widetilde{\Lambda}}(0,\lambda)$ is the Euler–Frobenius polynomial with respect to the symmetrized vector $\widetilde{\Lambda}=(\Lambda,-\Lambda)$. Furthermore we have for all $\xi\in\mathbb{R}$

$$|S_{Q_\Lambda}(-\pi)| \leq |S_\Lambda(\xi)| \leq |S_{Q_\Lambda}(0)|. \tag{26}$$

Proof: Note that the Fourier transform of Q_Λ can be estimated by $\left|\widehat{Q_\Lambda}(\xi)\right| \leq C\frac{1}{|\xi|^{N+1}}$. By Theorem 2.28 in [3], we conclude that (note that Q_Λ is real-valued)

$$S_{Q_\Lambda}(\xi) = \sum_{j=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} Q_\Lambda(y+j)\, Q_\Lambda(y)\, dy \right\} e^{-ij\xi}. \tag{27}$$

It is easy to see that $S_{Q_\Lambda}(-\xi) = S_{Q_\Lambda}(\xi)$, since

$$\int_{-\infty}^{\infty} Q_\Lambda(y-j)\, Q_\Lambda(y)\, dy = \int_{-\infty}^{\infty} Q_\Lambda(x)\, Q_\Lambda(x+j)\, dy.$$

An application of the Hermite-Genocchi formula (cf. Proposition 3 in the appendix) shows that the coefficients of the above Fourier series can be computed by

$$\int_{-\infty}^{\infty} Q_{\Lambda}(y+j)\, Q_{\Lambda}(y)\, dy = e^{-(\lambda_1+\cdots+\lambda_{N+1})} Q_{(\Lambda,-\Lambda)}(N+1+j).$$

This fact and (27) and definition (22) imply that

$$S_{Q_{\Lambda}}(\xi) = e^{-(\lambda_1+\cdots+\lambda_{N+1})} \Phi_{\widetilde{\Lambda}}\left(N+1, e^{-i\xi}\right). \tag{28}$$

By (23) we have $\Phi_{\widetilde{\Lambda}}(N+1,\lambda) = \lambda^{N+1}\Phi_{\widetilde{\Lambda}}(0,\lambda)$. Hence (24) applied to the symmetrized vector $\widetilde{\Lambda}$ (which has length $2N+2$) shows that

$$\Phi_{\widetilde{\Lambda}}(N+1,\lambda) = \lambda^{N+1}\frac{(-1)^{2N+1}}{\lambda^{2N+1}} \cdot \Pi_{\widetilde{\Lambda}}(0,\lambda). \tag{29}$$

Hence the identity (21) is proven if we take $\lambda = e^{-ix}$ recalling that $S_{Q_{\Lambda}}(-\xi) = S_{Q_{\Lambda}}(\xi)$.

It is well known that the polynomial $\lambda \longmapsto \Pi_{\widetilde{\Lambda}}(0,\lambda)$ has $2N$ zeros (since $\widetilde{\Lambda}$ has length $2N+2$) which are simple, real and negative, cf. [13]. If v is a zero then, by the symmetry of $\widetilde{\Lambda}$, v^{-1} is a zero as well. From this it follows that $\lambda=-1$ is not a zero of $\Pi_{\widetilde{\Lambda}}(0,\lambda)$ since otherwise $\lambda=-1$ is not a simple zero. It follows that $\Pi_{\widetilde{\Lambda}}\left(0,e^{i\xi}\right) \neq 0$ for all $\xi \in \mathbb{R}$.

By the above we can write

$$\Pi_{\widetilde{\Lambda}}(0,\lambda) = D\prod_{j=1}^{N}(\lambda - v_j)\left(\lambda - v_j^{-1}\right) \tag{30}$$

for a suitable constant D. A simple computation yields

$$\left|\Pi_{\widetilde{\Lambda}}\left(0,e^{i\xi}\right)\right| = D\prod_{j=1}^{N}\frac{\left|e^{i\xi}-v_j\right|^2}{|v_j|} = D\prod_{j=1}^{N}\frac{1-2v_j\cos\xi + v_j^2}{|v_j|}.$$

Since $v_j < 0$ we obtain that $\left|\Pi_{\widetilde{\Lambda}}\left(0,e^{i\xi}\right)\right| \leq \left|\Pi_{\widetilde{\Lambda}}(0,1)\right|$ and similarly we have $\left|\Pi_{\widetilde{\Lambda}}\left(0,e^{i\xi}\right)\right| \geq \left|\Pi_{\widetilde{\Lambda}}(0,-1)\right|$ for all $\xi \in \mathbb{R}$. □

§4. Proof of the Main Result

Let I be a countable (index) set. Then $l^2(I)$ is the Hilbert space of all sequences $c = \{c_j\}_{j\in I}$ for which the l^2-norm $\|c\|_{\ell_2(I)} = \left(\sum_{j\in I} |c_j|^2\right)^{1/2}$ is finite. Recall that a family of functions $f_j \in L^2(\mathbb{R})$, $j \in I$, is stable or satisfies the Riesz condition if there exist two constants $0 < A \le B < \infty$ such that

$$A\|c\|^2_{\ell_2(I)} \le \left\| \sum_{j=-\infty}^{\infty} c_j f_j(x) \right\|^2_{L_2(\mathbb{R})} \le B\|c\|^2_{\ell_2(I)}. \tag{31}$$

The optimal bounds A, B in (31) are called Riesz bounds. Now we formulate our main result.

Theorem 2. *Assume that $\Lambda \in \mathbb{R}^{N+1}$. Then there exist positive constants N_j such that the system of functions $N_j\psi_{2^{-j}\Lambda}\left(2^j x - l\right)$ $j, l \in \mathbb{Z}$, is stable.*

Proof: Define $f_{j,l}(x) := \psi_{2^{-j}\Lambda}\left(2^j x - l\right)$. Let N_j be numbers which will be specified later, and let $(c_{j,l})_{j,l\in\mathbb{Z}}$ be in $l^2(\mathbb{Z}\times\mathbb{Z})$. Then by the orthogonality of the different levels j

$$M := \left\| \sum_{j,l\in\mathbb{Z}} c_{j,l} N_j f_{j,l} \right\|^2 = \sum_{j\in\mathbb{Z}} N_j^2 \left\| \sum_{l\in\mathbb{Z}} c_{j,l} f_{j,l} \right\|^2. \tag{32}$$

Now let $0 < A_j \le B_j$ be Riesz bounds of the system of functions $f_{j,l}, l \in \mathbb{Z}$. It is not difficult to see that the system $N_j f_{j,l}$, $j, l \in \mathbb{Z}$, is stable if and only if B_j/A_j, $j \in \mathbb{Z}$, is bounded. In that case N_j can be choosen as $N_j = \frac{1}{\sqrt{A_j}}$.

Let us now consider the Riesz bounds $A_j \le B_j$ of the system $f_{j,l}(x) = \psi_{2^{-j}\Lambda}\left(2^j x - l\right), l \in \mathbb{Z}$. By a transformation of variables we see that $A_j = 2^{-j}\widetilde{A_j}$ where $\widetilde{A_j}$ is the lower Riesz bound of the system $\psi_{2^{-j}\Lambda}(\cdot - l), l \in \mathbb{Z}$. It is well known that a family $\varphi(x-l)$, $l \in \mathbb{Z}$, is stable (see Theorem 3.24 in [3, p. 76]) with bounds $A \le B$ in (31) if and only if $A \le |S_\varphi(\xi)| \le B$ for almost every $\xi \in \mathbb{R}$. The function $S_{\psi_{2^{-j}\Lambda}}$ can be estimated according to (16) and Theorem 1 by

$$\begin{aligned}\left(S_{Q_{\frac{1}{2}2^{-j}\Lambda}}(-\pi)\right)^2 S_{Q_{2^{-j}\Lambda}}(-\pi) &\le 4|S_{\psi_{2^{-j}\Lambda}}(\xi)| \\ &\le \left(S_{Q_{\frac{1}{2}2^{-j}\Lambda}}(0)\right)^2 S_{Q_{2^{-j}\Lambda}}(0).\end{aligned}$$

Hence it suffices to show that

$$\frac{S_{Q_{2^{-j}\Lambda}}(0)}{S_{Q_{2^{-j}\Lambda}}(-\pi)} \tag{33}$$

is bounded for $j \in \mathbb{Z}$. Let us put $w_j = 2^j$ for $j \in \mathbb{Z}$. Theorem 1 shows that

$$\left|S_{Q_{w_j\Lambda}}(\xi)\right| = e^{-w_j(\lambda_1+\cdots+\lambda_{N+1})}\left|\Pi_{\widetilde{w_j\Lambda}}\left(0, e^{i\xi}\right)\right|. \tag{34}$$

Note that $\widetilde{w_j\Lambda} = w_j\widetilde{\Lambda}$. In the first case we assume that $j \to -\infty$, hence $w_j \to 0$. It follows that $w_j\widetilde{\Lambda}$ converges to the zero vector 0. Lemma 1 (see below, applied to the symmetrized vector $\widetilde{\Lambda}$) shows that $\Pi_{w_j\widetilde{\Lambda}}(0,\lambda)$ converges to $\Pi_{0\widetilde{\Lambda}}(0,\lambda)$ for $w_j \to 0$. Theorem 1 tells us that $\Pi_{0\widetilde{\Lambda}}(0,\pm 1) \neq 0$, hence the quotient $\Pi_{w_j\widetilde{\Lambda}}(0,1)/\Pi_{w_j\widetilde{\Lambda}}(0,-1)$ is bounded for $w_j \to 0$.

In the second case we assume that $j \to \infty$, hence $w_j \to \infty$. Recall that $\Pi_{w\widetilde{\Lambda}}(0,\lambda) = r_{w\widetilde{\Lambda}}(\lambda)\, A_{w\widetilde{\Lambda}}(0,\lambda)$, cf. (20). By (34) and (33) it suffices to show that

$$\frac{A_{w\widetilde{\Lambda}}(0,1)}{A_{w\widetilde{\Lambda}}(0,-1)} \tag{35}$$

is bounded for all $w \geq 1$, since clearly

$$\frac{r_{w\widetilde{\Lambda}}(1)}{r_{w\widetilde{\Lambda}}(-1)} = \prod_{k=1}^{2N+2}\frac{\left(e^{w_j\lambda_k}-1\right)}{\left(e^{w_j\lambda_k}+1\right)}$$

is bounded: each factor is of the type $(x-1)/(x+1)$ with $x \in (0,\infty)$.

In the first subcase we assume that 0 does not occur in Λ. Then Lemma 2 applied to $\widetilde{\Lambda}$ shows that for $w \to \infty$

$$w^{2N+1}A_{w\widetilde{\Lambda}}(0,\lambda) \to \frac{-1}{\lambda}\frac{1}{2\pi i}\int_{\Gamma_1}\frac{1}{q_{\widetilde{\Lambda}}(z)}dz,$$

and Lemma 3 shows that the last number is non-zero. Hence the quotient (35) is bounded for $w \to \infty$.

In the second subcase, assume that the multiplicity of 0 is strictly positive. Then the symmetrized vector has multiplicity $m > 1$. Then Lemma 2 shows that $w^{2N+1-(m-1)}A_{w\widetilde{\Lambda}}(0,\lambda)$ converges to the constant $D_{\widetilde{\Lambda}}(\lambda)$ defined in (45), and it is easy to see that this constant is non-zero for $\lambda = -1$. Hence we have proved that the quotient (35) is bounded for all $j \in \mathbb{Z}$ and the theorem is proved. □

Lemma 1. *Suppose that $\Lambda_j \in \mathbb{R}^{N+1}$, $j \in \mathbb{N}$, converges to Λ in $\mathbb{R}^{N+1}$. Then $\Pi_{\Lambda_j}(0,\lambda)$ converges to $\Pi_\Lambda(0,\lambda)$ for $j \to \infty$ uniformly on compact sets.*

Proof: Recall that $\Pi_{\Lambda_j}(0,\lambda) = r_{\Lambda_j}(\lambda)\, A_{\Lambda_j}(0,\lambda)$ for all $\lambda \notin e^{\Lambda_j}$. Clearly $r_{\Lambda_j}(\lambda)$ converges to $r_\Lambda(\lambda)$ uniformly on compact sets. Now let $\lambda \neq e^{\lambda_j}$, where $\Lambda = (\lambda_1, \ldots, \lambda_{N+1})$, and let K be a compact neighborhood of λ

such that $K \cap e^{\Lambda}$ is empty. Then for large j we know that $K \cap e^{\Lambda_j}$ is empty, and we can consider

$$A_{\Lambda_j}(0,\lambda) = \frac{1}{2\pi i}\int_{\Gamma} \frac{1}{q_{\Lambda_j}(z)} \frac{1}{e^z - \lambda} dz \qquad (36)$$

for all $\lambda \in K$, where Γ is a cycle surrounding all λ's in Λ_j for large j and avoiding all zeros of $z \longmapsto e^z - \lambda$. Since $q_{\Lambda_j}(z)$ converges to $q_{\Lambda}(z)$ on Γ we conclude that $A_{\Lambda_j}(0,\lambda)$ converges to $A_{\Lambda}(0,\lambda)$ on K. Hence $\Pi_{\Lambda_j}(0,\lambda)$ converges to $\Pi_{\Lambda}(0,\lambda)$ on K which is disjoint to e^{Λ}. Since $\Pi_{\Lambda_j}(0,\lambda)$ and $\Pi_{\Lambda}(0,\lambda)$ are polynomials, the maximum principle shows that the convergence holds for all points $\lambda \in \mathbb{C}$. The proof is complete. □

Lemma 2. *Let $\Lambda = (\lambda_1, \dots, \lambda_{N+1})$ and $w > 0$. Let Γ_1 be a path only surrounding the negative values of Λ. Then*

1) *Suppose that $\lambda_j \neq 0$ for all $j = 1, \dots, N+1$. Then for $w \to \infty$,*

$$w^N A_{w\Lambda}(0,\lambda) \to \frac{-1}{\lambda}\frac{1}{2\pi i}\int_{\Gamma_1} \frac{1}{q_{\Lambda}(z)} dz. \qquad (37)$$

2) *If Λ contains 0 with multiplicity $m > 1$ (clearly $m \leq N+1$), then for $w \to \infty$*

$$w^{N-(m-1)} A_{w\Lambda}(0,\lambda) \to D_{\Lambda}(\lambda), \qquad (38)$$

where $D_{\Lambda}(\lambda)$ is defined in (45) below.

Proof: 1) Since $q_{w\Lambda}(z) = w^{N+1} q_{\Lambda}\left(\frac{z}{w}\right)$, a short computation shows that

$$A_{w\Lambda}(0,\lambda) = \frac{1}{w^N}\frac{1}{2\pi i}\int_{\Gamma} \frac{1}{q_{\Lambda}(z)} \frac{1}{e^{wz} - \lambda} dz. \qquad (39)$$

Let us split the integral in (39) into three terms: let Γ_1 be a suitable path contained in the left half plane such that it surrounds exactly the negative λ's in Λ. Further Γ_2 should be a path in the right half plane which surrounds the positive λ's in Λ, and finally let Γ_3 be a path surrounding zero. Define

$$R_j(w,\lambda) := \frac{1}{2\pi i}\int_{\Gamma_j} \frac{1}{q_{\Lambda}(z)} \frac{1}{e^{wz} - \lambda} dz. \qquad (40)$$

Hence we have

$$w^N A_{w\Lambda}(0,\lambda) = R_1(w,\lambda) + R_2(w,\lambda) + R_3(w,\lambda). \qquad (41)$$

2) The path Γ_1 is contained in the left half plane. Hence $1/(e^{wz} - \lambda)$ converges to $-1/\lambda$ for $w \to \infty$ uniformly for all $z \in \Gamma_1$ and for $w \to \infty$. This shows that

$$R_1(w,\lambda) \to -\frac{1}{\lambda}\frac{1}{2\pi i}\int_{\Gamma_1} \frac{1}{q_{\Lambda}(z)} dz. \qquad (42)$$

The path Γ_2 is contained in the right half plane, hence $\frac{1}{e^{wz}-\lambda} \to 0$ uniformly for all $z \in \Gamma_3$ and for $w \to \infty$. Hence $R_2(w,\lambda) \to 0$.

3) If $m = 0$ then by definition $R_3(w,\lambda)$ is equal to zero and the first claim is obvious.

4) Assume now that the multiplicity $m > 0$. For $j = 3$ we make the transformation $y = wz$ and obtain

$$R_3(w,\lambda) = \frac{1}{2\pi i} \int_{\Gamma_3} \frac{1}{q_\Lambda\left(\frac{y}{w}\right)} \frac{1}{e^y - \lambda} \frac{dy}{w}. \tag{43}$$

Let us write (if $m = N+1$ then put $f_\Lambda = 1$)

$$q_\Lambda\left(\frac{y}{w}\right) = \frac{y^m}{w^m} f_\Lambda(y), \quad \text{where } f_\Lambda(y) = \prod_{\lambda_j \neq 0} \left(\frac{y}{w} - \lambda_j\right). \tag{44}$$

It follows that for $w \to \infty$

$$w^{-(m-1)} R_3(w,\lambda) \to \frac{(-1)^{N+1-m}}{\prod_{\lambda_j \neq 0} \lambda_j} \frac{1}{2\pi i} \int_{\Gamma_3} \frac{1}{y^m} \frac{1}{e^y - \lambda} dy =: D_\Lambda(\lambda). \tag{45}$$

Finally we see that $w^{N-(m-1)} A_{w\Lambda}(0,\lambda)$, which is equal to

$$w^{-(m-1)} R_1(w,\lambda) + w^{-(m-1)} R_2(w,\lambda) + w^{-(m-1)} R_3(w,\lambda),$$

converges for $w \to \infty$ to the constant $D(\lambda)$ defined in (45) provided that $m > 1$ since then ($w^{-(m-1)} \to 0$). □

Lemma 3. *Suppose $\Lambda = (\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$ is given where $N \geq 0$ and $0 \neq \lambda_j$ for all $j = 1, \ldots, N+1$. Let Γ_1 be a path only surrounding the negative values of the symmetrized vector $\widetilde{\Lambda}$. Then the following identity holds:*

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{q_{\widetilde{\Lambda}}(z)} dz = \frac{(-1)^{N+1}}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\prod_{j=1}^{N+1} \left(x^2 + \lambda_j^2\right)} dx \neq 0. \tag{46}$$

Proof: We can assume that Γ_1 consists of the path γ_R from $-iR$ to iR defined by $\gamma_R(t) = it$ and the path ρ_R defined by $\rho_R(t) = R \cdot e^{it}$ with $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, where $R > 0$ is taken to be so large such that the path contains all negative values of $\widetilde{\Lambda}$. Since $q_{\widetilde{\Lambda}}(z)$ is a polynomial of degree ≥ 2, it is clear that

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\rho_R} \frac{1}{q_{\widetilde{\Lambda}}(z)} dz = 0. \tag{47}$$

For the second integral we obtain

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma_R} \frac{1}{q_{\widetilde{\Lambda}}(z)} dz = \frac{1}{2\pi} \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{\prod_{j=1}^{N+1} (ix + \lambda_j)(ix - \lambda_j)} dx.$$

The proof is complete. □

§5. Wavelet Analysis of Polysplines

A function $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called a cardinal polyspline (on spheres) of order p if S is $(2p-2)$-times continuously differentiable and the restriction of S to each open annulus

$$A_{0,l} := \left\{ x \in \mathbb{R}^n : e^l < |x| < e^{l+1} \right\} \tag{48}$$

is a polyharmonic function of order p. Recall that a function f defined on an open set U in Euclidean space $\mathbb{R}^n$ is polyharmonic of order p if f is $2p$–times continuously differentiable and $\Delta^p f(x) = 0$ for all $x \in U$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the Laplace operator and Δ^p its $p-$th iterate.

In analogy to the case of cardinal splines (cf. the definition of the scaling spaces $V_j(\Lambda)$ as the $L^2(\mathbb{R})$-closure of $\mathcal{S}_{2^{-j}\mathbb{Z}}(\Lambda) \cap L^2(\mathbb{R})$), we define P_j to be the set of all functions $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ which are $(2p-2)$-times continuously differentiable and whose restriction to each open annulus

$$A_{j,l} := \left\{ x \in \mathbb{R}^n : e^{2^{-j} l} < |x| < e^{2^{-j}(l+1)} \right\} \tag{49}$$

is a polyharmonic function of order p. Then we define for $j \in \mathbb{Z}$

$$PV_j := L^2 - \text{closure of } P_j \cap L^2(\mathbb{R}^n). \tag{50}$$

Our goal in this section is to prove the following result.

Theorem 3. *The sequence $(PV_j)_{j\in\mathbb{Z}}$ satisfies the following conditions: (i) $PV_j \subset PV_{j+1}$ for all $j \in \mathbb{Z}$, (ii) the set $\bigcup_{j\in\mathbb{Z}} PV_j$ is dense in $L_2(\mathbb{R}^n)$ and (iii) $\bigcap_{j\in\mathbb{Z}} PV_j = \{0\}$.*

In the terminology of [1], the sequence $(PV_j)_{j\in\mathbb{Z}}$, forms a multiresolution. In this general framework it is possible to define wavelet spaces PW_j, $j \in \mathbb{Z}$ as the orthogonal complement of PV_j in PV_{j+1}, i.e. that $PW_j := PV_{j+1} \ominus PV_j$. By property (ii) and (iii) we obtain the orthogonal decomposition

$$L_2(\mathbb{R}^n) = \bigoplus_{j\in\mathbb{Z}} PW_j. \tag{51}$$

We mention that PV_j is invariant under *rotations* while in the classical approaches, based on tensor products, box splines or radial basis function methods, the scaling spaces are shift-invariant under the discrete action $2^{-j}\mathbb{Z}^n$.

In the following we need some facts about polyharmonic functions: Each $x \in \mathbb{R}^n$ will be written in spherical coordinates $x = r\theta$ with $r \geq 0$ and $\theta \in S^{n-1}$. Recall that a function $Y : S^{n-1} \to \mathbb{C}$ is a spherical harmonic of degree $k \in \mathbb{N}_0$ if there exists a homogeneous harmonic polynomial $P(x)$

of degree k such that $P(\theta) = Y(\theta)$ for all $\theta \in S^{n-1}$. The set $\mathcal{H}_k$ of all spherical harmonics of degree exactly k is a linear space of dimension

$$a_k := \dim \mathcal{H}_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}. \tag{52}$$

We denote by $Y_{k,l}$ with $l = 1, 2, \ldots, a_k$ an orthonormal basis of $\mathcal{H}_k$ with respect to the usual inner product

$$\int_{S^{n-1}} f(\theta)\overline{g(\theta)}d\theta.$$

For a detailed account we refer to [15].

Let $u : (R_1, R_2) \to \mathbb{C}$ be infinitely differentiable and $Y_k \in \mathcal{H}_k$. Then it is well known that

$$\Delta(u(r) \cdot Y_k(\theta)) = Y_k(\theta) \cdot L_{(k)} u(r), \tag{53}$$

where we have put

$$L_{(k)} = \frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} - \frac{k(k+n-2)}{r^2}. \tag{54}$$

By iteration we have $\Delta^p(u(r)\, Y_k(\theta)) = Y_k(\theta) \cdot \left[L_{(k)}\right]^p u(r)$. Let us put for convenience

$$\Lambda_+(k,p) = (k, k+2, \ldots, k+2p-2) \tag{55}$$

$$\Lambda_-(k,p) = (-k-n+2, -k-n+4, \ldots, -k-n+2p) \tag{56}$$

The following result describes the solutions of the differential operator $[L_{(k)}]^p$ explicitly:

Proposition 2. *The space of solutions of the equation $L_{(k)}^p f(r) = 0$ which are C^∞ for $r > 0$ is generated by a simple basis subdivided into the following parts:*

$$r^j \quad \textit{for all } j \textit{ in } \Lambda_+(k,p) \textit{ or } \Lambda_-(k,p),$$

$$r^j \log r \quad \textit{for all } j \textit{ in } \Lambda_+(k,p) \textit{ and } \Lambda_-(k,p).$$

It will be convenient to make a transformation of variables from r to $v = \log r$. Then a solution of the form r^j as in Proposition 2 will be transformed to e^{jv}. A solution of the form $r^j \log r$ with j in $\Lambda_+(k,p)$ and $\Lambda_-(k,p)$ is transformed to ve^{jv}. We see immediately that all solutions to the equation $L_{(k)}^p f(r) = 0$ are transformed to solutions of the equation $M_{\Lambda(k)} g(v) = 0$, where $M_{\Lambda(k)}$ is defined by (3) with respect to the vector

$$\Lambda_k := (k, k+2, \ldots, k+2(p-1), -(k+n)+2, \ldots, -(k+n)+2p).$$

The dependence on the parameter p and n will be suppressed. Now we are able to formulate the following result.

Theorem 4. *Let $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a polyspline of order p. Then the function $S_{k,l} : \mathbb{R} \to \mathbb{R}$ defined by*

$$S_{k,l}(v) := \int_{S^{n-1}} S(e^v \theta) Y_{k,l}(\theta) d\theta \tag{57}$$

is a cardinal $L-$spline with respect to the linear differential operator $M_{\Lambda(k)}$.

Proof: Since S is $(2p-2)-$times continuously differentiable in $\mathbb{R}^n \setminus \{0\}$, it is clear that $S_{k,l}$ is $(2p-2)-$times continuously differentiable in $\mathbb{R}$ with respect to the variable v. Let $j \in \mathbb{Z}$. By definition S is polyharmonic on the open annulus $A_{0,l}$. Then (see [16] or [7]) there exist infinitely differentiable functions $\psi_{k_1,l_1} : (e^j, e^{j+1}) \to \mathbb{C}$, such that

$$S(r\theta) = \sum_{k_1=0}^{\infty} \sum_{l_1=1}^{a_{k_1}} \psi_{k_1,l_1}(r) Y_{k_1,l_1}(\theta) \tag{58}$$

with respect to convergence on compact subsets of the annulus $A_{0,l}$ defined in (48) and ψ_{k_1,l_1} are solutions of the linear differential equation $L^p_{(k_1)} \psi_{k_1,l_1}(r) = 0$ for $e^j < r < e^{j+1}$. Inserting (58) in (57) and interchanging integration and summation, we obtain for $v \in (j, j+1)$

$$S_{k,l}(v) = \sum_{k_1=0}^{\infty} \sum_{l_1=1}^{a_k} \psi_{k_1,l_1}(e^v) \int_{S^{n-1}} Y_{k_1,l_1}(\theta) Y_{k,l}(\theta) d\theta = \psi_{k,l}(e^v),$$

where for the last equality we have used the orthogonality relations for spherical harmonics. This shows that $\psi_{k,l}(r) = S_{k,l}(\log r)$ for all $r \in (e^j, e^{j+1})$, $j \in \mathbb{Z}$. Since $L^p_{(k)} \psi_{k,l}(r) = 0$ on the annulus, the function $S_{k,l}(v)$ is a solution of the equation $M_{\Lambda(k)}\left(\frac{d}{dv}\right) = 0$ for all $v \in (j, j+1)$ with $j \in \mathbb{Z}$. Hence $S_{k,l}$ is a cardinal $L-$spline with respect to $M_{\Lambda(k)}$. □

Recall that P_j is the set of all cardinal polysplines on the mesh defined in (49). By a straightforward modification of the proof of Theorem 4, it follows that for $S \in P_j$ the function $S_{k,l}$ defined in (57) is a cardinal $L-$spline on the mesh $2^{-j}\mathbb{Z}$ with respect to Λ_k.

We now want to characterize the $L^2(\mathbb{R}^n)$-closure of $P_j \cap L^2(\mathbb{R}^n)$ which we have denoted by PV_j. It is a temptation to assume that for $S \in PV_j$ the Fourier-Laplace coefficient defined through formula (57) will be in $V_j(\Lambda_k)$, i.e. in the closure of $\mathcal{S}_{2^{-j}\mathbb{Z}}(\Lambda_k) \cap L^2(\mathbb{R})$. This is not true since the transformation rule will give us an additional weight in the following formula which is valid for all $f \in L_2(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int_0^{\infty} \int_{S^{n-1}} |f(r\theta)|^2 r^{n-1} d\theta dr. \tag{59}$$

Fortunately, this problem can be easily remedied: let us define

$$\overline{\Lambda_k} = \left(\frac{n}{2}, \ldots, \frac{n}{2}\right) + \Lambda_k. \tag{60}$$

Theorem 5. *For each $k \in \mathbb{N}_0, l = 1, \ldots, a_k$, the map $S \longmapsto \overline{S}_{k,l}$, defined on the domain $P_j \cap L^2(\mathbb{R}^n)$ by*

$$\overline{S}_{k,l}(v) := e^{\frac{n}{2}v} \int_{S^{n-1}} S(e^v\theta) Y_{k,l}(\theta)\, d\theta, \tag{61}$$

maps into $\mathcal{S}_{2^{-j}\mathbb{Z}}\left(\overline{\Lambda_k}\right) \cap L^2(\mathbb{R}, dv)$ and is continuous with respect to the L^2-norms on $\mathbb{R}^n$ and $\mathbb{R}$, respectively. Hence there exists a continuous extension which maps

$$PV_j \to V_j\left(\overline{\Lambda_k}\right).$$

Proof: Recall that the set of spherical harmonics $Y_{k,l}(\theta)$, for $k \in \mathbb{N}_0$, $l = 1, \ldots, a_k$, is an orthonormal basis of the space $\mathcal{H}_k$ of all spherical harmonics with respect to $d\theta$. Let $S \in P_j \cap L^2(\mathbb{R}^n)$. For all $k \in \mathbb{N}_0$, and $l = 1, \ldots, a_k$ the Fourier–Laplace coefficients of the integrable function $\theta \longmapsto S(r\theta)$ are defined by

$$S_{k,l}(r) := \int_{S^{n-1}} S(r\theta) Y_{k,l}(\theta)\, d\theta. \tag{62}$$

Formula (59) and the orthonormality of the system $Y_{k,l}$ show that

$$\int_{\mathbb{R}^n} |S(x)|^2\, dx = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_0^\infty |S_{k,l}(r)|^2 r^{n-1} dr. \tag{63}$$

By the substitution $v = \log r$, we obtain the formula

$$\int_{\mathbb{R}^n} |S(x)|^2\, dx = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{-\infty}^\infty \left| e^{\frac{n}{2}} S_{k,l}(e^v)\right|^2 dv. \tag{64}$$

By the remark after Theorem 4, the function $S_{k,l}$ is in $\mathcal{S}_{2^{-j}\mathbb{Z}}(\Lambda_k)$. It is easy to see that $\overline{S}_{k,l}$ defined by $\overline{S}_{k,l}(v) = e^{\frac{n}{2}} S_{k,l}(e^v)$ is in $\mathcal{S}_{2^{-j}\mathbb{Z}}\left(\overline{\Lambda_k}\right)$, since it is clearly $2(p-1)$-times continuously differentiable and a solution of the differential operator $M_{\overline{\Lambda_k}}$ on the open interval $\left(2^{-j}l, 2^{-j}(l+1)\right)$ (we know already that $v \longmapsto S_{k,l}(e^v)$ is a solution of the differential operator M_{Λ_k}), hence it is in $\mathcal{S}_{2^{-j}\mathbb{Z}}\left(\overline{\Lambda_k}\right)$. Formula (64) shows that

$$\int_{\mathbb{R}^n} |S(x)|^2\, dx = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{-\infty}^\infty \left|\overline{S}_{k,l}(v)\right|^2 dv. \tag{65}$$

Hence $\overline{S}_{k,l}$ is square-integrable and clearly the map $S \longmapsto \overline{S}_{k,l}$ is continuous and linear. Hence it can be extended to PV_j. □

Theorem 6. *Let $V_j\left(\overline{\Lambda_k}\right), j \in \mathbb{Z}$, be the scaling spaces of the cardinal $L-$splines with respect to $M_{\overline{\Lambda_k}}$ and let $W_j\left(\overline{\Lambda_k}\right), j \in \mathbb{Z}$ be the associated wavelet spaces. Then the scaling spaces PV_j of polysplines of order p are isomorphic to*

$$V_j := \bigoplus_{k\in\mathbb{N}_0, l=1,\ldots,a_k} V_j\left(\overline{\Lambda_k}\right), \tag{66}$$

and PW_j is isomorphic to

$$W_j := \bigoplus_{k\in\mathbb{N}_0, l=1,\ldots,a_k} W_j\left(\overline{\Lambda_k}\right). \tag{67}$$

Proof: The claim is obvious from the above. □

The following characterization shows that the scaling spaces PV_j could have been introduced without using the term "polyharmonic", using only terms like spherical harmonics and cardinal $L-$splines (with respect to vectors $\Lambda_k \in \mathbb{R}^{2p}$).

Theorem 7. *Let $C_c(\mathbb{R}^n)$ be the set of all continuous functions with compact support in $\mathbb{R}^n$. For each $j \in \mathbb{Z}$ the set PV_j is equal to the $L^2(\mathbb{R}^n)$-closure of the following subspaces (where $\theta = \frac{x}{|x|}$):*

$$A_j = \left\{ \sum_{k=0}^{N} \sum_{l=1}^{a_k} S_{k,l}(\log|x|) Y_{k,l}(\theta) : N \in \mathbb{N}_0, \right.$$
$$\left. S_{k,l} \in \mathcal{S}_{2^{-j}\mathbb{Z}}(\Lambda_k) \cap C_c(\mathbb{R}) \right\},$$

$$B_j = \left\{ \sum_{k=0}^{N} \sum_{l=1}^{a_k} S_{k,l}(\log|x|) Y_{k,l}(\theta) : N \in \mathbb{N}_0, \right.$$
$$\left. S_{k,l} \in \mathcal{S}_{2^{-j}\mathbb{Z}}(\Lambda_k) \cap L^2(\mathbb{R}) \right\}.$$

Proof: The inclusion $A_j \subset B_j$ is trivial. Since each member $S \in B_j$ is clearly $2(p-1)$-times continuously differentiable, and since S is polyharmonic on annuli with radii $e^{l2^{-j}}$ and $e^{(l+1)2^{-j}}, l \in \mathbb{Z}$, it follows that B_j is contained in $P_j \cap L^2(\mathbb{R}^n)$, cf. the remarks after formula (53). Since PV_j is defined as the closure of the latter space, we conlcude that $\overline{A_j} \subset \overline{B_j} \subset PV_j$. Hence the proof is complete if we can show that each $S \in P_j \cap L^2(\mathbb{R}^n)$ can be approximated by some elements of A_j.

Let $S\in P_j\cap L^2(\mathbb{R}^n)$. Recall that $\overline{S}_{k,l}(v)=e^{\frac{n}{2}}S_{k,l}(e^v)$. By formula (65) we can find for each $\varepsilon>0$ a natural number K such that

$$I_\varepsilon := \left|\int_{\mathbb{R}^n}|S(x)|^2\,dx-\sum_{k=0}^{K}\sum_{l=1}^{a_k}\int_{-\infty}^{\infty}\left|\overline{S}_{k,l}(v)\right|^2dv\right|\leq\frac{1}{2}\varepsilon. \qquad (68)$$

Define $M=\sum_{k=0}^{K}\sum_{l=1}^{a_k}1$. By well known properties of the basic cardinal $L-$spline, there exists for each $\overline{S}_{k,l}$ an $L-$spline $T_{k,l}\in\mathcal{S}_{2^{-j}\mathbb{Z}}\left(\overline{\Lambda_k}\right)\cap C_c(\mathbb{R})$ such that

$$\left\|\overline{S}_{k,l}-T_{k,l}\right\|_{L^2(\mathbb{R})}\leq\frac{1}{2M}\varepsilon. \qquad (69)$$

Define $T_K(x):=\sum_{k=0}^{K}\sum_{l=1}^{a_k}e^{-\frac{n}{2}}T_{k,l}(\log|x|)\,Y_{k,l}(\theta)$. As before, it is easy to see that $v\mapsto e^{-\frac{n}{2}}T_{k,l}(v)$ is in $\mathcal{S}_{2^{-j}\mathbb{Z}}(\Lambda_k)$, hence T_k is in A_j. Define $S_K(x)=\sum_{k=0}^{K}\sum_{l=1}^{a_k}S_{k,l}(\log|x|)\,Y_{k,l}(\theta)$. Then

$$A:=\|S-T_K\|_{L^2(\mathbb{R}^N)}\leq\|S_K-T_K\|_{L^2(\mathbb{R}^N)}+\|S_K-T_K\|_{L^2(\mathbb{R}^N)}.$$

The transformation rule now shows that the first summand is less than or equal to $I_\varepsilon\leq\frac{1}{2}\varepsilon$. Similarly, by (69) and the definition of M, the second summand is less than or equal to $\frac{1}{2}\varepsilon$. □

Finally we are able to give

Proof of Theorem 3:. This says that the sequence of scaling spaces $(PV_j)_{j\in\mathbb{Z}}$ forms a multiresolution. The first statement $PV_j\subset PV_{j+1}$ follows directly from the definition. Assume now that $S\in PV_j$ for all $j\in\mathbb{Z}$. Then $\overline{S_{k,l}}\in V_j(\overline{\Lambda_k})$ for all $j\in\mathbb{Z}$. Since we already know that $V_j\left(\overline{\Lambda_k}\right),j\in\mathbb{Z}$, forms a multiresolution we conclude that $\overline{S_{k,l}}=0$. It follows that $S=0$. Finally we have to show that the union of all PV_j is dense in $L^2(\mathbb{R}^n)$. But it is easy to see that already the union of all subspaces $A_j,j\in\mathbb{Z}$, is dense. The proof is complete. □

Theorem 6 shows that the wavelet space PW_j of the scaling spaces PV_j can be decomposed into a direct sum of wavelet spaces $W_j(\overline{\Lambda_k})$, where $k=0,1,2,..$ and $l=1,\ldots,a_k$. The space $W_j(\overline{\Lambda_k})$ is generated by the stable system $\psi_{j,m}$, $m\in\mathbb{Z}$, as we have seen in Section 3. But it is a fact that the Riesz bounds $0<A_{j,k}\leq B_{j,k}$ depend on k and $j\in\mathbb{Z}$. An important result in [7] states that for each $j\in\mathbb{Z}$ the sequence $B_{j,k}/A_{j,k}$ is bounded (for the variable $k\in\mathbb{Z}$). This fact allows one to carry over the decomposition and reconstruction algorithm of the component spaces $W_j(\overline{\Lambda_k}),j\in\mathbb{Z}$, to the spaces PW_j for $j\in\mathbb{Z}$. For further details we refer to [7].

§6. Appendix: The Hermite-Genocchi Formula

The Hermite Genocchi formula for cardinal $L-$splines was already derived in [5]. For our purposes we need a variant of this formula, and for convenience of the reader we give a quick proof. The formula for $\widehat{Q_\Lambda}$ in (5) shows that $\widehat{Q_{\{\lambda_1\}}}(\xi)$ is equal to

$$\frac{e^{-\lambda_1}-e^{-i\xi}}{i\xi-\lambda_1}=e^{-\lambda_1}\frac{e^{\lambda_1-i\xi}-1}{\lambda_1-i\xi}=e^{-\lambda_1}\int_0^1 e^{\lambda_1 x}e^{-i\xi x}dx. \tag{70}$$

Let $\Lambda_2=(\mu_1,\ldots,\mu_{M+1})$ and set $(\Lambda_1,\Lambda_2):=(\lambda_1,\ldots,\lambda_{N+1},\mu_1,\ldots,\mu_{M+1})$. By formula (5) we obtain

$$\widehat{Q_{(\Lambda_1,\Lambda_2)}}(\xi)=\widehat{Q_{\Lambda_1}}(\xi)\cdot\widehat{Q_{\Lambda_2}}(\xi)=\widehat{Q_{\Lambda_1}*Q_{\Lambda_2}}(\xi). \tag{71}$$

The inverse Fourier transform yields $Q_{(\Lambda_1,\Lambda_2)}(x)=Q_{\Lambda_1}*Q_{\Lambda_2}(x)$. This implies the recursive formula

$$Q_{\Lambda\cup\{\mu\}}(x)=e^{-\mu}\int_0^1 Q_\Lambda(x-y)\,e^{\mu y}dy. \tag{72}$$

Theorem 8. *Let $f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ and Q_Λ be the basic spline with respect to $\Lambda=(\lambda_1,\ldots,\lambda_{N+1})$. Define $C_\Lambda=e^{-(\lambda_1+\cdots+\lambda_{N+1})}$. Then*

$$\int_{-\infty}^{\infty} f(x)Q_\Lambda(x)\,dx$$
$$=C_\Lambda\int_0^1\cdots\int_0^1 f(x_1+\cdots+x_{N+1})\prod_{j=1}^{N+1}e^{\lambda_j x_j}dx_1\cdots dx_{N+1}.$$

Proof: Recall that Q_Λ is real-valued. The Parseval identity implies that

$$I:=\int_{-\infty}^{\infty} f(x)\,Q_\Lambda(x)\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\xi)\cdot\overline{\widehat{Q_\Lambda}(\xi)}d\xi \tag{73}$$

Formula (70) and (71) show that

$$\overline{\widehat{Q_\Lambda}(\xi)}=\prod_{j=1}^{N+1}e^{-\lambda_j}\cdot\prod_{j=1}^{N+1}\int_0^1 e^{\lambda_j x}e^{i\xi x}dx. \tag{74}$$

It follows that I is equal to

$$\frac{C_\Lambda}{2\pi}\int_0^1\cdots\int_0^1\int_{-\infty}^{\infty}\widehat{f}(\xi)\cdot e^{i\xi(x_1+\cdots+x_{N+1})}\prod_{j=1}^{N+1}e^{\lambda_j x_j}d\xi dx_1\cdots dx_{N+1}.$$

The proof is accomplished by using the inverse Fourier transform formula for the function f. □

Proposition 3. *Let* $\Lambda = (\lambda_1, \ldots, \lambda_{N+1})$. *Then the following identity holds:*

$$\int_{-\infty}^{\infty} Q_\Lambda (x+j)\, Q_\Lambda (x)\, dx = e^{-(\lambda_1+\cdots+\lambda_{N+1})} Q_{(\Lambda,-\Lambda)} (N+1+j) \qquad (75)$$

Proof: Apply Theorem 8 to $f(x) = Q_\Lambda (x+j)$. Then the right-hand side of the Hermite-Genocchi formula is equal to

$$e^{-(\lambda_1+\cdots+\lambda_{N+1})} \int_{[0,1]^{N+1}} \prod_{j=1}^{N+1} e^{\lambda_j x_j} \cdot Q_\Lambda (x_1 + \cdots + x_{N+1} + j)\, dx_1 \cdots dx_{N+1}.$$

By substituting $x_j = 1 - y_j$ for $j = 1, \ldots, N+1$, we obtain the expression

$$\int_0^1 \cdots \int_0^1 \prod_{j=1}^{N+1} e^{-\lambda_j y_j} \cdot Q_\Lambda (N+1+j-y_1-\cdots-y_{N+1})\, dy_1 \cdots dy_{N+1}.$$

Apply now inductively formula (72) to the last expression. □

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