Peano Kernel
for Harmonicity Differences of Order p

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In the present paper we consider linear functionals L which vanish on polyharmonic functions of a fixed order $p \geq 1$ in a bounded domain $D \subset \mathbb{R}^d$, i. e. on the kernel of the polyharmonic operator Δ^p, where Δ denotes the Laplacian differential operator. For such functionals we prove a theorem of Peano type (see Peano [13], Davis [6], Sard [15]) which states that

$$L(f) = \int_D \mathcal{P}(x) \Delta^p f(x) dx$$

where $\mathcal{P}$ is the corresponding Peano kernel (cf. Theorem 1.1). This formula shows a full analogy with the classical one–dimensional case where (see Davis [6, p. 69])

$$L(g) = \int_a^b \mathcal{P}_0(t) g^{(p+1)}(t) dt$$

and where L is a functional vanishing on the polynomials of degree p, i.e. on the kernel of the operator $\dfrac{d^{p+1}}{dt^{p+1}}$.

Another well–known one–dimensional result is that the Peano kernel of the finite difference operator of order $p+1$ is a univariate B–spline (cf. Tchakaloff [20], Curry–Schoenberg [5], Schumaker [17]).

The main purpose of the present paper is to introduce the concept of the *harmonicity difference of order* p and to study the properties of the Peano kernel associated with it. The harmonicity difference of order p naturally arises from a mean value property of the polyharmonic functions of order p (cf. Bramble–Payne [3], Cheng [4], Picone [14]).

The harmonicity difference of order p enjoys many of the properties of the one–dimensional finite difference operator. In particular, we prove that its Peano kernel is (i) compactly supported (ii) radially symmetric with high order smoothness, and (iii) satisfying an extremal problem which resembles Holladay's theorem for one–dimensional splines, see Ahlberg–Nilson–Walsh [1, Ch. III, §3.1].

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1. Peano Type Theorem

Let $D \subset \mathbb{R}^d, d \in \mathbb{N}$, be a bounded domain, i.e. an open and connected set. We denote by $H_p(D) \subset C^{2p}(D)$ the set of all polyharmonic functions h of order $p \in \mathbb{N}$, i.e. for which

$$\Delta^p h = 0 \qquad \text{in } D,$$

where Δ^p is the p–th power of the Laplacian operator Δ with $\Delta^0 := id$.

We shall prove a multivariate Peano type theorem for functionals L vanishing on $H_p(D)$. This theorem corresponds to the classical univariate Peano theorem for functionals vanishing on polynomials (see e.g. Davis [6]).

We remark that in the multivariate case there are quite few results for Peano kernels in general domains. The difficulty for constructing such are pointed out in Shapiro [18] who proved a Peano type theorem for functionals vanishing on multivariate polynomials.

Denote by R_p the fundamental solution for the operator Δ^p in $\mathbb{R}^d$, $p \in \mathbb{N}$, which is given by (cf. Aronszajn–Creese–Lipkin [2])

$$R_p(x) = R_p(|x|) = r^{2p-d}(A_{p,d}\log(r) + B_{p,d}) \tag{1.1}$$

where $r := |x|$, and $A_{p,d}$ and $B_{p,d}$ are appropriate constants with $A_{p,d} \neq 0$ only for even d and $p \geq \frac{d}{2}$. Note that throughout this paper we use the notation $f(x) = f(|x|)$ to indicate the radial symmetry of a function f.

The function R_p has the property that

$$\Delta^k R_p = R_{p-k} \quad \text{for } 0 \leq k < p. \tag{1.2}$$

For details we refer to [2].

We consider linear functionals L on $C^{2p}(D)$ which consist of terms of the following type

$$L_{S,\alpha,\mu}(f) := \int_S D^\alpha f(x) d\mu(x), \qquad (f \in C^{2p}(D)), \tag{1.3}$$

where $S \subset D$ is a q–dimensional compact manifold ($1 \leq q \leq d$), μ is a finite Borel measure on S, and where D^α denotes the partial derivative of order $|\alpha| = \sum_{i=1}^{d} \alpha_i \leq 2p$ where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$.

Then we put

$$L(f) := \sum_{k=1}^{K} L_{S_k,\alpha_{(k)},\mu_k}(f) + \sum_{j=1}^{J} \sum_{|\alpha|<2p} c_{\alpha j} \cdot D^\alpha f(x_j), \qquad (f \in C^{2p}(D)), \tag{1.4}$$

where $x_j \in D$, $1 \leq j \leq J$, and $|\alpha_{(k)}| < 2p$, $k = 1, ..., K$.

For the following we define a constant Ω_d by

$$\Omega_d = \begin{cases} \dfrac{1}{(d-2)\sigma_d} & \text{for } d \geq 3, \\ \dfrac{1}{\sigma_d} & \text{for } d = 2, \end{cases} \tag{1.5}$$

where $\sigma_d := \dfrac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ denotes the area of the $(d-1)$–sphere $S_{d-1} := \{\xi \in \mathbb{R}^d \ : \ |\xi| = 1\}$. In addition, we define $S_{d-1}(\tau) := \{\xi \in \mathbb{R}^d \ : \ |\xi| = \tau\}$.

Now we can state the Peano theorem for linear functionals vanishing on polyharmonic functions:

Theorem 1.1. *Let the bounded domain D be such that ∂D is a Lipschitzian graph manifold (see [2, p. 8]). Let L be a linear functional of type (1.4) such that $L(g) = 0$ for $g \in H_p(D)$. Then for all $f \in C^{2p}$ in an open neighbourhood $U(\overline{D})$ of $\overline{D}$, we have the representation*

$$L(f) = \int_D \mathcal{P}(x) \Delta^p f(x) dx,$$

where $\mathcal{P}(x) := -\Omega_d K(x) := -\Omega_d L_y(R_p(x-y))$.

Here the notation L_y means that the functional L is applied with respect to the y–variable.

In order to prove Theorem 1.1, we need the following

Lemma 1.2.
(i) If $|\alpha| < 2p$, then

$$\int_D \left(\int_S D_x^\alpha R_p(x-y) d\mu(x) \right) g(y) dy \ = \ \int_S \left(D_x^\alpha \int_D R_p(x-y) g(y) dy \right) d\mu(x)$$

for every $g \in C(\overline{D})$.

(ii) If $|\alpha| < 2p$ then

$$\int_D D_x^\alpha R_p(x-y) g(y) dy \ = \ D_x^\alpha \int_D R_p(x-y) g(y) dy$$

for every $g \in C(\overline{D})$ and $x \in D$.

Proof: For the fundamental solution R_p given by (1.1) we obtain the following inequalities

$$|D^\alpha R_p(x)| \ \leq \ C_1 \cdot |x|^{2p-d-|\alpha|} \cdot (|\log|x|\,| + C_2) \tag{1.6}$$

for all $x \in \mathbb{R}^d \backslash \{0\}$ (cf. [2, p. 16]). The log term appears only in the case of even dimension d and $p \geq d/2$.

Let us put $N = 2p - d - |\alpha|$. The integrals in (i) exist if

$$I := \int_S \left(\int_D |D_x^\alpha R_p(x-y) g(y)| dy \right) d|\mu|(x) < \infty.$$

Since $g \in C(\overline{D})$, we obtain

$$\begin{aligned} I &\leq C_3 \int_S \left(\int_D |D_x^\alpha R_p(x-y)| dy \right) d|\mu|(x) \\ &\leq C_4 \int_S \int_{B_M(0)} |z|^N (|\log |z|| + C_2) dz d|\mu|(x), \end{aligned}$$

where $z = x - y$ and $M > 0$ is sufficiently large depending on the diameter of D. Passing to spherical coordinates we get

$$I \leq C_5 \int_0^M \rho^{N+d-1} (|\log \rho| + C_2) d\rho.$$

The last integral is finite if $N + d - 1 > -1$, i.e. $2p - d - |\alpha| + d > 0$, or $|\alpha| < 2p$ which holds by assumption, hence the theorem of Fubini and Lebesgue's theorem of dominated convergence yield the desired result.

A similar argument proves (ii). □

Proof of Theorem 1.1: We apply the second Green formula with respect to D (cf. [2, p.10]) to the function f which is possible since f is C^{2p} in an open neighbourhood of $\overline{D}$:

$$\Omega_d \cdot \sum_{l=0}^{p-1} \int_{\partial D} \left(\Delta^l f(x) \frac{\partial}{\partial n_x} R_{l+1}(x-y) - R_{l+1}(x-y) \frac{\partial}{\partial n_x} \Delta^l f(x) \right) d\sigma(x)$$

$$(1.7) \qquad - \Omega_d \int_D R_p(x-y) \Delta^p f(x) dx = \begin{cases} f(y) & \text{if } y \in D, \\ 0 & \text{if } y \notin \overline{D}. \end{cases}$$

Here n_x is the inner normal vector to ∂D, and σ is the area measure on ∂D.

The function

$$y \to u(y) = \Omega_d \sum_{l=0}^{p-1} \int_{\partial D} \left(\Delta^l f(x) \frac{\partial}{\partial n_x} R_{l+1}(x-y) - R_{l+1}(x-y) \frac{\partial}{\partial n_x} \Delta^l f(x) \right) d\sigma(x)$$

is in $C^\infty(D)$ and $\Delta^p u(y) = 0$, since for $x \in \partial D$ we have

$$\Delta^p_{(y)} R_{l+1}(x-y) = 0, \qquad \text{for } l+1 \leq p.$$

Here $\Delta_{(y)}$ means the Laplace operator with respect to the y–variable. (Note that R_{l+1} is the fundamental solution for Δ^{l+1}, hence $\Delta^{l+1} R_{l+1}(x-y) = 0$ since $\partial D \ni x \neq y \in D$.)

Hence from (1.7) we have for $y \in D$

$$f(y) = u(y) - \Omega_d \int_D R_p(x-y)\Delta^p f(x)dx. \tag{1.8}$$

Now let us apply L to both sides of (1.8):

$$L(f) = L(u) - \Omega_d L_y \left(\int_D R_p(x-y)\Delta^p f(x)dx \right). \tag{1.9}$$

Since $u \in H_p(D)$ we have $Lu = 0$.

The second term on the right hand side of (1.9) is a sum of repeated integrals of the type appearing in Lemma 1.2, hence it is possible to interchange the order of integration, i.e. we have from (1.9)

$$L(f) = -\Omega_d \int_D L_y\left(R_p(x-y)\right)\Delta^p f(x)dx = \int_D \mathcal{P}(x)\Delta^p f(x)dx$$

with $\mathcal{P}(x) = -\Omega_d L_y(R_p(x-y))$. $\square$

Remark. (i) For simplicity sake, further in this paper we will refer to the function K as to the Peano kernel rather than to its multiple $\mathcal{P}$, which is $\mathcal{P} = -\Omega_d \dot{K}$.

(ii) Examples of linear functionals vanishing on harmonic functions are provided by Gauss' mean value theorem. E. g., if $x_0 \in D$, then the functional

$$Lf \ = \ f(x_0) \ - \ \frac{1}{\sigma_d} \int_{\partial B_h(x_0)} f(x_0 + h\xi)d\sigma_\xi \tag{1.10}$$

vanishes for all harmonic functions in D provided that $\overline{B_h(x_0)}$ is contained in D.

In this paper we consider the harmonicity difference of order p which is a generalization of the functional (1.10) vanishing on the space of polyharmonic functions.

On the other hand, the balayage principle considered in Schulze–Wildenhain [16, p. 259] generates functionals of type (1.4) vanishing also on polyharmonic functions.

2. Harmonicity Differences of Order p

We introduce the *harmonicity difference of order* p as a natural generalization of the classical univariate finite difference of even order. For this purpose we need the following notation:

For a domain $D \subset \mathbb{R}^d$, $d \geq 2$, for $f \in C(D)$, $x \in D$ and $h > 0$ such that the open ball $B_h(x) := \{y \in \mathbb{R}^d : |x - y| < h\}$ is strictly contained in D, i. e. $\overline{B_h(x)} \subset D$, we denote by

$$\mu(f, x; h) := \frac{1}{\sigma_d} \int_{S_{d-1}} f(x + h\xi) d\sigma_\xi = \frac{1}{\sigma_d h^{d-1}} \int_{S_{d-1}(h)} f(x + y) d\sigma(y)$$

the surface mean of f over $\partial B_h(x)$. Here $d\sigma_\xi$ denotes the area element of the $(d-1)$–sphere S_{d-1} and $d\sigma(y)$ is the area element of the sphere $S_{d-1}(h)$. For $h = 0$ we have $\mu(f, x; 0) = f(x)$.

In [9] the *harmonicity difference (of order 1)* was defined by

$$\mathbb{\Delta}_h f(x) := \mu(f, x; h) - f(x) = \frac{1}{\sigma_d} \int_{S_{d-1}} (f(x + h\xi) - f(x)) \, d\sigma_\xi.$$

By the mean value property of harmonic functions it is obvious that $\mathbb{\Delta}_h f(x) = 0$ for every harmonic function f. This is also true for $d = 1$. In this case harmonic functions are polynomials of degree at most 1. Note that in the case $d = 1$, where we have $\sigma_1 = 2$ and the 0–sphere $S_0 = \{-1, 1\}$, we get

$$\mathbb{\Delta}_h f(x) = \frac{1}{2}(f(x + h) - 2f(x) + f(x - h)) = \frac{1}{2}\Delta_h^2 f(x), \tag{2.1}$$

where Δ_h^2 is the classical univariate symmetric difference of order 2.

Every (univariate) symmetric difference $\Delta_h^{2p} f(x) = (\Delta_h^2)^p f(x)$ of even order $2p$ can be expressed as

$$\begin{aligned} (\Delta_h^2)^p f(x) &= \Delta_h^{2p} f(x) = \sum_{j=0}^{2p} (-1)^j \binom{2p}{j} f(x + (p - j)h) \\ &= \sum_{j=1}^{p} (-1)^{p+j} \binom{2p}{p + j} \Delta_{jh}^2 f(x). \end{aligned} \tag{2.2}$$

Motivated by (2.1) and (2.2), we replace Δ_h^2 by $\mathbb{\Delta}_h$ and we define (with the appropriate normalization factor $\frac{2}{\binom{2p}{p}}$) the *harmonicity difference of order* p by

$$(2.3)\qquad \mathbb{\Delta}_h^p f(x) \;:=\; \frac{2}{\binom{2p}{p}} \sum_{j=1}^{p} (-1)^{p+j} \binom{2p}{p+j} \mathbb{\Delta}_{jh} f(x) \;=:\; \sum_{j=1}^{p} \lambda_j \mathbb{\Delta}_{jh} f(x).$$

Note that for $x \in D$, $h > 0$ has to be sufficiently small such that the ball $\overline{B_{ph}(x)}$ is contained in D.

Remark. According to (2.2) and (2.3) we have for radially symmetric functions $f(x) = f(|x|)$ the equation $\Delta_h^{2p} f(0) = \binom{2p}{p} \mathbb{\Delta}_h^p f(0)$.

As is well known, the univariate difference $\Delta_h^{2p} f(x)$ vanishes for polynomials of degree at most $2p-1$. Similarly, the harmonicity difference $\mathbb{\Delta}_h^p f(x)$ vanishes for polyharmonic functions of order p. This is stated in

Theorem 2.1. *Let $p \in \mathbb{N}$, $h > 0$ and $x \in D$ such that $\overline{B_{ph}(x)} \subset D$. Then for any $f \in C^{2p}(D)$ satisfying $\Delta^p f = 0$ we have*

$$\mathbb{\Delta}_h^p f(x) \;= 0.$$

Throughout this paper we shall use simultaneously the standard notations Δ for the Laplacian differential operator and Δ_h for the univariate difference operator with stepsize h, and $\mathbb{\Delta}_h$ for the harmonicity difference. There should be no confusion.

In order to prove Theorem 2.1 we shall use the *Pizzetti–Nicolescu formula* (see Nicolescu [12]). This formula provides a representation of the mean value $\mu(f, x; h)$ for a function $f \in C^{2p}(D)$, $p \in \mathbb{N}$, in terms of $\Delta^j f$, $1 \le j \le p$. To state the Pizzetti–Nicolescu formula we need the linear integral operator J defined for a given $h > 0$ by

$$J(\phi; t) \;:=\; \int_0^t \left(r - \frac{r^{d-1}}{t^{d-2}} \right) \phi(r) dr \qquad \text{for } \phi \in C[0,h],\ 0 \le t \le h$$

if $d \ge 3$. In the case $d = 2$, J is defined by

$$J(\phi; t) \;:=\; \int_0^t r \log \frac{t}{r} \phi(r) dr \qquad \text{for } \phi \in C[0,h],\ 0 \le t \le h.$$

The powers of J are defined as usual by $J^1 := J$, and $J^{k+1} := J(J^k)$ for $k \ge 1$. We mention that, when $\phi = 1$ is the constant function, then

$$J^p(1; t) := \begin{cases} a_{dp}(d-2)^p t^{2p} & \text{for } d \ge 3, \\ \dfrac{1}{4^p (p!)^2} t^{2p} & \text{for } d = 2, \end{cases}$$

where $p \geq 1$ and

$$a_{dp} := \frac{1}{2^p p! d(d+2)...(d+2p-2)} \qquad (p \geq 1,\ d \geq 2).$$

Note that $a_{2p} = \dfrac{1}{4^p (p!)^2}$. We put $a_{d0} = 1$.

Now we can give a proof of the Pizzetti–Nicolescu formula (see also Nicolescu [12]).

Proposition 2.2.
(i) *Suppose $f \in C^{2p}(D)$. Then for any ball $B_h(x)$ strictly contained in D, the following equation holds*

$$\mu(f,x;h) = f(x) + \sum_{j=1}^{p-1} a_{dj} h^{2j} \Delta^j f(x) + J^p(\mu(\Delta^p f, x; \cdot); h) \cdot \begin{cases} \dfrac{1}{(d-2)^p} & \text{for } d \geq 3, \\ 1 & \text{for } d = 2. \end{cases}$$

(ii) *The remainder in the Pizzetti–Nicolescu formula can be written as*

$$\begin{aligned} J^p(\mu(\Delta^p f, x; \cdot); h) &= \mu(\Delta^p f, x; \vartheta_p h) \cdot J^p(1; h) \\ &= \Delta^p f(\xi_{x,p}) \cdot \begin{cases} a_{dp} h^{2p} (d-2)^p & \text{for } d \geq 3, \\ a_{2p} h^{2p} & \text{for } d = 2, \end{cases} \end{aligned}$$

for a suitable $\xi_{x,p} \in B_h(x)$.

Proof : (i) By the Poisson formula (cf. Aronszajn–Creese–Lipkin [2, formula (2.16′)]) we have

$$\text{(2.4)} \qquad f(y) = \frac{1}{\sigma_d} \int_{\partial B_h(y)} \frac{1}{h^{d-1}} f(x) d\sigma(x) - \Omega_d \int_{B_h(y)} G(x,y) \Delta f(x) dx,$$

where G is the Green function of the ball $B_h(y)$ given by (see [2, formula (2.12)])

$$\begin{aligned} \text{(2.5)} \qquad G(x,y) &= R_1(x-y) - R_1\Big(h \cdot \frac{x-y}{|x-y|}\Big) \\ &= R_1(x-y) - R_1(h) = \begin{cases} R_1(x-y) - \dfrac{1}{h^{d-2}} & \text{for } d \geq 3, \\ -\log|x-y| + \log h & \text{for } d = 2, \end{cases} \end{aligned}$$

due to the fact that (see [2])

$$R_1(x) \;=\; \begin{cases} |x|^{2-d} & \text{for } d \geq 3, \\ -\log\ |x| & \text{for } d = 2. \end{cases}$$

The first integral in (2.4) is equal to $\mu(f, y; h)$, and the second integral becomes (after a change of variables)

(2.6)
$$\Omega_d \int_{B_h(y)} G(x, y)\Delta f(x)dx = \Omega_d \int_{S_{d-1}} \int_0^h G(y + r\xi, y)\Delta f(y + r\xi)r^{d-1}drd\sigma_\xi.$$

Since by (2.5) we see that

$$G(y + r\xi, y) = \begin{cases} \dfrac{1}{r^{d-2}} - \dfrac{1}{h^{d-2}} & \text{for } d \geq 3, \\ -\log\ r + \log\ h & \text{for } d = 2, \end{cases}$$

depends only on r. Put $G_0(r) := G(y + r\xi, y)$. Then the integral on the right hand side of (2.6) becomes

$$\Omega_d \int_0^h G_0(r) \int_{S_{d-1}} \Delta f(y + r\xi)d\sigma_\xi r^{d-1}dr = \Omega_d\sigma_d \int_0^h G_0(r)r^{d-1}\mu(\Delta f, y; r)dr.$$

By the definition of J for both cases $d \geq 3$ and $d = 2$, we see that

$$\mu(f, y; h) = f(y) + \Omega_d\sigma_d J(\mu(\Delta f, y; \cdot), h).$$

This is the assertion for $p = 1$; the case $p \geq 2$ is proved by induction. For the calculation of the constants we use the fact that

$$\Omega_d\sigma_d \;=\; \begin{cases} \dfrac{1}{d-2} & \text{for } d \geq 3, \\ 1 & \text{for } d = 2. \end{cases}$$

(ii) follows from (i) using the mean value theorem of integration. □

For the proof of Theorem 2.1 we need, in addition, the following elementary property of the binomial coefficients:

Proposition 2.3. *For $0 \leq k \leq p - 1$ we have*

(2.7)
$$\frac{2}{\binom{2p}{p}} \sum_{j=1}^{p} (-1)^{p+j} \binom{2p}{p+j} j^{2k} \;=\; (-1)^{p+1}\delta_{0k},$$

where δ_{0k} denotes the Kronecker symbol.

Proof : First we consider $k = 0$. Here we have

$$\sum_{j=1}^{p}(-1)^{p+j}\binom{2p}{p+j} \;=\; \sum_{j=1}^{p}(-1)^{p-j}\binom{2p}{p-j}$$

$$= \;\frac{1}{2}\Big(\sum_{j=0}^{2p}(-1)^{j}\binom{2p}{j} \;-\; (-1)^{p}\binom{2p}{p}\Big) \;=\; (-1)^{p+1}\frac{\binom{2p}{p}}{2},$$

i.e. (2.7).

For $k > 0$, $k \leq p - 1$, we apply (2.2) with $f(x) = x^{2k}$ and $h = 1$. Since f is a polynomial of degree less than $2p$ we have

$$0 \;=\; \Delta_1^{2p} f(0) \;=\; \sum_{j=1}^{p}(-1)^{p+j}\binom{2p}{p+j}\Delta_j^2 f(0)$$

$$= \;\sum_{j=1}^{p}(-1)^{p+j}\binom{2p}{p+j}2j^{2k},$$

which gives (2.7). □

Proof of Theorem 2.1: Consider

$$\mathbb{\Delta}_h^p f(x) \;=\; \frac{2}{\binom{2p}{p}}\sum_{j=1}^{p}(-1)^{p+j}\binom{2p}{p+j}\mathbb{\Delta}_{jh} f(x)$$

$$= \;\frac{2}{\binom{2p}{p}}\sum_{j=1}^{p}(-1)^{p+j}\binom{2p}{p+j}(\mu(f, x; jh) \;-\; f(x))$$

$$= \;\frac{2}{\binom{2p}{p}}\sum_{j=1}^{p}(-1)^{p+j}\binom{2p}{p+j}\sum_{i=1}^{p-1} a_{di}(jh)^{2i}\Delta^i f(x)$$

$$+ \;\frac{2}{\binom{2p}{p}}\sum_{j=1}^{p}(-1)^{p+j}\binom{2p}{p+j}J^p(\mu(\Delta^p f, x; \cdot); jh)\cdot\begin{cases}\dfrac{1}{(d-2)^p} & \text{for } d \geq 3,\\[2mm] 1 & \text{for } d = 2.\end{cases}$$

Here we have used Proposition 2.2 (i). The first term vanishes by Proposition 2.3, hence

$$\mathbb{\Delta}_h^p f(x) \;=\; \frac{2}{\binom{2p}{p}}\sum_{j=1}^{p}(-1)^{p+j}\binom{2p}{p+j}J^p(\mu(\Delta^p f, x; \cdot); jh)\cdot\begin{cases}\dfrac{1}{(d-2)^p} & \text{for } d \geq 3,\\[2mm] 1 & \text{for } d = 2.\end{cases}$$

Thus $\mathbb{\Delta}_h^p f(x) = 0$ for every function polyharmonic of order p. □

3. Peano Kernel for the Harmonicity Difference Operator

We will need the following

Lemma 3.1. *Let $\tau > 0$. For $p \geq 1$ the mean value integral*

$$I_p(y,\tau) := \mu(R_p, y; \tau) = \frac{1}{\sigma_d} \int\limits_{S_{d-1}} R_p(\tau\xi - y) d\sigma_\xi \;=\; \frac{1}{\sigma_d \tau^{d-1}} \int\limits_{S_{d-1}(\tau)} R_p(x-y) d\sigma(x)$$

is a spherically symmetric function of y, and has the following properties:

(i) $\Delta^p I_p(y,\tau) = 0$ *for $|y| \neq \tau$;*

(ii) $\Delta^k I_p(y,\tau) = I_{p-k}(y,\tau)$ *for every y and for $0 \leq k \leq p-1$;*

(iii) *The functions*

$$y \to \Delta^k I_p(y,\tau) \qquad (k = 0, \ldots, p-1),$$

and

$$y \to \frac{\partial}{\partial |y|} \Delta^k I_p(y,\tau) \qquad (k = 0, \ldots, p-2)$$

are continuous of the variable $y \in \mathbb{R}^d$. After putting $t := |y|$, we get

(iii′)
$$t \to \frac{\partial^k}{\partial t^k} I_p(t,\tau) \qquad (k = 0, \ldots, 2p-2)$$

is continuous for every $t \geq 0$;

(iv)
$$\Delta^{p-1} I_p(y,\tau) = I_1(y,\tau)$$

is equal to a constant for $|y| \leq \tau$, and is equal to $R_1(y)$ for $|y| \geq \tau$.

Proof : The spherical symmetry follows since $R_p(x) = R_p(|x|), x \in \mathbb{R}^d \backslash \{0\}$. Let $|y| = |y_1|$ and A be an orthogonal transformation such that $Ay_1 = y$. Then we have $d\sigma_\xi = d\sigma_\eta$, where $A\eta = \xi$, and $|\tau\xi - y| = |\tau A\eta - Ay_1| = |A(\tau\eta - y_1)| = |\tau\eta - y_1|$. Hence $I_p(y_1,\tau) := I_p(y,\tau)$.

Property (i) follows from $\Delta^p_{(y)} R_p(x-y) = 0$ for $x \neq y$. In addition, (ii) is implied by (1.2).

Property (iii) follows from the fact that the function

$$x \to D^\alpha_y R_p(x-y)$$

has a weak singularity at the points $y \in S_{d-1}(\tau)$ for every multi–index α, $|\alpha| \leq 2p-2$.

Indeed, for a small ball $B_\varepsilon(y)$, $y \in S_{d-1}(\tau)$ we have a diffeomorphism

$$F : B_\varepsilon(y) \to B_{\varepsilon_1}(0)$$

such that $F(y) = 0$ and

$$F(B_\varepsilon(y) \cap S_{d-1}(\tau)) \subset B_{\varepsilon_2}^{(d-1)}(0) \subset \mathbb{R}^{d-1} \subset \mathbb{R}^d$$

with

$$C|x-y| \le |F(x) - F(y)| \le C_1|x-y| \quad \text{for } x \in B_\varepsilon(y). \tag{3.1}$$

Note that $B_r(y)$ denotes a d–dimensional ball of radius $r > 0$ centered at y, whereas $B_r^{(d-1)}(y)$ denotes a $(d-1)$–dimensional ball.

From this we get with (1.6)

$$|D_y^\alpha R_p(x-y)| \le C_2|F(x) - F(y)|^{2p-d-|\alpha|} \cdot (|\ \log|F(x) - F(y)|\ | + C_3).$$

Indeed, by taking $\varepsilon > 0$ small enough, for an exponent $2p - d - |\alpha| \ge 0$ we have

$$|x-y|^{2p-d-|\alpha|} \le K_1 \cdot |F(x) - F(y)|^{2p-d-|\alpha|}$$

due to the left–hand side inequality in (3.1), and for an exponent $2p - d - |\alpha| < 0$ we have

$$|x-y|^{2p-d-|\alpha|} \le K_2 \cdot |F(x) - F(y)|^{2p-d-|\alpha|}$$

due to the right–hand side inequality in (3.1). We also have

$$|\ \log|x-y|\ | \le K_3 \cdot (|\ \log|F(x) - F(y)|\ | + K_4)$$

due to the right–hand side inequality in (3.1) with positive constants $K_1, ..., K_4$. By changing the variables we obtain

$$J := \left| \int_{B_\epsilon(y)\cap S_{d-1}(\tau)} D_y^\alpha R_p(x-y) d\sigma(x) \right| = \left| \tau^{d-1} \cdot \int_{B_{\frac{\epsilon}{\tau}}(\frac{y}{\tau})\cap S_{d-1}} D_y^\alpha R_p(\tau\xi - y) d\sigma_\xi \right|$$

$$\le C_4 \int_{B_{\varepsilon_2}^{(d-1)}(0)} |u|^{2p-d-|\alpha|}(|\ \log|u|\ | + C_5) du.$$

Passing to spherical coordinates we get

$$J \le C_6 \int_0^{\varepsilon_2} \rho^{2p-d-|\alpha|} \rho^{d-2}(|\ \log\, \rho\ | + C_5) d\rho < \infty, \tag{3.2}$$

for $2p - d - |\alpha| + d - 2 > -1$, i.e. for $2p - |\alpha| > 1$. Since $\Delta^k I_p \quad (k = 0, \dots, p-1)$, resp. $\dfrac{\partial}{\partial|y|}\Delta^k I_p$ for $k = 0, \dots, p-2$, are linear combinations of derivatives of

I_p (with respect to y) of order $\leq 2p-2$, it follows by (3.2) that they are continuous.

Property (iii') will follow from (iii) using the spherical symmetry and

$$\Delta_{(y)} = \frac{\partial^2}{\partial |y|^2} + \frac{(d-1)}{|y|}\frac{\partial}{\partial |y|}. \tag{3.3}$$

Indeed, this implies

$$\frac{\partial^2}{\partial |y|^2}(\Delta_{(y)}^{s-1} I_p) = \Delta_{(y)}(\Delta_{(y)}^{s-1} I_p) - \frac{(d-1)}{|y|}\frac{\partial}{\partial |y|}(\Delta_{(y)}^{s-1} I_p).$$

For $s = 1$ this yields the result for $k = 2$.
Then we proceed by induction on s. (For $k = 1$ the result follows from (iii)).

Property (iv) is a classical property of the single layer potential (see e.g. Sobolev [19, p. 212] for the case $d = 3$). Indeed, for $y \in B_\tau(0)$ we have

$$\Delta I_1(y,\tau) = 0 \quad \text{ for } y \in B_\tau(0),$$

from (i), i.e. I_1 is a harmonic function inside the ball $B_\tau(0)$. Since it is a constant for $|y| = \tau$, it follows that I_1 is equal to $I_1(\tau,\tau)$ in $B_\tau(0)$.

For every $y \notin B_\tau(0)$, the function

$$x \to R_1(x-y) \text{ is harmonic in } B_\tau(0).$$

According to the mean value theorem for harmonic functions we get

$$I_1(y,\tau) = R_1(0-y) = R_1(y). \quad \square$$

We can write the function I_p in an explicit way:

Proposition 3.2. *The integrals $I_p(y,\tau)$ satisfy (for $\tau > 0,\ p \geq 1$)*

$$I_p(y,\tau) \quad = \quad \begin{cases} \sum_{k=0}^{p-1} d_{pk}|y|^{2k} & \text{for } |y| \leq \tau, \\ \sum_{k=0}^{p-1} a_{dk}\tau^{2k} R_{p-k}(|y|) & \text{for } |y| \geq \tau, \end{cases} \tag{3.4}$$

where

$$d_{pk} = \frac{1}{\Delta^k |y|^{2k}} R_{p-k}(\tau) = \frac{\Gamma(\frac{d}{2})}{2^{2k}\Gamma(k+1)\Gamma(k+\frac{d}{2})} R_{p-k}(\tau),$$

for $k = 0,\ldots,p$, and the coefficients a_{dk} are those of Proposition 2.2.

Proof : Since $\Delta^p I_p(y,\tau) = 0$ for $|y| < \tau$ (see Lemma 3.1,(i)), and $I_p(y,\tau)$ is spherically symmetric, we shall see that $I_p(y,\tau)$ is a polynomial in $|y|^2$, i.e.

$$I_p(y,\tau) = \sum_{k=0}^{p-1} d_{pk}|y|^{2k} \quad \text{ for } |y| < \tau. \tag{3.5}$$

In order to prove (3.5), we proceed by induction: For $p = 1$, I_1 is a harmonic function by Lemma 3.1(i). Since I_1 is spherically symmetric, it is constant by the maximum principle.

Now assume that the statement is true for $p-1$. Since $\Delta I_p(y) = I_{p-1}(y)$, by induction hypothesis we have

$$\Delta I_p(y) = \sum_{k=0}^{p-2} c_k |y|^{2k}, \qquad c_k \in \mathbb{R}.$$

Since $\Delta |y|^{2l} = (2l)(2l-2+d)|y|^{2l-2}$, we get that the function

$$\phi(y) := \sum_{k=1}^{p-1} \frac{c_{k-1}}{2k(2k-2+d)} |y|^{2k},$$

satisfies $\Delta\phi = \Delta I_p$. Hence $\Delta(\phi - I_p) = 0$. Since $\phi - I_p$ is spherically symmetric, it follows that $\phi - I_p$ is equal to a constant, hence I_p is of the desired form. Due to the continuity of $I_p(y,\tau)$ the formula holds for $|y| = \tau$ as well. In order to compute the constants d_{pk} consider

$$\Delta^k \Big(\sum_{l=0}^{p-1} d_{pl} |y|^{2l}\Big)\Big|_{y=0} = d_{pk} \Delta^k(|y|^{2k}) = I_{p-k}(0,\tau)$$

$$= \frac{1}{\sigma_d} \int_{S_{d-1}} R_{p-k}(\tau\xi) d\sigma_\xi = R_{p-k}(\tau) \cdot \frac{1}{\sigma_d} \int_{S_{d-1}} d\sigma_\xi = R_{p-k}(\tau).$$

The explicit value of d_{pk} follows from [2, p. 2].

For $|y| \geq \tau$ we compute $I_p(y,\tau)$ by the Pizzetti–Nicolescu formula (Proposition 2.2). □

Lemma 3.3. *Using the notation as in Lemma* 3.1 *we get: The derivative*

$$\frac{\partial}{\partial |y|} \Delta^{p-1} I_p(y,\tau) = \frac{\partial}{\partial |y|} I_1(y,\tau) = \frac{\partial}{\partial t} I_1(t,\tau)$$

where $t = |y|$, *satisfies*

$$\frac{\partial}{\partial t} I_1(\tau+0,\tau) - \frac{\partial}{\partial t} I_1(\tau-0,\tau) = -\frac{1}{\tau},$$

for $d = 2$, *and*

$$\frac{\partial}{\partial t} I_1(\tau+0,\tau) - \frac{\partial}{\partial t} I_1(\tau-0,\tau) = (2-d)\tau^{1-d},$$

for $d \geq 3$.

Proof: In order to prove this we differentiate the formula (3.4). For $|y| \leq \tau$ we have

$$\Delta_{(y)}^{p-1} I_p(|y|, \tau) = \Delta_{(y)}^{p-1}(\sum_{k=0}^{p-1} d_{pk}|y|^{2k}) = d_{p,p-1} \cdot C,$$

hence

$$\frac{\partial}{\partial t} I_1(\tau - 0, \tau) = 0.$$

On the other hand for $|y| \geq \tau$ we get (using $\Delta^j R_{p-k} = R_{p-k-j}$ if $k+j \leq p-1$, and $\Delta^j R_{p-k} = 0$ for $k+j > p-1$):

$$\Delta_{(y)}^{p-1}(\sum_{k=0}^{p-1} a_{dk}\tau^{2k} R_{p-k}(|y|))$$

$$= a_{d0} R_1(|y|) = R_1(|y|).$$

Hence

$$\frac{\partial}{\partial t} I_1(\tau + 0, \tau) = \frac{\partial}{\partial t} R_1(t)\Big|_{t=\tau},$$

which is equal to

$$\frac{\partial}{\partial t}(-\log t)\Big|_{t=\tau} = -\frac{1}{\tau} \quad \text{ for } d = 2,$$

and to

$$\frac{\partial}{\partial t}(t^{2-d})\Big|_{t=\tau} = (2-d)\tau^{1-d} \quad \text{ for } d > 2.$$

This concludes the proof. □

Theorem 3.4. *The Peano kernel for the harmonicity difference operator (at the point $x = 0$) is equal to $\mathcal{P}_p = -\Omega_d \cdot K_p$, where*

$$K_p(y) = \mathbb{\Delta}_h^p(R_p(\cdot - y))\Big|_{x=0} = \lambda_0 R_p(y) + \sum_{j=1}^{p} \lambda_j I_p(y, jh) \tag{3.6}$$

with $\lambda_j = \frac{2}{\binom{2p}{p}}(-1)^{p+j}\binom{2p}{p+j}$, *for* $1 \leq j \leq p$, *and*

$$\lambda_0 := -\sum_{j=1}^{p} \lambda_j = (-1)^p.$$

The kernel K_p (and $\mathcal{P}_p$) is spherically symmetric and satisfies

(i) $\Delta^p K_p(y) = 0$ *for* $|y| \neq jh,\ j = 0, ..., p.$

(ii) *The functions*

$$\Delta^k K_p(y) \quad (k = 0, \ldots, p-1), \text{ and } \frac{\partial}{\partial |y|}\Delta^k K_p(y) \quad (k = 0, \ldots, p-2)$$

are continuous for $y \neq 0$*.*

(iii) *Let* $j = 1, ..., p$*. Then for* $\tau = jh$*, we have*

$$\frac{\partial}{\partial |y|} \Delta^{p-1} K_p(y)\big|_{|y|=jh+0} - \frac{\partial}{\partial |y|} \Delta^{p-1} K_p(y)\big|$$

$$= \lambda_j \begin{cases} -\dfrac{1}{\tau} & \textit{for } d = 2, \\ (2-d)\tau^{1-d} & \textit{for } d \geq 3. \end{cases}$$

Proof: The value of λ_0 comes directly from Proposition 2.3, while the spherical symmetry follows from Lemma 3.1. Properties (i) and (ii) come from Lemma 3.1, and (iii) follows by Lemma 3.3. □

4. B–spline Properties of the Peano Kernel

We will prove that for $p = 2q$, $q \in \mathbb{N}$, the Peano kernel K_{2q} corresponding to $\mathbb{\Delta}^{2q}$ is a solution to an extremal problem which is characteristic for splines, and also satisfies boundary conditions typical for B–splines.
Let us restrict ourselves to the case $4q > d$, for which K_{2q} is continuous. Choose a number $h > 0$ and put

$$\eta_j := K_{2q}(jh), \quad j = 0, \ldots, 2q.$$

Theorem 4.1. *Let $q \in \mathbb{N}$ and $h > 0$.*
(i) *The kernel K_{2q} has compact support:*

$$supp\ K_{2q} \subset \overline{B_{2qh}(0)} = \{y \in \mathbb{R}^d | \quad |y| \le 2qh\};$$

(ii) For $4q > d$, the kernel K_{2q} is a solution of the following extremal problem

$$\min \int_{B_{2qh}(0)} (\Delta^q u(x))^2 dx \tag{4.1}$$

where the functions u range in the Sobolev space $H^{2q}(B_{2qh}(0))$ (see Lions–Magenes [11]) subject to the interior interpolation conditions

$$u(x) = \eta_j \qquad |x| = jh, \quad 0 \le j \le 2q \tag{4.2}$$

and to the boundary conditions

$$\Delta^k u(x) = 0, \qquad |x| = 2qh, \quad 1 \le k \le q-1, \tag{4.3}$$

$$\frac{\partial}{\partial n}\Delta^k u(x) = 0, \qquad |x| = 2qh, \quad 0 \le k \le q-1,$$

where $\frac{\partial}{\partial n}$ is the inner normal derivative at the points of the sphere $S(0, 2qh)$.

Proof : The proof of (i) follows from the fact that for every y with $|y| > 2qh$ the function $R_{2q}(x-y)$ – as a function of x – is polyharmonic of order $2q$ in the ball $B_{2qh}(0)$. Since $K_{2q} = \mathbb{\Delta}_h^{2q} R_{2q}(\cdot - y)$, we can apply Theorem 2.1, hence $K_{2q}(x) = 0$ for $|y| > 2qh$.
Let us remark that (i) is true not only for $p = 2q$ but for all p.

From (i) and Lemma 3.1 (iii) it follows that K_{2q} satisfies the boundary conditions (4.3). We remark that (i) implies $\eta_{2q} = 0$.

Now let us prove (ii):

1. Let $g \in H^{2q}(B_{2qh}(0))$ and consider

(4.4)
$$\int_{B_{2qh}(0)} \Delta^q K_{2q}(x) \Delta^q g(x) dx.$$

Due to formula (3.6), we have $\Delta^q K_{2q} = \lambda_0 R_q(y) + \sum_{j=1}^{2q} \lambda_j I_q(y, jh)$. The first term on the right hand side is in $L^2(B_{2qh+\varepsilon}(0))$, while the $I_q(y, jh)$ are continuous in $\mathbb{R}^d$ by Lemma 3.1 (iii). Hence $\Delta^q K_{2q} \in H^0(B_{2qh+\varepsilon}(0))$. Due to the regularity theorem for elliptic operators (see Lions–Magenes [11, p. 125]) it follows that $K_{2q} \in H^{2q}_{loc}(B_{2qh+\varepsilon}(0))$. This implies $K_{2q} \in H^{2q}(B_{2qh}(0))$.
We evaluate the integral in (4.4) using the Green formula (cf. [2, (2.9)] - which is also valid for functions in the relevant Sobolev spaces, see Lions-Magenes [11, Remark 2.2 on p. 120]).

We apply the Green formula to every spherical layer $B_{(j+1)h}(0) \backslash B_{jh}(0) =: D_j, 1 \leq j \leq 2q-1$:

$$\int_{D_j} \Delta^q K_{2q}(x) \Delta^q g(x) dx$$

$$= \sum_{l=0}^{q-1} \int_{\partial D_j} \left(\Delta^l g(x) \frac{\partial}{\partial n} \Delta^{2q-1-l} K_{2q}(x) - \Delta^{2q-1-l} K_{2q}(x) \frac{\partial}{\partial n} \Delta^l g(x) \right) d\sigma(x)$$

since according to Lemma 3.1 (ii) $\Delta^{2q} K_{2q}(x) = 0$ in D_j.

2. For the ball $D_0 := B_g(0)$ we have by definition

$$\int_{D_0} \Delta^q K_{2q}(x) \Delta^q g(x) dx$$
$$= \lambda_0 \int_{D_0} \Delta^q R_{2q}(x) \Delta^q g(x) dx + \sum_{j=1}^{2q} \lambda_j \int_{D_0} \Delta^q I_{2q}(x, jh) \Delta^q g(x)) dx.$$

The first term can be computed with the aid of the second Green formula (see [2, p. 10]) which is

$$\lambda_0 \sum_{l=0}^{q-1} \int_{\partial D_0} \left(\Delta^l g(x) \frac{\partial}{\partial n} \Delta^{2q-1-l} R_{2q}(x) - \Delta^{2q-1-l} R_{2q}(x) \frac{\partial}{\partial n} \Delta^l g(x) \right) d\sigma(x)$$

(4.5)
$$-\lambda_0 \int_{D_0} \Delta^q R_{2q}(x) \Delta^q g(x) dx = \frac{\lambda_0}{\Omega_d} g(0)$$

(with Ω_d as in (1.5)).

To the second term we apply the first Green formula, which yields

$$\sum_{j=1}^{2q}\lambda_j\int_{D_0}\Delta^q I_{2q}(x,jh)\Delta^q g(x))dx \tag{4.6}$$

$$=\sum_{j=1}^{2q}\lambda_j\sum_{l=0}^{q-1}\int_{\partial D_0}\left(\Delta^l g(x)\frac{\partial}{\partial n}\Delta^{2q-1-l}I_{2q}(x,jh)-\Delta^{2q-1-l}I_{2q}(x,jh)\frac{\partial}{\partial n}\Delta^l g(x)\right)d\sigma(x).$$

Summing up (4.5) and (4.6) we obtain

$$\int_{D_0}\Delta^q K_{2q}(x)\Delta^q g(x)dx$$

$$=\sum_{l=0}^{q-1}\int_{\partial D_0}\left(\Delta^l g(x)\frac{\partial}{\partial n}\Delta^{2q-1-l}K_{2q}(x)-\Delta^{2q-1-l}K_{2q}(x,jh)\frac{\partial}{\partial n}\Delta^l g(x)\right)d\sigma(x)$$

$$-\frac{1}{\Omega_d}g(0),$$

according to Theorem 3.4 for the constant λ_0.

3. Now we sum up the formulas obtained in 1. and 2. for all D_j ($j=0,\ldots,2q-1$). Due to Lemma 3.1 (iii) we get

$$\int_{B_{2qh}(0)}\Delta^q K_{2q}(x)\Delta^q g(x)dx \tag{4.7}$$

$$=-\frac{g(0)}{\Omega_d}+\sum_{j=1}^{2q-1}\int_{S(0,jh)}\{\frac{\partial}{\partial n}\Delta^{2q-1}K_{2q}(x)\}_j\cdot g(x)d\sigma(x)$$

$$-\sum_{l=0}^{q-1}\left(\int_{S(0,2qh)}(\Delta^{q+l}K_{2q}(x,jh)\frac{\partial}{\partial n}\Delta^{q-1-l}g(x)-\frac{\partial}{\partial n}\Delta^{q+l}K_{2q}(x)\Delta^{q-1-l}g(x))d\sigma(x)\right),$$

where $\{\frac{\partial}{\partial n}\Delta^{2q-1}K_{2q}(x)\}_j$ denotes the jump of the normal derivative at the point $x\in S(0,jh)$, which is equal to

$$\frac{\partial}{\partial n}\Delta^{2q-1}K_{2q}(y)|_{|y|=jh-0}-\frac{\partial}{\partial n}\Delta^{2q-1}K_{2q}(y)|_{|y|=jh+0}$$

$$=\frac{\partial}{\partial|y|}\Delta^{2q-1}K_{2q}(y)|_{|y|=jh-0}-\frac{\partial}{\partial|y|}\Delta^{2q-1}K_{2q}(y)|_{|y|=jh+0},$$

which is a certain constant (independent of y) due to the radial symmetry of K_{2q}.

4. Finally, we consider all functions $u\in H^{2q}(B_{2qh}(0))$ which also satisfy (4.2) and (4.3). Then we have

$$\int_{B_{2qh}(0)}(\Delta^q K_{2q}(x)-\Delta^q u(x))^2dx$$

$$= \int_{B_{2qh}(0)} (\Delta^q u(x))^2 dx - 2 \int_{B_{2qh}(0)} (\Delta^q u(x) - \Delta^q K_{2q}(x)) \Delta^q K_{2q}(x) dx$$
$$- \int_{B_{2qh}(0)} (\Delta^q K_{2q}(x))^2 dx.$$

Let us now apply (4.7) to the second term on the right–hand side by putting $g = \Delta^q u(x) - \Delta^q K_{2q}(x)$. since Note that the function K_{2q} belongs to $H^{2q}(B_{2qh}(0))$ as seen in part 1. of this proof. We obtain

$$\int_{B_{2qh}(0)} (\Delta^q u(x) - \Delta^q K_{2q}(x)) \Delta^q K_{2q}(x) dx$$
$$= -\frac{1}{\Omega_d}(u(0) - K_{2q}(0))$$
$$+ \sum_{j=1}^{2q-1} \int_{S(0,jh)} \{\frac{\partial}{\partial n} \Delta^{2q-1} K_{2q}(x)\}_j (u(x) - K_{2q}(x)) d\sigma_j(x)$$
$$- \sum_{l=0}^{q-1} \int_{S(0,2qh)} (\Delta^{q+l} K_{2q}(x) \frac{\partial}{\partial n} \Delta^{q-1-l}(u(x) - K_{2q}(x))$$
$$- \frac{\partial}{\partial n} \Delta^{q+l} K_{2q}(x) \Delta^{q-1-l}(u(x) - K_{2q}(x))) d\sigma(x).$$

All terms on the right side are zero since K_{2q} and u satisfy the matching interpolation conditions (4.2) and (4.3).

Thus we obtain

$$\int_{B_{2qh}(0)} (\Delta^q K_{2q}(x))^2 dx = \int_{B_{2qh}(0)} (\Delta^q u(x))^2 dx - \int_{B_{2qh}(0)} (\Delta^q u(x) - \Delta^q K_{2q}(x))^2 dx \tag{4.8}$$

which proves the minimum property. □

Theorem 4.2. *The solution to problem* (4.1–4.3) *is unique.*

Proof. 1. Let $f \in H^{2q}(B_{2qh}(0))$ be another solution to problem (4.1–4.3). Let us put $v = f - K_{2q}$. Then by (4.8) we obtain

$$\int_{B_{2qh}(0)} (\Delta^q v(x))^2 dx = 0. \tag{4.9}$$

2. On the other hand from (4.3) it follows that

$$\Delta^k v(x) = 0, \qquad |x| = 2qh, \quad 0 \le k \le q-1, \tag{4.10}$$

$$\frac{\partial}{\partial n}\Delta^k v(x) = 0, \qquad |x| = 2qh, \quad 0 \le k \le q-1.$$

Also, by Theorem 3.4 we have that the functions $\Delta^k v(x)$, $0 \le k \le q-1$, and $(\partial/\partial n)\Delta^k v(x)$, $0 \le k \le q-1$, have the same boundary values (traces in the sense of Lions–Magenes [11, Ch.I.7]) when taken from both sides of the sphere $|x| = jh$ for $j = 1, 2, \ldots, 2q-1$.

3. For every domain $D_j, 0 \le j \le 2q-1$, the second Green formula yields:

$$\Omega_d \sum_{l=0}^{q-1} \int_{\partial D_j} \Big(\Delta^l v(x) \frac{\partial}{\partial n} R_{l+1}(x-y) - R_{l+1}(x-y) \frac{\partial}{\partial n} \Delta^l v(x)\Big) d\sigma(x)$$

$$-\Omega_d \int_{D_j} R_q(x-y)\Delta^q v(x) dx = \begin{cases} v(y) & \text{if } y \in D_j \\ 0 & \text{if } y \notin \overline{D}_j \end{cases}$$

We sum up these formulas for $j = 0, \ldots, 2q-1$. Due to the statements about the traces made above, we obtain:

$$-\Omega_d \int_{B_{2qh}} R_q(x-y)\Delta^q v(x) dx = \begin{cases} v(y) & \text{if } y \in \tilde{D} \\ 0 & \text{if } y \notin B_{2qh}(0), \end{cases}$$

where $\tilde{D} = B_{2qh}(0) \setminus \bigcup_{j=1}^{2q} S_{jh}(0)$.

Equality (4.9) implies that $v(y) = 0, y \in \tilde{D}$, due to the inequalities

$$|v(y)| \ \le \ \Big| \int_{B_{2qh}} R_q(x-y)\Delta^q v(x) dx \Big|$$

$$\le \ \sqrt{\int_{B_{2qh}} R_q^2(x-y) dx \cdot \int_{B_{2qh}} (\Delta^q v(x))^2 dx} \ = \ 0.$$

Since $v(x) = 0$, for $|x| = jh$, $j = 1, \ldots, 2q$, we obtain that v is identically zero in $B_{2qh}(0)$. This finishes the proof. □

Remark. The kernel K_{2q} is an example of a polyspline. These were introduced as solutions of a problem of type (4.1) – (4.3) and studied in [7, 8, 10].

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