Cardinal interpolation with biharmonic polysplines on strips

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Abstract

Biharmonic polysplines on strips are natural generalizations of univariate cubic splines. Our construction of cardinal interpolation for such polysplines is based on cardinal interpolation with parameter-dependent univariate $L-$splines and establishes a multivariable version of a classical result of Schoenberg.

1 Introduction

The theory of polysplines was conceived as a multivariate generalization of the theory of one–dimensional splines (see [5]). For every natural $p \geq 1$, a polyspline of order p is a piecewise polyharmonic function of several variables whose pieces are functions polyharmonic of order p. They join on the interfaces smoothly up to the order C^{2p-2}. For an extensive development of the theory of polysplines we refer to the monograph [6]. An important special case of the polyspline concept arises if the interfaces are parallel hyperplanes (in two dimensions – parallel straight lines). Polysplines defined on strips determined by a *finite* number of parallel hyperplanes have been studied in [6]. In the present paper, by analogy with cardinal splines of one variable, the polysplines whose interfaces are infinitely many equidistant hyperplanes will be called *cardinal polysplines on strips.* Their theory is potentially interesting not only from the point of view of Multivariable Approximation, but also from that of Wavelet Analysis.

Another important case occurs when the interfaces are concentric spheres—for infinitely many such spheres of radii $r_j = ab^j$, with $j \in \mathbb{Z}$, the corresponding *cardinal polysplines on annuli* have been exhaustively studied in [6]–[11]. The main difference is that the theory of cardinal polysplines on annuli relies heavily on the intricate theory of spherical harmonics; by comparison, the strips case will allow a simpler and more elegant treatment in the following.

The present paper is devoted to studying cardinal polysplines on strips in the biharmonic case $p = 2$. Specifically, we consider the so–called *interpolation polysplines,* which interpolate data functions prescribed on the parallel hyperplanes. Our main result is a multivariate analog to the following theorem of Schoenberg [17] for one–dimensional cardinal cubic splines: Let the data

$\{f_j\} \subset \mathbb{R}$ satisfy the power growth condition

$$|f_j| \leq D\,(1+|j|)^{\gamma} \qquad \text{for all } j \in \mathbb{Z},$$

for some $D, \gamma \geq 0$. Then there exists a unique cardinal cubic spline s (with knots at the integers $\mathbb{Z}$) interpolating the data, i.e.

$$s\,(j) = f_j \qquad \text{for all } j \in \mathbb{Z},$$

and having the same power growth, i.e.

$$|s\,(t)| \leq C\,(1+|t|)^{\gamma} \qquad \text{for all } t \in \mathbb{R},$$

for some constant C independent of the data $\{f_j\}$. Further, this cardinal spline s is given by the locally uniformly convergent Lagrange series

$$s\,(t) = \sum_{j=-\infty}^{\infty} f_j L\,(t-j) \qquad \text{for all } t \in \mathbb{R}, \tag{1}$$

where the "Lagrange" or "fundamental function" L is the unique cardinal cubic spline, bounded on $\mathbb{R}$, that satisfies

$$L\,(j) = \begin{cases} 1, & j = 0, \\ 0, & j \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

In our multivariable generalization, the set of integer knots will be replaced by the set of hyperplanes $\{\Gamma_j : j \in \mathbb{Z}\}$, where

$$\Gamma_j = \left\{(t,y) \in \mathbb{R}^{n+1} : t = j \text{ and } y \in \mathbb{R}^n\right\}.$$

For $j \in \mathbb{Z}$, let Ω_j be the strip

$$\Omega_j = \left\{(t,y) \in \mathbb{R}^{n+1} : j < t < j+1 \text{ and } y \in \mathbb{R}^n\right\},$$

and denote by $\overline{\Omega_j}$ the topological closure of Ω_j.

Definition 1 *For each $j \in \mathbb{Z}$, let $S_j : \overline{\Omega_j} \to \mathbb{C}$ be a locally integrable function defined on the closed strip $\overline{\Omega_j}$. If the following conditions are satisfied:*

1. *Each function S_j is biharmonic in the open strip Ω_j, i.e.*

$$\Delta^2 S_j = 0 \qquad \text{in } \Omega_j,$$

where Δ is the Laplace operator in $\mathbb{R}^{n+1}$, acting in the distributional sense;

2. *Each function S_j belongs to the smoothness class $C^2\left(\overline{\Omega_j}\right)$, i.e. S_j is C^2-smooth up to the boundary of Ω_j;*

3. For every $j \in \mathbb{Z}$ the following equalities (called matching conditions*) hold*

$$\begin{cases} S_j(j+1,y) = S_{j+1}(j+1,y) \\ \frac{\partial S_j}{\partial t}(j+1,y) = \frac{\partial S_{j+1}}{\partial t}(j+1,y) \\ \frac{\partial^2 S_j}{\partial t^2}(j+1,y) = \frac{\partial^2 S_{j+1}}{\partial t^2}(j+1,y) \end{cases} \qquad \text{for all } y \in \mathbb{R}^n, \tag{2}$$

(note that the derivatives exist on the boundary by the previous condition);

then the function $S : \mathbb{R}^{n+1} \to \mathbb{C}$ whose restriction on each strip $\overline{\Omega_j}$ coincides with S_j, will be called ***a biharmonic cardinal polyspline on strips.*** *The hyperplanes Γ_j will be called* ***knot surfaces*** *or* ***break surfaces*** *of S.*

Assume that for every $j \in \mathbb{Z}$ some data function $f_j \in C^2(\mathbb{R}^n)$ is given. The above cardinal polyspline S will be called ***an interpolation polyspline for the data*** *$\{f_j\}$ if*

$$S(j,y) = f_j(y) \qquad \text{for all } j \in \mathbb{Z} \text{ and } y \in \mathbb{R}^n. \tag{3}$$

Remark 2 *1. According to the regularity theorem for elliptic operators (see Corollary to Theorem 8.12 in [14]) it follows that the function $S_j \in C^\infty(\Omega_j)$.*

2. According to the above definition, the biharmonic cardinal polysplines on strips are functions $S \in C^2\left(\mathbb{R}^{n+1}\right)$.

The main result of this paper (Theorem 10) establishes the existence of a biharmonic cardinal polyspline S on strips verifying the interpolation conditions (3), under the assumption that, in a certain norm, the set of data functions $\{f_j\}$ satisfies a growth restriction with respect to j, and the functions f_j have smoothness measured in Hölder spaces.

We have restricted our attention to the nontrivial and practically interesting biharmonic case $p = 2$ since the proofs are simpler than in the case of general $p \geq 1$. The case of arbitrary polyharmonic order p will be treated elsewhere. Note that for $p = 1$ we have harmonic polysplines. The existence of *cardinal interpolation harmonic polysplines* is clearly reduced to the existence of solutions to the Dirichlet problem in each strip (similarly, in the one–dimensional case the existence of interpolation linear splines is trivial); for that reason such a problem is not interesting.

Outline of the paper

The construction of cardinal polysplines on parallel strips will be based on the fundamental idea of separation of variables. This motivates the introduction of the Fourier transform with respect to the variable y,

$$\widehat{S}(t,\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} S(t,y)\, dy \qquad t \in \mathbb{R},\ \xi \in \mathbb{R}^n, \tag{4}$$

understood either in a classical or distributional sense ($\langle \cdot, \cdot \rangle$ denotes the dot product).

Using the Fourier transform, the biharmonic functions in a strip can be characterized in terms of an ordinary differential equation which depends on the parameter ξ.

Proposition 3 *For $a < b$, let S be a tempered distribution on the strip $\Omega_{a,b} = \left\{(t, y) \in \mathbb{R}^{n+1} : a < t < b, y \in \mathbb{R}^n\right\}$. Then S is biharmonic, i.e.*

$$\Delta^2 S(t, y) = 0 \qquad \text{in } \Omega_{a,b},$$

if and only if its Fourier transform with respect to the variable y satisfies

$$\left(\frac{\partial^2}{\partial t^2} - |\xi|^2\right)^2 \widehat{S}(t, \xi) = 0 \qquad \text{for } a < t < b, \quad \text{and } \xi \in \mathbb{R}^n,$$

where the equations are understood in the distributional sense.

On the basis of this result, it is possible to reduce the construction of a biharmonic polyspline S on a *finite* number of adjacent strips to that of the family of univariate functions $\widehat{S}(\cdot, \xi)$ depending on the parameter $\xi \in \mathbb{R}^n$, where, for each ξ, the function $\widehat{S}(t, \xi)$ with respect to the variable t is a so-called "$L-$spline" for the ordinary differential operator $\left(\frac{\partial^2}{\partial t^2} - |\xi|^2\right)^2$. This approach was illustrated in [6, Theorem 9.4] for polysplines of arbitrary order on a finite number of strips.

In the same wisdom, in **Section 3** of the present paper we construct *cardinal* interpolation for biharmonic polysplines on strips by means of a family of "cardinal $L-$splines" for the ordinary differential operator $\left(\frac{\partial^2}{\partial t^2} - |\xi|^2\right)^2$ depending on the parameter $\xi \in \mathbb{R}^n$. The associated parameter-dependent Lagrange schemes of the type (1) rely upon the more general theory for cardinal interpolation with $L-$splines developed by Micchelli in [12]. Of crucial importance to our analysis are uniform estimates with respect to the parameter ξ for the fundamental functions of cardinal interpolation with parameter $L-$splines. In **Section 2** we obtain such estimates by employing the Fourier integral representation of these fundamental functions (for polynomial splines, this representation was first proposed by Schoenberg in [15]). By comparison, the construction of cardinal polysplines on annuli given in [6], [7] has been based on B-spline type representations of the corresponding $L-$spline fundamental functions.

2 Cardinal interpolation with parameter $L-$splines

2.1 The Euler–Schoenberg function

For each $\xi \in \mathbb{R}^n$, we define the polynomial

$$q(z) = \left(z^2 - |\xi|^2\right)^2. \tag{5}$$

As suggested by Proposition 3, the operator

$$q\left(\frac{d}{dt}\right)=\left(\frac{d^2}{dt^2}-|\xi|^2\right)^2 \tag{6}$$

will play an essential role in our considerations. Note that on any open interval the kernel space of this ordinary differential operator is spanned by the set of four functions $\left\{e^{|\xi|t}, te^{|\xi|t}, e^{-|\xi|t}, te^{-|\xi|t}\right\}$ if $\xi\neq 0$, or $\left\{1,t,t^2,t^3\right\}$ if $\xi=0$.

In order to construct the fundamental functions of cardinal $L-$spline interpolation with respect to the operator $q\left(\frac{d}{dt}\right)$, we shall follow the steps of the general theory developed by Micchelli [12], which extends Schoenberg's theory [17] of cardinal interpolation with polynomial splines (see [6, Ch. 13] for a concise exposition). The so-called **Euler-Schoenberg function** with respect to q is defined by

$$\begin{aligned} A(x,\lambda) &:= \frac{1}{2\pi i}\int_\Gamma \frac{1}{q(z)}\frac{e^{xz}}{e^z-\lambda}dz \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{\left(z^2-|\xi|^2\right)^2}\frac{e^{xz}}{e^z-\lambda}dz \end{aligned} \tag{7}$$

where Γ is a closed contour in the complex plane surrounding the points $\{-|\xi|,|\xi|\}$ and leaving out the zeros of the function $e^z-\lambda$. Obviously, it is well defined for all $x\in\mathbb{R}$ and all $\lambda\notin\left\{e^{|\xi|t},e^{-|\xi|t}\right\}$.

Applying the residue theorem we compute

$$\begin{aligned} A(0;\lambda) &= \frac{1}{2\pi i}\int\limits_\Gamma \frac{1}{(z-|\xi|)^2(z+|\xi|)^2}\frac{1}{e^z-\lambda}dz \\ &= \frac{-1}{4|\xi|^3}\left(\frac{e^{-|\xi|}-\lambda-e^{-|\xi|}|\xi|}{-\left(-e^{-|\xi|}+\lambda\right)^2}+\frac{e^{|\xi|}-\lambda+e^{|\xi|}|\xi|}{\left(-e^{|\xi|}+\lambda\right)^2}\right). \end{aligned} \tag{8}$$

Next, define the polynomial

$$r(\lambda):=\left(-e^{-|\xi|}+\lambda\right)^2\left(-e^{|\xi|}+\lambda\right)^2.$$

Then, for each ξ, the **Euler–Frobenius polynomial**

$$\Pi(\lambda):=r(\lambda)A(0;\lambda) \tag{9}$$

is a quadratic polynomial in λ, hence

$$\Pi(\lambda)=C^\xi(\lambda-\mu_1)(\lambda-\mu_2),$$

where C^ξ denotes its leading coefficient, and μ_1, μ_2 are its roots.

From formulas (8) and (9) we have

$$\begin{aligned} \Pi(\lambda) &= \frac{e^{-2|\xi|}}{4|\xi|^3}\left(-e^{3|\xi|}+e^{3|\xi|}|\xi|+e^{|\xi|}+e^{|\xi|}|\xi|\right)\times \\ &\times\left[1+\frac{\left(e^{4|\xi|}-4e^{2|\xi|}|\xi|-1\right)}{\left(-e^{3|\xi|}+e^{3|\xi|}|\xi|+e^{|\xi|}+e^{|\xi|}|\xi|\right)}\lambda+\lambda^2\right], \end{aligned} \tag{10}$$

which shows that

$$C^{\xi}=\frac{e^{-2|\xi|}}{4\left|\xi\right|^{3}}\left(e^{3|\xi|}\left|\xi\right|+e^{|\xi|}\left|\xi\right|+e^{|\xi|}-e^{3|\xi|}\right). \tag{11}$$

2.2 The fundamental function for cardinal $L-$ spline interpolation

For each parameter ξ, the fundamental function $L^{\xi}\in C^{2}\left(\mathbb{R}\right)$ for cardinal $L-$spline interpolation with respect to the operator $q\left(\frac{d}{dt}\right)$ is piecewise spanned by the kernel $\mathcal{K}^{\xi}$ of this operator on each interval $\left(j,j+1\right)$, and satisfies the "Lagrange" conditions

$$\begin{cases} L^{\xi}\left(0\right)=1, & \\ L^{\xi}\left(j\right)=0, & \text{for } j\in\mathbb{Z}\backslash\{0\}. \end{cases} \tag{12}$$

It is uniquely determined if required to satisfy the additional condition of having at most polynomial growth at ∞ (see [17], [12]). Micchelli's construction of L^{ξ} employs the above Euler-Schoenberg function A, as follows.

Since the integral of $\frac{1}{q(z)}\frac{e^{xz}}{e^{z}-\lambda}$ over a large contour tends obviously to zero, from (7) we obtain

$$A\left(x;\lambda\right)=\sum_{k=-\infty}^{\infty}\frac{e^{(x-1)\delta_{k}}}{q\left(\delta_{k}\right)}=\lambda^{x-1}\sum_{k=-\infty}^{\infty}\frac{e^{2\pi ikx}}{q\left(\delta_{k}\right)},$$

where

$$\begin{aligned} \lambda&=\tau e^{iu}, \qquad \text{for } -\pi<u\leq\pi,\\ \delta_{k}&=\log\tau+i\left(u+2\pi k\right). \end{aligned}$$

In particular,

$$A\left(0;\lambda\right)=\lambda^{-1}\sum_{k=-\infty}^{\infty}\frac{1}{q\left(\delta_{k}\right)}.$$

Next, for $\lambda\notin\{-\left|\xi\right|,\left|\xi\right|\}$ and $A\left(0;\lambda\right)\neq0$, we define as in [12, p. 222, formula (22)]

$$E\left(x;\lambda\right):=\frac{A\left(x;\lambda\right)}{A\left(0;\lambda\right)} \qquad \text{for } 0\leq x\leq1.$$

This is the only cardinal $L-$spline with respect to $q\left(\frac{d}{dt}\right)$ which satisfies the functional equation

$$\begin{aligned} E\left(x+1;\lambda\right)&=\lambda E\left(x;\lambda\right) \qquad \text{with the norming}\\ E\left(0;\lambda\right)&=1. \end{aligned}$$

Letting

$$\Omega\left(x;\lambda\right):=\lambda^{-x}E\left(x;\lambda\right),$$

by the above we obtain

$$\Omega(x;\lambda) = \frac{\sum_{k=-\infty}^{\infty} \frac{e^{2\pi ikx}}{q(\delta_k)}}{\sum_{k=-\infty}^{\infty} \frac{1}{q(\delta_k)}}.$$

Finally, as in [12, formula (26)], the fundamental cardinal $L-$spline with respect to the operator $q\left(\frac{d}{dt}\right)$ is defined by

$$\begin{aligned} L^{\xi}(t) &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} E\left(t; e^{iu}\right) du \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iut} \Omega\left(t; e^{iu}\right) du. \end{aligned}$$

Due to the properties of the function E, the above expression for L^{ξ} satisfies all the properties stated at the beginning of this subsection, which characterize uniquely the fundamental $L-$spline.

Note that by substituting Ω and employing a classical periodization argument, we obtain (as in [12])

$$\begin{aligned} L^{\xi}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iut} \frac{1}{q(iu)\, e^{iu} A\left(0; e^{iu}\right)} du \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iut} \frac{\omega^{\xi}\left(e^{iu}\right)}{q(iu)} du, \end{aligned} \tag{13}$$

where we have put

$$\omega^{\xi}(w) := \frac{1}{wA(0; w)}. \tag{14}$$

In order to use this Fourier integral representation and estimate the behaviour of $L^{\xi}(t)$ for large t, we need to investigate the dependence of the zeros of Π on the parameter ξ.

2.3 The zeros of $\Pi(\lambda)$ and their asymptotics

The general theory in [12] guarantees that the two zeros μ_1 and μ_2 of Π are distinct and satisfy

$$\mu_2 < \mu_1 < 0.$$

The following result studies their dependence on the parameter ξ.

Theorem 4 *For every $\xi \in \mathbb{R}^n$, we may choose the roots $\mu_1 = \mu_1(\xi)$ and $\mu_2 = \mu_2(\xi)$ of Π so that $\mu_1(\xi)$ and $\mu_2(\xi)$ are continuous functions of the variable ξ and satisfy*

$$\sup_{\xi} \mu_2(\xi) < -1 < \inf_{\xi} \mu_1(\xi).$$

Proof. 1. First of all it is obvious from (10) that

$$\mu_1 \mu_2 = 1, \tag{15}$$

$$\mu_1 + \mu_2 = -\frac{\left(e^{4|\xi|} - 4e^{2|\xi|}\,|\xi| - 1\right)}{\left(-e^{3|\xi|} + e^{3|\xi|}\,|\xi| + e^{|\xi|} + e^{|\xi|}\,|\xi|\right)}. \tag{16}$$

A triple application of *L'Hopital*'s rule shows

$$\mu_1 + \mu_2 \overset{|\xi| \longrightarrow 0}{\longrightarrow} -4.$$

Note that for $\xi = 0$, we may let

$$\mu_1(0) := -2 + \sqrt{3}, \quad \mu_2(0) := -2 - \sqrt{3}.$$

2. Now let us prove that we may find two branches of the solutions of $\Pi(\lambda) = 0$ which are continuous functions of ξ. We may find the roots of $\Pi(\lambda)$ explicitely, by solving the quadratic equation

$$1 + \frac{\left(e^{4|\xi|} - 4e^{2|\xi|}\,|\xi| - 1\right)}{\left(-e^{3|\xi|} + e^{3|\xi|}\,|\xi| + e^{|\xi|} + e^{|\xi|}\,|\xi|\right)}\lambda + \lambda^2 = 0.$$

Letting

$$D := e^{8|\xi|} + 6e^{4|\xi|} + 8e^{4|\xi|}\,|\xi|^2 + 1 - 4e^{6|\xi|} - 4e^{6|\xi|}\,|\xi|^2 - 4e^{2|\xi|} - 4e^{2|\xi|}\,|\xi|^2,$$

we have that

$$\mu_1(\xi) := \frac{-e^{4|\xi|} + 4e^{2|\xi|}\,|\xi| + 1 + \sqrt{D}}{2\left(-e^{3|\xi|} + e^{3|\xi|}\,|\xi| + e^{|\xi|} + e^{|\xi|}\,|\xi|\right)},$$
$$\mu_2(\xi) := \frac{-e^{4|\xi|} + 4e^{2|\xi|}\,|\xi| + 1 - \sqrt{D}}{2\left(-e^{3|\xi|} + e^{3|\xi|}\,|\xi| + e^{|\xi|} + e^{|\xi|}\,|\xi|\right)},$$

satisfy

$$\mu_2(\xi) < \mu_1(\xi).$$

Let us prove that these are continuous functions of ξ. For this purpose we check that the function (the denominator in μ_1 and μ_2)

$$g(\rho) := -e^{3\rho} + e^{3\rho}\rho + e^{\rho} + e^{\rho}\rho$$

does not have zeros for $\rho > 0$. Indeed, it is evident that for $\rho \geq 1$ we have $g(\rho) > 0$. In the interval $[0,1]$ we need more detailed study. It is easy to check that

$$\begin{aligned}
g(0) &= 0, \quad g(1) > 0;\\
g'(\rho) &= -2e^{3\rho} + 3e^{3\rho}\rho + 2e^{\rho} + \rho e^{\rho}\\
g'(0) &= 0, \quad g'(1) > 0;\\
g''(\xi) &= -3e^{3\rho} + 9e^{3\rho}\rho + 3e^{\rho} + \rho e^{\rho}\\
g''(0) &= 0, \quad g''(1) > 0;\\
g'''(\xi) &= 27e^{3\rho}\rho + 4e^{\rho} + \rho e^{\rho} > 0.
\end{aligned}$$

A triple application of Rolle's theorem shows that $g(\rho)$ has no zero in the interval $(0,1)$. Thus we see that the only zero of g is for $\rho = 0$.

We deduce that μ_1 and μ_2 are continuous functions of $|\xi|$.

3. For $|\xi| \longrightarrow \infty$ we have the following asymptotics

$$\mu_2 \approx \mu_1 + \mu_2 = -\frac{\left(e^{4|\xi|} - 4e^{2|\xi|}|\xi| - 1\right)}{\left(-e^{3|\xi|} + e^{3|\xi|}|\xi| + e^{|\xi|} + e^{|\xi|}|\xi|\right)} \approx -\frac{e^{4|\xi|}}{e^{3|\xi|}|\xi|} = -\frac{e^{|\xi|}}{|\xi|}. \quad (17)$$

This also implies

$$\mu_1 = \frac{1}{\mu_2} \approx -\frac{|\xi|}{e^{|\xi|}}.$$

Since we have $\mu_2(\xi) < \mu_1(\xi) < 0$, if we assume that $\mu_2\left(\xi^1\right) \geq -1$ for some ξ^1, then $\mu_1\left(\xi^1\right) = 1/\mu_2\left(\xi^2\right) \leq -1$, which is a contradiction. Due to the asymptotics for $|\xi| \longrightarrow \infty$ and the continuity of μ_2 and μ_1, it follows that $\sup_\xi \mu_2(\xi) < -1 < \inf_\xi \mu_1(\xi)$. ■

We define

$$\eta_0 := -\ln\left|\inf_\xi \mu_1(\xi)\right| > 0, \quad (18)$$

hence $e^{-\eta_0} = |\inf_\xi \mu_1(\xi)|$ and

$$|\mu_1(\xi)| \leq e^{-\eta_0} < 1 < e^{\eta_0} \leq |\mu_2(\xi)|.$$

Remark 5 *Note that the asymptotics of μ_1 and μ_2 is the same as the asymptotics of the roots of the corresponding Euler-Frobenius polynomial in the case of biharmonic polysplines on annuli, cf. [6, p. 300], [7].*

2.4 Estimates for the fundamental function L^ξ

In this subsection we prove uniform estimates in ξ for L^ξ and its derivatives, which are crucial for the construction of biharmonic polysplines on strips. To this aim, we rely on the integral representation (13) of L^ξ. We note that this approach differs from the one based on the $B-$spline representation of L^ξ which has been followed in [6], [7].

Theorem 6 *There exist positive constants C_0 and η_1, both independent of ξ, such that for all $t \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$, we have the estimate*

$$\left|L^\xi(t)\right| \leq C_0 e^{-\eta_1|t|}. \quad (19)$$

We may let η_1 be an arbitrary number satisfying $0 < \eta_1 < \eta_0$, where η_0 is defined in (18). Also, for $m \in \{1,2\}$, there exists a constant C_m such that for all $\xi \in \mathbb{R}^n$, we have the estimate

$$\left|\frac{d^m}{dt^m}L^\xi(t)\right| \leq C_m\left(1 + |\xi|^m\right)e^{-\eta_1|t|}, \qquad \textit{for } t \in \mathbb{R}. \quad (20)$$

Proof. 1. After substituting the function A in formula (14) for $\omega^{\xi}(w)$, we obtain

$$\omega^{\xi}(w)=\frac{\left(-e^{-|\xi|}+w\right)^{2}\left(-e^{|\xi|}+w\right)^{2}}{C^{\xi}w\left(w-\mu_{1}\right)\left(w-\mu_{2}\right)}. \tag{21}$$

Above we have seen that for every ξ, there are no zeros of the polynomial $(w-\mu_1)(w-\mu_2)$ in the annulus $e^{-\eta_0}<|w|<e^{\eta_0}$, and consequently we have a Laurent expansion

$$\omega^{\xi}(w)=\sum_{j=-\infty}^{\infty}\gamma_j w^j \qquad \text{for } e^{-\eta_0}<|w|<e^{\eta_0}.$$

2. Now we are ready to estimate the function L^{ξ} using the integral representation (13), which first appeared in the original paper of Schoenberg [15] in the polynomial case ($\xi=0$). We will employ the change of variable

$$u\longrightarrow u+i\eta_2$$

in the integral (13) where, for each t, the constant η_2 is chosen such that $\eta_2 t>0$.

For the sake of clarity, consider the case $t<0$. We define

$$\eta_2:=-\eta_0+\varepsilon,$$

where $\varepsilon>0$ is an arbitrary small number such that $\varepsilon<\eta_0$. Let us note that in the final result in (19) we take in fact $\eta_1=|\eta_2|$. The case $t\geq 0$ is treated in a similar way.

3. For every fixed parameter ξ, by (21) and (5) we have

$$\begin{aligned}\frac{\omega^{\xi}\left(e^{iz}\right)}{q\left(iz\right)}&=\frac{\left(-e^{-|\xi|}+e^{iz}\right)^{2}\left(-e^{|\xi|}+e^{iz}\right)^{2}}{C^{\xi}e^{iz}\left(e^{iz}-\mu_{1}\right)\left(e^{iz}-\mu_{2}\right)\left(z^{2}+|\xi|^{2}\right)^{2}}\\&=-\frac{e^{2|\xi|}\mu_{1}(\xi)}{C^{\xi}\left(1-\mu_{1}e^{iz}\right)\left(1-\mu_{1}e^{-iz}\right)}\left(\frac{1-e^{iz-|\xi|}}{iz-|\xi|}\right)^{2}\left(\frac{1-e^{-iz-|\xi|}}{-iz-|\xi|}\right)^{2}.\end{aligned} \tag{22}$$

Since $\left(1-\mu_1 e^{iz}\right)^{-1}\left(1-\mu_1 e^{-iz}\right)^{-1}$ is analytic for $z\in\mathbb{R}\times[\eta_2,0]$, and $\frac{1-e^{Z}}{Z}$ is an entire function, it follows that the integrand of (13) is analytic on the strip $\mathbb{R}\times[\eta_2,0]$. Thus, for any $N>0$, we can apply Cauchy's formula to the integral

$$\int_{\Gamma}e^{izt}\frac{\omega^{\xi}\left(e^{iz}\right)}{q\left(iz\right)}dz,$$

where Γ is the oriented boundary of the rectangle $[-N,N]\times[\eta_2,0]$, and we note that the side integrals on each of the intervals

$$-N+i\tau \qquad \text{and} \qquad N+i\tau, \qquad \text{for } \eta_2\leq\tau\leq 0,$$

tend to zero as $N \longrightarrow \infty$, due to the growth of $q\left(i\left(\pm N+i\tau\right)\right)$ for $N \longrightarrow \infty$ and to the fact that $\omega^{\xi}\left(e^{iz}\right)$ is periodic and continuous (hence bounded) for $z \in \mathbb{R}\times\left[\eta_2, 0\right]$. Therefore, letting $N \longrightarrow \infty$ in Cauchy's formula, we obtain

$$L^{\xi}\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\left(u+i\eta_2\right)t}\frac{\omega^{\xi}\left(e^{i\left(u+i\eta_2\right)}\right)}{q\left(i\left(u+i\eta_2\right)\right)}du. \tag{23}$$

4. To estimate the last integral, consider first the case $\left|\xi\right| > -2\eta_2$. Since ω^{ξ} is bounded on $\mathbb{R}\times\left[\eta_2, 0\right]$, there exists a constant M^{ξ} independent of u, such that

$$\left|\omega^{\xi}\left(e^{i\left(u+i\eta_2\right)}\right)\right| \leq M^{\xi}, \qquad \forall\, u \in \mathbb{R}.$$

Thus from (23) we obtain

$$\left|L^{\xi}\left(t\right)\right| \leq \frac{1}{2\pi}M^{\xi}e^{-\eta_2 t}I\left(\xi\right) \tag{24}$$

where

$$I\left(\xi\right):=\int_{-\infty}^{\infty}\frac{1}{\left|q\left(i\left(u+i\eta_2\right)\right)\right|}du.$$

5. To check the asymptotics of the constant M^{ξ} for $\left|\xi\right| \longrightarrow \infty$, we use formula (11) for C^{ξ} and the asymptotics (17) of μ_2, and from (21) we obtain

$$M^{\xi} \approx \frac{4\left|\xi\right|^4 e^{3\left|\xi\right|}}{\left(e^{3\left|\xi\right|}\left|\xi\right|+e^{\left|\xi\right|}\left|\xi\right|+e^{\left|\xi\right|}-e^{3\left|\xi\right|}\right)} \approx \left|\xi\right|^3, \quad \left|\xi\right| \longrightarrow \infty.$$

In order to compute the asymptotics of the integral $I\left(\xi\right)$ for $\left|\xi\right| \longrightarrow \infty$, note that (5) implies

$$\begin{aligned}\frac{1}{\left|q\left(i\left(u+i\eta_2\right)\right)\right|} &= \frac{1}{\left|\left(u+i\eta_2\right)^2+\left|\xi\right|^2\right|^2}\\ &= \frac{1}{\left(u^2+\left(\eta_2-\left|\xi\right|\right)^2\right)\left(u^2+\left(\eta_2+\left|\xi\right|\right)^2\right)}\\ &= \frac{1}{4\left|\xi\right|\eta_2}\left(\frac{1}{u^2+\left(\eta_2-\left|\xi\right|\right)^2}-\frac{1}{u^2+\left(\eta_2+\left|\xi\right|\right)^2}\right).\end{aligned}$$

Since $\int_{-\infty}^{\infty}\frac{1}{v^2+1}dv=\pi$, after changing the variables we obtain

$$\begin{aligned}I\left(\xi\right) &= \frac{1}{4\left|\xi\right|\eta_2}\int_{-\infty}^{\infty}\left(\frac{1}{u^2+\left(\eta_2-\left|\xi\right|\right)^2}-\frac{1}{u^2+\left(\eta_2+\left|\xi\right|\right)^2}\right)du\\ &= \frac{\pi}{4\left|\xi\right|\eta_2}\left(\frac{\left|\eta_2+\left|\xi\right|\right|-\left|\eta_2-\left|\xi\right|\right|}{\left|\eta_2-\left|\xi\right|\right|\cdot\left|\eta_2+\left|\xi\right|\right|}\right).\end{aligned}$$

Thus for $|\xi| \longrightarrow \infty$ we have

$$\begin{aligned} I(\xi) &= \frac{\pi}{4|\xi|\eta_2} \frac{2\eta_2}{(|\xi|-\eta_2)(\eta_2+|\xi|)} \\ &= \frac{\pi}{2|\xi|} \frac{1}{(|\xi|-\eta_2)(\eta_2+|\xi|)} \approx \frac{1}{|\xi|^3}. \end{aligned}$$

The above asymptotics and the estimate (24) show that (19) holds with $\eta_1 := |\eta_2|$, whenever $|\xi| > -2\eta_2$.

6. It remains to estimate (23) for $0 \le |\xi| \le -2\eta_2$. In this case, since C^{ξ} and $\mu_1(\xi)$ are continuous functions of ξ, using (22) in (23), we obtain

$$\left|L^{\xi}(t)\right| \le c e^{-\eta_2 t} J(\xi), \tag{25}$$

for some absolute constant $c > 0$, where

$$J(\xi) := \int_{-\infty}^{\infty} \left| \left(\frac{1-e^{iu-\eta_2-|\xi|}}{iu-\eta_2-|\xi|} \right)^2 \left(\frac{1-e^{-iu+\eta_2-|\xi|}}{-iu+\eta_2-|\xi|} \right)^2 \right| du.$$

We will show that this integral is bounded above by a constant independent of ξ, for $0 \le |\xi| \le -2\eta_2$. To this end, we use the splitting

$$J(\xi) = J_1(\xi) + J_2(\xi) := \int_{|u|\le 1} + \int_{|u|>1}.$$

Using the estimates

$$\left|1-e^{\pm iu \mp \eta_2 - |\xi|}\right| \le 1 + e^{\mp \eta_2},$$

$$|\pm iu \mp \eta_2 - |\xi||^2 = u^2 + (\eta_2 \pm |\xi|)^2 \ge u^2,$$

we have

$$J_2(\xi) \le c' \int_{|u|>1} u^{-4} du = \text{const}.$$

On the other hand, if $|u| \le 1$, then

$$|\pm iu \mp \eta_2 - |\xi|| \le |u| + |\eta_2| + |\xi| \le 1 - 3\eta_2,$$

and since the entire function $\frac{1-e^Z}{Z}$ is bounded for $|Z| \le 1 - 3\eta_2$, we also obtain

$$J_1(\xi) \le \text{const}.$$

Hence there exists an absolute constant $c'' > 0$ such that $J(\xi) \le c''$, for all ξ with $0 \le |\xi| \le -2\eta_2$, and combining this with the estimate (25) implies inequality (19), as required. Therefore the first part of the theorem is proved.

7. Turning to (20), for the sake of simplicity consider only the highest derivative $m = 2$. After differentiating under the integral in representation (13), we can use the change of variable $u \longrightarrow u + i\eta_2$ as in part **3** above to obtain a representation similar to (23) for $\frac{d^2}{dt^2}L^{\xi}(t)$ when $t<0$. Thus, if $|\xi| > -2\eta_2$, we obtain the estimate

$$\left|\frac{d^2}{dt^2}L^{\xi}(t)\right| \leq \frac{1}{2\pi}M^{\xi}e^{-\eta_2 t}\int_{-\infty}^{\infty}\frac{u^2+\eta_2^2}{|q(i(u+i\eta_2))|}du,$$

where M^{ξ} is the constant from part **4** of the proof. Since

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{u^2}{|q(i(u+i\eta_2))|}du\\
&=\frac{1}{4|\xi|\eta_2}\int_{-\infty}^{\infty}\left(\frac{u^2}{u^2+(\eta_2-|\xi|)^2}-\frac{u^2}{u^2+(\eta_2+|\xi|)^2}\right)du\\
&=\frac{1}{4|\xi|\eta_2}\int_{-\infty}^{\infty}\left(\frac{-(\eta_2-|\xi|)^2}{u^2+(\eta_2-|\xi|)^2}+\frac{(\eta_2+|\xi|)^2}{u^2+(\eta_2+|\xi|)^2}\right)du\\
&=\frac{\pi}{4|\xi|\eta_2}\left(-|\eta_2-|\xi||+|\eta_2+|\xi||\right)=\frac{\pi}{2|\xi|},
\end{aligned}$$

the asymptotics obtained in part **5** imply the desired estimate (20) in this case. The case $0\leq|\xi|\leq-2\eta_2$ is treated as in part **6** above, showing that (20) holds for all $\xi\in\mathbb{R}^n$. ■

3 Cardinal interpolation polysplines

3.1 The fundamental cardinal polyspline

Let the data function f be given on the hyperplane Γ_0. In this subsection we consider the **problem** of finding a biharmonic cardinal polyspline L_f in the sense of Definition 1, satisfying the interpolation conditions

$$\begin{cases} L_f(0,y)=f(y), & \text{for } y\in\mathbb{R}^n,\\ L_f(j,y)=0, & \text{for } y\in\mathbb{R}^n, \quad \text{for all } j\in\mathbb{Z}\setminus\{0\}. \end{cases} \tag{26}$$

By analogy with the one–dimensional case we will call such a polyspline (if it exists) “fundamental interpolation polyspline”. As shown in the next subsection, polysplines which satisfy general interpolation conditions on the hyperplanes Γ_j can be constructed as Lagrange–type representations based on the fundamental polysplines.

Following the separation of variables principle described in the introduction, by formal Fourier transform with respect to the variable y the above interpolation conditions become

$$\begin{cases} \widehat{L_f}(0,\xi)=\widehat{f}(\xi) & \text{for } \xi\in\mathbb{R}^n,\\ \widehat{L_f}(j,\xi)=0 & \text{for } \xi\in\mathbb{R}^n, \quad \text{for all } j\in\mathbb{Z}\setminus\{0\}. \end{cases}$$

For every $\xi\in\mathbb{R}^n$, these conditions are clearly verified if

$$\widehat{L_f}(t,\xi)=\widehat{f}(\xi)L^{\xi}(t),$$

where $L^{\xi}(t)$ is the fundamental cardinal $L-$spline with respect to the operator (6) which satisfies (12). Along with Proposition 3, this suggests the formal definition by inverse Fourier transform

$$L_f(t,y) := \mathcal{F}^{-1}_{\xi\longrightarrow y}\left[\widehat{f}(\xi)\,L^{\xi}(t)\right](y).$$

Our next theorem derives all the required properties of the fundamental polyspline L_f from this definition, which will be justified in the classical sense.

In the following, we use the spaces $B_s(\mathbb{R}^n)$ of all tempered distributions f whose Fourier transforms $\widehat{f}$ are measurable functions and satisfy

$$\|f\|_s := \int_{\mathbb{R}^n}\left|\widehat{f}(\xi)\right|(1+|\xi|^s)\,d\xi < \infty$$

(see Definition 10.1.6 in Hörmander [4], vol. 2).

Theorem 7 *Let the function $f\in L_1(\mathbb{R}^n)\cap B_2(\mathbb{R}^n)\cap C^2(\mathbb{R}^n)$.*

(i). There exists a biharmonic polyspline L_f in the sense of Definition 1 and satisfying conditions (26).

(ii). In addition, for every multi-index α such that $|\alpha|\leq 2$, we have the decay estimate

$$|D^{\alpha}L_f(t,y)|\leq C_{\alpha}e^{-\eta_1|t|}\|f\|_{|\alpha|}\qquad \text{for } t\in\mathbb{R},$$

for some constant $C_{\alpha}>0$, where η_1 is the constant of Theorem 6.

Proof. 1. According to Theorem 6 (i), the function

$$L_f(t,y) := \mathcal{F}^{-1}_{\xi\longrightarrow y}\left[\widehat{f}(\xi)\,L^{\xi}(t)\right] = \int_{\mathbb{R}^n} e^{i\langle y,\xi\rangle}\widehat{f}(\xi)\,L^{\xi}(t)\,d\xi \tag{27}$$

is well defined, since $L^{\xi}(t)$ is bounded as a function of ξ, and $f\in B_2(\mathbb{R}^n)$ implies $\widehat{f}\in L_1(\mathbb{R}^n)$

2. Let us prove the decay property for $\alpha=0$. By estimate (19) we obtain

$$\begin{aligned}|L_f(t,y)| &\leq \int_{\mathbb{R}^n}\left|\widehat{f}(\xi)\right|\left|L^{\xi}(t)\right|d\xi\\ &\leq C_0e^{-\eta_1|t|}\int_{\mathbb{R}^n}\left|\widehat{f}(\xi)\right|d\xi = C_0e^{-\eta_1|t|}\|f\|_0.\end{aligned}$$

3. To prove the "matching conditions" (2) of Definition 1, we restrict attention to the second derivative $\frac{\partial^2 L_f(t,y)}{\partial t^2}$, i.e. the case $m=2$. Indeed, differentiating formally under the integral sign, thanks to Theorem 6 (ii), we obtain the estimate

$$\begin{aligned}\left|\frac{\partial^2 L_f(t,y)}{\partial t^2}\right| &\leq \int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right| \left|\frac{\partial^2}{\partial t^2} L^\xi(t)\right| d\xi \\ &\leq C_2 e^{-\eta_1|t|} \int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right| \left(1+|\xi|^2\right) d\xi \\ &= C_2 e^{-\eta_1|t|} \|f\|_2\end{aligned}$$

Since $\|f\|_2 < \infty$ by hypothesis, it follows that $\frac{\partial^2 L_f(t,y)}{\partial t^2}$ exists and is continuous in $\mathbb{R}^{n+1}$. In a similar way, all derivatives of second order $D^\alpha L_f(t,y)$, $|\alpha| \leq 2$, exist, are continuous and satisfy the decay estimates of (ii).

4. We check that L_f is a biharmonic function in every strip Ω_j. By the definition of $L^\xi(t)$, for every $j \in \mathbb{Z}$ we have

$$\left(\frac{\partial^2}{\partial t^2} - |\xi|^2\right)^2 L^\xi(t) = 0 \qquad \text{for } t \in (j, j+1), \ \xi \in \mathbb{R}^n.$$

On the other hand, since the integral representation of L_f in (27) is absolutely convergent, for every j we have the following uniform convergence on any compact subset K of Ω_j:

$$h_k(t,y) := \int_{|\xi|\leq k} e^{i\langle y,\xi\rangle} \widehat{f}(\xi) L^\xi(t)\, d\xi \overset{k\longrightarrow\infty}{\longrightarrow} L_f(t,y), \qquad \text{for } (t,y) \in K \subset \Omega_j,$$

Note that

$$\begin{aligned}\Delta^2 h_k(t,y) &= \int_{|\xi|\leq k} e^{i\langle y,\xi\rangle} \widehat{f}(\xi) \left(\frac{\partial^2}{\partial t^2} - |\xi|^2\right)^2 L^\xi(t)\, d\xi \\ &= 0 \qquad \text{for } (t,y) \in \Omega_j.\end{aligned}$$

Thus in every strip Ω_j we have a sequence of biharmonic functions which converges uniformly on compact sets to L_f. According to the classical results of Nicolescu [13] on mean value properties for polyharmonic functions (see also Avanissian [2]), it follows that L_f is biharmonic in the strip Ω_j, as required. Thus L_f satisfies all properties of a polyspline according to Definition 1.

5. The interpolation conditions (26) follow from the integral representation (27) and the inversion formula for the Fourier transform which is valid since f and $\widehat{f}$ are in L_1.

■

Remark 8 *1. Let us note that if we impose more regularity on f (e.g. in the sense of Hölder space smoothness) by requiring $f \in C^{4+r}(\mathbb{R}^n)$ for some non–integer r, then using $L_f \in C^2\left(\mathbb{R}^{n+1}\right)$ and applying the classical "regularity* *up to the boundary" results in [1] (also Theorem 6.19 in [3]) we will obtain* $L_f \in C^{4+r}\left(\overline{S_j}\right).$

2. Let us note that an uniqueness result in Theorem 7 needs further specification of the class of polysplines to be considered and would represent another serious piece of research. This is mainly due to the non–trivial character of the uniqueness theorems for PDE boundary value problems in unbounded domains.

3.2 The Lagrange scheme

In this subsection we assume that the sequence of C^2-continuous functions $f_j : \mathbb{R}^n \to \mathbb{C}$, for $j \in \mathbb{Z}$, is given, and we are looking for a cardinal biharmonic polyspline S on strips which solves the interpolation problem

$$S(j, y) = f_j(y) \qquad \text{for } y \in \mathbb{R}^n, \quad j \in \mathbb{Z}. \tag{28}$$

The main result below proves the existence of such an interpolation biharmonic polyspline under some growth conditions for the data functions, by means of the Lagrange-type representation

$$S(t, y) = \sum_{j=-\infty}^{\infty} L_{f_j}(t - j, y), \tag{29}$$

where each function L_{f_j} is a fundamental biharmonic polyspline as in the previous subsection.

We will need the following technical result found for example in Schoenberg [16], or [6, p. 297, Lemma 15.3].

Proposition 9 *Let $\gamma \geq 0$ and $\varepsilon > 0$. Then there exists a constant $D(\varepsilon, \gamma) > 0$ and a constant R_0 such that for all $x \in \mathbb{R}$ with $|x| \geq R_0$ the following inequality holds:*

$$\sum_{j=-\infty}^{\infty} |j|^{\gamma} e^{-\varepsilon|x-j|} \leq D(\varepsilon, \gamma) |x|^{\gamma}.$$

We come to the polyspline generalisation of the Schoenberg one–dimensional result.

Theorem 10 *Let the data functions f_j be given such that $f_j \in B_2(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, and assume that the following growth condition holds,*

$$\|f_j\|_2 \leq C\left(1 + |j|^{\gamma}\right) \qquad \textit{for all } j \in \mathbb{Z}, \tag{30}$$

for some $\gamma \geq 0$. Then there exists a biharmonic polyspline S on strips in the sense of Definition 1, satisfying the interpolation conditions (28), as well as the growth estimate

$$|S(t, y)| \leq D\left(1 + |t|^{\gamma}\right) \qquad \textit{for all } y \in \mathbb{R}^n.$$

Proof. **1.** For each j, define the fundamental biharmonic polyspline $L_{f_j}(t,y)$ by Theorem 7, and let S be given by the series in the right-hand side of (29). We have to see that the series is absolutely convergent. Applying the inequality of Theorem 7, we obtain the estimate

$$|S(t,y)| \leq \sum_{j=-\infty}^{\infty} \left|L_{f_j}(t-j,y)\right| \leq C \sum_{j=-\infty}^{\infty} e^{-\eta_1|t-j|} \|f_j\|_2 .$$

Now applying Proposition 9, the growth assumption (30) implies

$$|S(t,y)| \leq CC_2 \sum_{j=-\infty}^{\infty} e^{-\eta_1|t-j|} \left(1+|j|^{\gamma}\right) \leq CD(\eta_1,s)\cdot|t|^{\gamma} .$$

The last estimate proves the absolute convergence of the defining series for S; since this convergence is locally uniform, it proves that S is continuous and polyharmonic on every strip Ω_j.

2. We have to prove the differentiability of S up to the order 2. For every multi–index α with $|\alpha| \leq 2$ we differentiate formally the series and apply Theorem 7 (ii); we obtain

$$\begin{aligned}\left|D^{\alpha}_{t,y}S(t,y)\right| &\leq \sum_{j=-\infty}^{\infty} \left|D^{\alpha}_{t,y}L_{f_j}(t-j,y)\right| \leq C \sum_{j=-\infty}^{\infty} e^{-\eta_1|t-j|} \|f_j\|_2 \\ &\leq C_3 \sum_{j=-\infty}^{\infty} e^{-\eta_1|t-j|}\left(1+|j|^{\gamma}\right) \\ &\leq C_4\left(1+|t|^{\gamma}\right).\end{aligned}$$

The last proves the convergence of the integral representing the derivative. ■

By imposing more regularity on the data, we obtain more regular polysplines as in Remark 8.

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