MARKET DYNAMICS

MATHEMATICAL MARKET MODELS

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Contents

1 Preliminaries

Our goal is to build a mathematical model of a market. That means that we should associate with what we observe on the market some mathematical objects and relations and build a theory in such a way that there should be no contradiction between the predictions of the theory and the observations.

We start with an intuitive analysis of what we observe on the market in order to get some suggestion about how to postulate later the mathematical model. We consider the society as constituted from some units that exchange some items. The units are individuals, households, firms, public structures, etc. We shall call them *traders*. The items are commodities, services, documents, etc. We shall call them *assets*. They are measured by weight, volume, duration, counting, etc. Exchange means change of the ownership of the assets. The ownership of something means the *right* to handle with this thing. We take the notion of *ownership* as basic. We consider the set $\mathfrak{M}$ of all assets. If two assets are exchangeable (by all traders) we consider them being in a relation which is obviously an equivalence relation, thus it foliates $\mathfrak{M}$ into equivalence classes. The assets being measurable, we observe on the market that there exists some asset say e such that any asset of $\mathfrak{M}$ is exchangeable with some amount of e. This enables us to establish an one to one correspondence between the equivalence classes and the real numbers. The number corresponding to a given class will be called *price* of the assets in this class.

The exchange and the prices we observe on the market depend, however, on multiple circumstances and hidden parameters that we cannot measure and take into account exactly. That is why an adequate description of the market should be stochastic. Then the prices of the assets become random variables on some probability space and *two assets are exchangeable when the expectations of their prices are equal.*

Let us now add to our considerations *the time*, i.e. consider the dynamics of the market. We start with the deterministic study, i.e. we take away the randomness. There should be some variables characterizing the change in time of the prices and their interrelations. Such variable is the market interest short rate $r(t)$ which is the mean percentage speed of change of the prices. Then the price $s(t_1)$ of an asset at time t_1 is the discounted price $s(t_2)$ of the asset at a future time t_2, i.e.

$$s(t_1) = \exp\left[-\int_{t_1}^{t_2} r(\tau)\,d\tau\right] s(t_2), \quad t_1 < t_2$$

in continuous time assessment.

In stochastic consideration the prices become random variables depending on the time, revealed by an associated information flow, i.e. stochastic processes adapted to a filtration $\{\mathcal{F}_t\}$ of the probability space. At a moment t_1 the price $s(t_1)$ is revealed by the information $\mathcal{F}_{t_1}$, so

$$\mathbf{E}[s(t_1)|\mathcal{F}_{t_1}] = s(t_1)\,.$$

However the discounted price $\exp\left[-\int_{t_1}^{t_2} r(\tau)\,d\tau\right] s(t_2)$ is random with respect to $\mathcal{F}_{t_1}$ with expectation $\mathbf{E}\left[\exp\left[-\int_{t_1}^{t_2} r(\tau)\,d\tau\right] s(t_2)|\mathcal{F}_{t_1}\right]$. Since $s(t_1)$ and $\exp\left[-\int_{t_1}^{t_2} r(\tau)\,d\tau\right] s(t_2)$ are exchangeable their expectations should be equal (as we mentioned above), i.e.

$$\mathbf{E}[s(t_1)|\mathcal{F}_{t_1}] = s(t_1) = \mathbf{E}\left[\exp\left[-\int_{t_1}^{t_2} r(\tau)\,d\tau\right] s(t_2)|\mathcal{F}_{t_1}\right]\,.$$

This is equivalent to

$$\mathbf{E}\left[\exp\left[-\int_{0}^{t_2} r(\tau)\,d\tau\right] s(t_2)|\mathcal{F}_{t_1}\right] = \exp\left[-\int_{0}^{t_1} r(\tau)\,d\tau\right] s(t_1),\ t_1 < t_2$$

which means that the *process* $\exp\left[-\int_0^t r(\tau)\,d\tau\right] s(t)$ *is a martingale.*

Thus we came to the conclusion that the model of the market should be based on some "intrinsic" probability measure Q such that the discounted price process $\exp\left[-\int_0^t r(\tau)\,d\tau\right] s(t)$ be a martingale with respect to Q. There is no need however the measure Q to coincide with the statistically observed measure P. We shall only require that Q and P be equivalent.

It is possible to build a model using one or other class of random processes. The model we are going to study is based on Ito processes. Then the discounted price process will be an Ito process which is a martingale with respect to Q but which may not be a martingale with respect to P. Thus the question arises: when an Ito process may be transformed into a martingale by an equivalent change of measure? The answer, as we know, is given by the Girsanov's theorem. The two measures should be connected by the Girsanov transformation, i.e. the Radon–Nicodym derivative of Q w.r.t. P should be a stochastic exponent.

Let us finally formulate the results of our preliminary considerations.

The model of a market should be based on two equivalent probability measures P and Q and some stochastic processes satisfying the following conditions:

1. P is statistically observed and the Radon–Nicodym derivative of Q w.r.t. P is a stochastic exponent.

2. There are two types of processes on the market—the price processes and the processes characterizing the state of the market (like, for instance, the short rate process). We suppose both of them are Ito processes.

3. The basic interrelation involving all the ingredients of the market is that the discounted price processes are martingales with respect to Q.

2 General Market Model

We point out the mathematical objects and relations inhering the model.

1. A measurable space $(\Omega, \mathcal{F})$ and two equivalent probability measures P and Q on it. We call the measure P *statistical* or *observable* and the measure Q—*risk-neutral* or *martingale*. The two measures are connected by the relation

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left[-\int_0^T \eta(s)\,\mathrm{d}B^P(s) - \frac{1}{2}\int_0^T \eta(s)\cdot\eta(s)\,\mathrm{d}s\right] \tag{2.1}$$

where $B^P(t) = (B_1^p(t), \ldots, B_d^p(t))$, $d \geq 1$, is a Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ in the time interval $[0, T]$

generating the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$; $\eta(t) = (\eta_1(t), \dots, \eta_d(t))$, $t \in [0, T]$, is a $\mathcal{F}_t$-adapted stochastic peocess called *risk-premium* or *market price of risk* and satisfying the condition (Novikov)

$$\mathbf{E}^p \left[\exp \left(\frac{1}{2} \int_0^T \eta(s) \cdot \eta(s) \, \mathrm{d}s \right) \right] < \infty. \tag{2.2}$$

2. An Ito process $\widetilde{Z}(t) = (r(t), Z_1(t), \dots, Z_k(t)) = (r(t), Z(t))$, $t \in [0, T]$, called *state-process.* The process $r(t)$ for which $\int_0^T r(t) \, \mathrm{d}t < \infty$ is called *short-rate process.*

3. An Ito process $\widetilde{S}(t) = (s_0(t), s_1(t), \dots, s_n(t)) = (s_0(t), s(t))$ called *price process.* $S(t)$ satisfies the equation

$$\mathrm{d}S(t) = \mu(t) \, \mathrm{d}t + \sigma(t) \, \mathrm{d}B^P(t) \tag{2.3}$$

   where $\mu : [0, T] \times \Omega \to \mathbb{R}^n$, $\sigma : [0, T] \times \Omega \to \mathbb{R}^{n \times d}$.

   $S_0(t)$ satisfies the equation

$$\mathrm{d}S_0(t) = -r(t) S_0(t) \, \mathrm{d}t \, . \tag{2.4}$$

   Thus

$$S_0(t) = S_0(0) \exp \left[- \int_0^t r(\tau) \, \mathrm{d}\tau \right] \, .$$

4. The process $S_0(t)S(t) = (S_0 S_1, \dots, S_0 S_n)$ is a *Q-martingale.*

These are the postulates defining the model. The basic relation in the model is 4. There are, of course, two questions to be answered:

1. Do there exist mathematical objects satisfying these postulates, i.e is our model consistent?

2. How do we establish the correspondence between the mathematical model and the observed reality?

Before discussing these questions we shall prove two propositions.

Proposition 1 *The process*

$$B^Q(t) \doteqdot \int_0^t \eta(\tau)\mathrm{d}\tau + B^P(t) \tag{2.5}$$

is a Q*-Brownian motion (consequently a* Q*-martingale).*

Proof. It follows from Girsanov's theorem.

Proposition 2 *The following equality holds*

$$\mu(t) - r(t)S(t) = \sigma(t)\eta(t)\,. \tag{2.6}$$

Proof. We apply the Ito formula to the process $Y \doteqdot S_0 S$ which is a Q-martingale

$$\begin{aligned} \mathrm{d}Y &= S\,\mathrm{d}S_0 + S_0\,\mathrm{d}S + \mathrm{d}S_0\,\mathrm{d}S = -SrS_0\,\mathrm{d}t + S_0\,\mathrm{d}S - rS_0\,\mathrm{d}t\,\mathrm{d}S \\ &= -SrS_0\,\mathrm{d}t + S_0(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}B^P) - rS_0(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}B^P)\,\mathrm{d}t \\ &= S_0\left[(\mu - rS)\,\mathrm{d}t + \sigma\,\mathrm{d}B^P\right]\,. \end{aligned} \tag{2.7}$$

We substitute $\mathrm{d}B^P = \mathrm{d}B^Q - \eta\,\mathrm{d}t$ (following from (2.5)) into (2.7) and obtain

$$\mathrm{d}Y = S_0[(\mu - rS - \sigma\eta)\,\mathrm{d}t + \sigma\,\mathrm{d}B^Q]\,.$$

Since Y is a Q-martingale its drift should be zero, so

$$\mu - rS - \sigma\eta = 0\,. \quad \blacksquare$$

Thus we obtain that a necessary condition for our model to be consistent is that the equation (2.6) has a solution satisfying (2.2). (In fact this is also sufficient.)

Let us now see how the equation (2.6) looks out in a well-known particular case. Let $B(t)$ be one-dimensional Brownian motion and let $S(t) = (S_1(t), \ldots, S_n(t))$ satisfies the equations

$$\mathrm{d}S_i = \mu_i S_i\,\mathrm{d}t + \sigma_i S_i\,\mathrm{d}B^P, \qquad i = 1, \ldots, n\,,$$

where μ_i, σ_i, $i = 1, \ldots, n$, are one-dimensional processes. The equation (2.6) becomes

$$\mu_i S_i - rS_i = S_i\sigma_i\eta\,, \qquad i = 1, \ldots, n,$$

η being one-dimensional process. Here from

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \cdots = \frac{\mu_n - r}{\sigma_n} = \eta\,.$$

These are the well-known non-arbitrage relations of CAPM, η being the risk premium (market price of risk).

Let us now discuss the question we asked above: how do we establish the correspondence between the model and the observed reality. The price processes and the state processes are observed on the market and may be statistically measured, i.e. their frequency characteristics can be found out. In this way we find μ, σ, S, r and P. Then, solving the equation (2.6) (if possible), we find η. The relation (2.1) gives us the measure Q.

3 Trading Strategies

We are going to consider collections of assets calling them *portfolios*. Let n assets whose price processes $S(t) = (S_1(t), \ldots, S_n(t))$ are Ito processes in some probability space $(\Omega, \mathcal{F}, P)$ be given, i.e.

$$S(t) = S(0) + \int_0^t \mu(\tau)\mathrm{d}\tau + \int_0^t \sigma(\tau)\mathrm{d}B^P(\tau), \qquad t \in [0, T]\,.$$

Let $\{\mathcal{F}_t\}$ be the filtration generated by the Brownian motion $B^P(t)$. Suppose $\theta(t) = (\theta_1(t), \ldots, \theta_n(t))$ is an $\mathcal{F}_t$-adapted process. We can think that $\theta(t, \omega)$ specifies at each state ω and time t the number of units of the assets hold in a portfolio. That is why we call $\theta(t)$ *trading strategy*. We admit that

$$\theta \in H^2(0, T) \doteqdot \left\{ \theta : \int_0^T \theta^2(t)\,\mathrm{d}t < \infty \text{ a.s.},\ \mathbf{E}\left[\int_0^T \theta^2(t)\,\mathrm{d}t\right] < \infty \right\}.$$

We define the stochastic integral $\int_0^t \theta(\tau)\,\mathrm{d}S(\tau)$ as the Ito process given by

$$G(t) \doteqdot \int_0^t \theta(\tau)\,\mathrm{d}S(\tau) \doteqdot \int_0^t \theta(\tau)\mu(\tau)\,\mathrm{d}\tau + \int_0^t \theta(\tau)\sigma(\tau)\,\mathrm{d}B(\tau)\,.$$

We call this process *gain process* of the strategy $\theta(t)$. In the particular case when $\theta(t)$ is piecewise constant the gain process is

$$G(t) = \sum_{j=0}^{n-1} \theta(t_j)(S(t_{j+1}) - S(t_j))$$

where

$$0 < t_0 < t_1 < \cdots < t_n = T .$$

The *value* at time t (value process) of the strategy (portfolio) $\theta(t)$ is defined by

$$V(t) \doteqdot \theta(t) \cdot S(t) = \sum_{j=1}^{n} \theta_j(t) S_j(t) .$$

A trading strategy $\theta(t)$ is called *self-financing* iff

$$V(t) = V(0) + \int_0^t \theta(\tau)\, \mathrm{d}S(t) . \tag{3.1}$$

A self-financing strategy $\theta(t)$ is called *arbitrage* iff

$$V(0) = 0, \qquad V(T) \geq 0 \text{ a.s.}, \qquad \mathbf{P}(V(T) > 0) > 0 .$$

We want to prove that there are no arbitrages in our model but first we prove an auxiliary proposition.

Denote the discounted price process $S_0(t)S(t)$ by $\overline{S}(t)$.

Proposition *A trading strategy $\theta(t)$ is self-financing w.r.t. $S(t)$ iff it is self-financing w.r.t. $\overline{S}(t)$.*

Proof. Let $\theta(t)$ be self-financing w.r.t. $S(t)$, hence

$$\mathrm{d}V(t) = \theta(t) \cdot \mathrm{d}S(t)$$

and let $\overline{V}(t) \doteqdot S_0(t)V(t)$. We apply the Ito formula

$$\begin{aligned} \mathrm{d}\overline{V}(t) &= V(t)\, \mathrm{d}S_0(t) + S_0(t)\, \mathrm{d}V(t) + \mathrm{d}S_0(t)\, \mathrm{d}V(t) \\ &= \theta(t) \cdot S(t)\, \mathrm{d}S_0(t) + S_0(t)\theta(t) \cdot \mathrm{d}S(t) + \mathrm{d}S_0(t)\, \mathrm{d}V(t) . \end{aligned} \tag{3.2}$$

Then we have

$$\mathrm{d}S_0 = -rS_0\,\mathrm{d}t, \quad \mathrm{d}S = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B(t), \quad \mathrm{d}S\mathrm{d}S_0 = 0,$$
$$\mathrm{d}V = \theta_0\,\mathrm{d}S, \quad \mathrm{d}V\mathrm{d}S_0 = \theta.\mathrm{d}S\mathrm{d}S_0 = 0,$$
$$\mathrm{d}\overline{V} = \theta.(S\,\mathrm{d}S_0 + S_0\,\mathrm{d}S) = \theta.\mathrm{d}(S_0S) = \theta.\mathrm{d}\overline{S}.$$

Thus, $\theta(t)$ is self-financing w.r.t. $\overline{S}(t)$. We proved that if $\theta(t)$ is self-financing w.r.t. $S(t)$ then it is self-financing w.r.t. $\overline{S}(t)$. Since $S(t) = S_0^{-1}(t)\overline{S}(t)$ the reverse is also true.

Corollary *A trading strategy is an arbitrage w.r.t.* $S(t)$ *iff it is an arbitrage w.r.t.* $\overline{S}(t)$.

Proof. Let θ be an arbitrage w.r.t. $S(t)$. Then

$$V(0) = 0, \qquad V(T) \geq 0 \text{ a.s.}, \qquad \mathbf{P}(V(T) > 0) > 0\,.$$

Hence

$$S_0(0)V(0) = 0, \qquad S_0(T)V(T) \geq 0 \text{ a.s.}, \qquad \mathbf{P}(S_0(T)V(T) > 0) > 0\,.$$

i.e.

$$\overline{V}(0) = 0, \qquad \overline{V}(T) \geq 0 \text{ a.s.}, \qquad \mathbf{P}(\overline{V}(T) > 0) > 0\,.$$

Hence, θ is an arbitrage w.r.t. $\overline{S}(t)$.

Theorem *There is no arbitrage in the market-model defined above.*

Proof. Under the martingale measure Q the discounted price process is

$$\overline{S}(t) = \overline{S}(0) + \int_0^t \overline{\sigma}(\tau)\,\mathrm{d}B^Q(\tau),$$

$\overline{\sigma}$ being the volatility of $\overline{S}$. (The drift is zero since $\overline{S}(t)$ is a Q-martingale.) Hence, the gain process is

$$\int_0^t \theta(\tau)\,\mathrm{d}\overline{S}(\tau) = \int_0^t \theta(\tau)\overline{\sigma}(\tau)\,\mathrm{d}B^Q(\tau)\,.$$

Consequently

$$\mathbf{E}^Q\left[\int_0^T \theta(\tau)\,\mathrm{d}\overline{S}(\tau)\right] = 0 \tag{3.3}$$

since the Ito integral $\int_0^t \theta(\tau)\sigma(\tau)\,\mathrm{d}B^Q(\tau)$ is a martingale. From (3.1) and (3.3) we obtain

$$(3.4)\quad \overline{V}(0) = \mathbf{E}^Q[\overline{V}(0)] = \mathbf{E}^Q\left[\overline{V}(T) - \int_0^T \theta(\tau)\,\mathrm{d}\overline{S}(\tau)\right] = \mathbf{E}^Q[\overline{V}(T)]\,.$$

If θ is an arbitrage w.r.t. $\overline{S}(t)$ we should have

$$(3.5)\qquad \overline{V}(0) = 0, \qquad \overline{V}(T) \geq 0 \text{ a.s.}, \qquad \mathbf{P}(\overline{V}(T) > 0) > 0\,.$$

Relations (3.4) and (3.5) imply

$$\mathbf{E}^Q[\overline{V}(T)] = \overline{V}(0) = 0\,.$$

Thus, $\overline{V}(T)$ should observe all three relations

$$\overline{V}(T) \geq 0, \qquad \mathbf{P}(\overline{V}(T) > 0) > 0, \qquad \mathbf{E}^Q[\overline{V}(T)] = 0$$

what is not possible since P and Q are equivalent. There is no arbitrage w.r.t. $\overline{S}(t)$. Hence, there is no arbitrage w.r.t. $S(t)$.

This theorem shows that if there exists a martingale measure Q equivalent to the statistically observable measure P there is no arbitrage on the market. More or less the reverse is also true but we shall not dwell on it.

In what follows we consider some particular cases of the general market model we outlined above.

4 Black–Scholes model

We specify the objects and relations of the market model in the following way.

The short rate process r is constant.

There are three securities whose price process $(S_0, S) = (S_0, S_1, S_2)$ satisfies the equations

$$(4.1)\qquad \begin{aligned} \mathrm{d}S_0 &= -rS_0\,\mathrm{d}t, \\ \mathrm{d}S_1 &= \mu S_1\,\mathrm{d}t + \sigma S_1\,\mathrm{d}B^P \end{aligned}$$

where μ, σ are constants and B^P is one-dimensional Brownian motion. S_2 is a call option on S_1.

It follows from (2.6) that

$$\mu S_1 - rS_1 = \sigma S_1 \eta$$

hence the risk premium is

$$\eta = \frac{\mu - r}{\sigma}\,. \tag{4.2}$$

This is the Black–Scholes market model. Now we are going to deduce a formula for the price of the option $S_2(t)$.

Suppose the expiration of the option S_2 is T and the strike is K. Then

$$S_2(T) = (S_1(T) - K)^+.$$

Because S_0S_2 is a martingale under Q we have

(4.3)

$$S_2(t) = S_0^{-1}\mathbf{E}^Q[S_0(T)S_2(T)|\mathcal{F}_t] = \exp[-r(T-t)]\mathbf{E}^Q[(S_1(T)-K)^+|\mathcal{F}_t]\,.$$

From (4.1) we get

$$S_1 = \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma B^P\right]\,. \tag{4.4}$$

(This is verified by the Ito formula

$$\mathrm{d}S_1 = \frac{\partial g}{\partial t}\,\mathrm{d}t + \frac{\partial g}{\partial x}\,\mathrm{d}X + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(\mathrm{d}X)^2\,;$$

taking

$$X(t) = B^P(t), \quad g(t,x) = \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma x\right], \quad S_1(t) = g(X(t),t),$$

we prove that $S_1(t)$ satisfies (4.1).)

We substitute $B^P(t) = B^Q(t) - \int_0^t \eta\,\mathrm{d}\tau = B^Q(t) - \eta t$ into (4.4) and obtain, in view of (4.2)

$$\begin{aligned} S_1(t) &= \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma\left(B^Q(t) - \eta t\right)\right] \\ &= \exp\left[\left(r - \tfrac{1}{2}\sigma^2\right)t + \sigma B^Q(t)\right]\,. \end{aligned} \tag{4.5}$$

Here from

$$\begin{aligned} &\mathbf{E}^Q[(S_1(T) - K)^+|\mathcal{F}_t] \\ &= \mathbf{E}^Q\left[\left(\exp\left(\left(r - \tfrac{1}{2}\sigma^2\right)T + \sigma B^Q(T)\right) - K\right)^+ \middle|\mathcal{F}_t\right] \\ &= \mathbf{E}^Q\left[\left(\exp\left(\left(r - \tfrac{1}{2}\sigma^2\right)T + \sigma B^Q(T)\right) - K\right)^+ \middle|B^Q(t) = y\right]\,. \end{aligned} \tag{4.6}$$

We have

$$(4.7)\quad \text{distribution density of } B^Q(T)|_{B^Q(t)=y} = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left[-\frac{(x-y)^2}{2(T-t)}\right] .$$

From (4.3), (4.6) and (4.7), recalling the formula for the expectation of a function of some random variable when the distribution density of the variable is given, we obtain

$$(4.8)\quad S_2(t) = \mathrm{e}^{-r(T-t)} \times \int_{-\infty}^{\infty} \left(\exp\left[\left(r - \tfrac{1}{2}\sigma^2\right) T + \sigma x\right] - K\right)^+ \frac{1}{\sqrt{2\pi(T-t)}} \exp\left[-\frac{(x-y)^2}{2(T-t)}\right] \mathrm{d}x .$$

From (4.5) we have

$$S_1(t)|_{B^Q(t)=y} = \exp\left[\left(r - \tfrac{1}{2}\sigma^2\right) t + \sigma y\right] .$$

Here from

$$(4.9)\qquad y = \frac{1}{\sigma}\left[\ln S_1(t) - \left(r - \tfrac{1}{2}\sigma^2\right) t\right] .$$

Thus, the price of the option $S_2(t)$ is given by the expression (4.8) where y is (4.9). In what follows we shall transform this formula by pure analytical calculations.

First of all, we mention that the integrand in (4.8) differs from zero only for

$$x \geq \frac{1}{\sigma}\left[\ln K + \left(\tfrac{1}{2}\sigma^2 - r\right) T\right]$$

Thus, equation (4.8) gives

$$(4.10)\qquad S_2(t) = I_1 - I_2$$

where

(4.11)

$$I_1 = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left[rt - \tfrac{1}{2}\sigma^2 T\right] \int\limits_{\frac{1}{\sigma}\left[\ln K + \left(\frac{1}{2}\sigma^2 - r\right)T\right]}^{\infty} \exp\left[\sigma x - \frac{(x-y)^2}{2(T-t)}\right] \mathrm{d}x \,,$$

(4.12)

$$I_2 = \frac{K}{\sqrt{2\pi(T-t)}} \exp[-r(T-t)] \int\limits_{\frac{1}{\sigma}\left[\ln K + \left(\frac{1}{2}\sigma^2 - r\right)T\right]}^{\infty} \exp\left[-\frac{(x-y)^2}{2(T-t)}\right] \mathrm{d}x \,.$$

Changing the variables in the integrals in (4.11) and (4.12) we shall try to express $S_2(t)$ by the normal distribution function

$$N(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\left[-\frac{z^2}{2}\right] \mathrm{d}z \,.$$

Starting with I_1 we verify the algebraic identity

$$\text{(4.13)} \qquad \sigma x - \frac{(x-y)^2}{2(T-t)} = \sigma y + \frac{1}{2}\sigma^2(T-t) - \frac{(x-y-\sigma(T-t))^2}{2(T-t)} \,.$$

We substitute (4.13) in (4.11) and obtain

$$I_1 = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left[\left(r - \tfrac{1}{2}\sigma^2\right)t + \sigma y\right] \times \int_{\frac{1}{\sigma}\left[\ln K + \left(\frac{1}{2}\sigma^2 - r\right)T\right]}^{\infty} \exp\left[\frac{-[x-y-\sigma(T-t)]^2}{2(T-t)}\right] \mathrm{d}x \,.$$

Now we change the variable x to z by

$$\frac{x-y-\sigma(T-t)}{\sqrt{T-t}} = z \,, \qquad \mathrm{d}x = \sqrt{T-t}\,\mathrm{d}z \,,$$

$$x = \frac{1}{\sigma}\left[\ln K - \left(r - \tfrac{1}{2}\sigma^2\right)T\right] \sim z = \frac{1}{\sigma\sqrt{T-t}}\left[\ln K - \left(r - \tfrac{1}{2}\sigma^2\right)T\right] - \frac{y}{\sqrt{T-t}} - \sigma\sqrt{T-t} \,,$$

$$I_1 = \frac{1}{\sqrt{2\pi}} \exp\left[\left(r - \tfrac{1}{2}\sigma^2\right) t + \sigma y\right] \times \int_{\frac{1}{\sigma\sqrt{T-t}}\left[\ln K - \left(r - \frac{1}{2}\sigma^2\right) T\right] - \frac{y}{\sqrt{T-t}} - \sigma\sqrt{T-t}}^{\infty} \exp\left[-\frac{z^2}{2}\right] \mathrm{d}z \,. \tag{4.14}$$

We substitute y from (4.9) into (4.14) and obtain

$$I_1 = S_1(t) N \left(\frac{1}{\sigma\sqrt{T-t}} \left[\ln \frac{S_1(t)}{K} + \left(r + \tfrac{1}{2}\sigma^2\right)(T-t) \right] \right) . \tag{4.15}$$

Now we deal with I_2. We change the variable x to z by

$$\frac{x-y}{\sqrt{T-t}} = z \,, \qquad \mathrm{d}x = \sqrt{T-t}\, \mathrm{d}z \,,$$

$$x = \frac{1}{\sigma}\left[\ln K + \left(\tfrac{1}{2}\sigma^2 - r\right) T\right] \sim z = \frac{1}{\sigma\sqrt{T-t}} \left[\ln \frac{K}{S_1(t)} - \left(r - \tfrac{1}{2}\sigma^2\right)(T-t) \right]$$

(here we used (4.9)) and obtain

$$\begin{aligned} I_2 &= K \mathrm{e}^{-r(t-t)} \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sigma\sqrt{T-t}}\left[\ln \frac{K}{S_1(t)} - \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right]}^{\infty} \exp\left[-\frac{z^2}{2}\right] \\ &= K \mathrm{e}^{-r(T-t)} N \left(\frac{1}{\sigma\sqrt{T-t}} \left[\ln \frac{S_1(t)}{K} + \left(r - \tfrac{1}{2}\sigma^2\right)(T-t) \right] \right) . \end{aligned} \tag{4.16}$$

We substitute (4.15), (4.16) into (4.10) and finally obtain

$$\begin{aligned} S_2(t) = S_1(t) N &\left(\frac{\ln \frac{S_1(t)}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \right) \\ &- K \mathrm{e}^{-r(T-t)} N \left(\frac{\ln \frac{S_1(t)}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \right) . \end{aligned} \tag{4.17}$$

This is the famous Black–Scholes formula for option pricing!

Because of the connection (4.5) between $S_1(t)$ and $B^Q(t)$ we can replace conditioning to $B^Q(t) = y$ in the expectations with conditioning to $S_1(t) = s$. Thus, we obtain from (4.3)

$$\begin{aligned} S_2(t) &= \mathrm{e}^{-r(T-t)} \mathbf{E}^Q \left[S_1(T) - K)^+ |_{B^Q(t)=y} \right] \\ &= \mathrm{e}^{-r(T-t)} \mathbf{E}^Q \left[S_1(T) - K)^+ |_{S_1(t)=S} \right] \\ &= \mathrm{e}^{-r(T-t)} \mathbf{E}^Q_{s,t} \left[(S_1(T) - K)^+ \right] \stackrel{\text{def}}{=} u(t,s) \,. \end{aligned}$$

The relation (4.5) implies that the process $S_1(t)$ which satisfies the stochastic differential equation (4.1) with respect to the measure P satisfies also a similar equation with respect to the measure Q

$$\mathrm{d}S_1 = rS_1\,\mathrm{d}t + \sigma S_1\,\mathrm{d}B^Q\,. \tag{4.18}$$

Thus, we conclude that the function $u(t,s)$ (the price of the option as a function of the time t and of the price s of the underlying asset at that time) is the Feynman–Kac solution of the PDE

$$\frac{\partial u}{\partial t} + rs\frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} - ru = 0 \tag{4.19}$$

satisfying the Cauchy condition

$$u(T,s) = (s-K)^+\,. \tag{4.20}$$

This is the famous Black–Scholes equation!

It is possible to solve the Cauchy problem (4.19), (4.20) in the usual analytical way (separating and changing variables) and we shall, of course, obtain again the Black–Scholes formula (4.17).

5 Term Structure of Interest Rates. One-Factor Models

At this point we shall specify the state process of the general market model and especially the short rate $r(t)$.

Let a market model as described above be given. The process (infinite dimensional) $\{\Lambda^\tau\}_{\tau\in[0,\infty)}$ where

$$\Lambda^\tau(t) \doteqdot \mathbf{E}^Q\left[\exp\left(-\int_t^\tau r(u)\,\mathrm{d}u\right)\middle|\,\mathcal{F}_t\right] = \mathbf{E}_t^Q\left[\exp\left(-\int_t^\tau r(u)\,\mathrm{d}u\right)\right] \tag{5.1}$$

is called *discount factor* or *term structure of interest rate*. $\Lambda^\tau(t)$ is the price at time t of a zero coupon bond paying one unit at maturity τ. The process

$$y_t(s) = -\frac{\log \Lambda^{t+s}(t)}{s}$$

is called the *yield curve*.

Obviously $\Lambda^\tau(t)$ and the short rate are closely related so in what follows we shall study various specifications of $r(t)$.

One-Factor Term-Structure Models

The general model is

$$\mathrm{d}r(t) = \mu(r(t),t)\,\mathrm{d}t + \sigma(r(t),t)\,\mathrm{d}B^Q(t) \tag{5.2}$$

where $\mu : \mathbb{R}\times[0,T] \to \mathbb{R}$, $\sigma : \mathbb{R}\times[0,T] \to \mathbb{R}^d$, $B^Q(t)$ is the standard Brownian motion in $\mathbb{R}^d$ under Q.

The one-factor models are so named because the short rate $r(t)$ is the only state variable or "factor" on which the yield curve depends.

We apply the Feynman–Kac formula for $h(x,s) \equiv 0$, $g(x) \equiv 1$, $R(x,t) \equiv x$ and obtain that the function

$$u(x,t;\tau) = \mathbf{E}^Q_{x,t}\left[\exp\left(-\int_t^\tau r(u)\,\mathrm{d}u\right)\right] \tag{5.3}$$

solves the equation

$$\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma(x,t)\sigma(x,t)^T\frac{\partial^2 u}{\partial x^2} - xu = 0 \tag{5.4}$$

with the boundary condition $(x,t) \in \mathbb{R}\times[0,\tau)$,

$$u(x,t;\tau)|_{t=\tau} = u(x,\tau;\tau) \equiv 1. \tag{5.5}$$

Comparing (5.3) and (5.1) we see that

$$\begin{aligned} u(x,t;\tau) &= \mathbf{E}^Q_{x,t}\left[\exp\left(-\int_t^\tau r(u)\,\mathrm{d}u\right)\right] \\ &= \mathbf{E}^Q\left[\exp\left(-\int_t^\tau r(u)\,\mathrm{d}u\right)\middle| r(t) = x\right] \\ &= \mathbf{E}^Q\left[\exp\left(-\int_t^\tau r(u)\,\mathrm{d}u\right)\middle| \mathcal{F}(t)\right] = \Lambda^\tau(t)\,. \end{aligned}$$

That is $\Lambda^\tau(t) = u(r(t),t;\tau)$.

Thus, for the one-factor models the term-structure can be computed by solving the differential equation problem (5.4)–(5.5). In order to exist a solution it is enough that μ and σ satisfy Lipschitz conditions in x and have derivative μ_x, σ_x, μ_{xx} and σ_{xx} that are continuous and satisfy growth conditions in x.

Examples (subclasses) of models:

$$\mathrm{d}r(t) = [K_0(t) + K_1(t)r(t) + K_2(t)r(t)\log r(t)]\ \mathrm{d}t \\ + [H_0(t) + H_1(t)r(t)]^{\nu}\,\mathrm{d}B^Q(t)$$

where K_0, K_1, K_2, H_0, H_1 are continuous functions $[0,T] \to \mathbb{R}$ and $\nu \in [0.5, 1.5]$, $d = 1$.

Model	K_0	K_1	K_2	H_0	H_1	ν
Cox–Ingersoll-Ross	◦	◦			◦	0.5
Pearson–Sun	◦	◦		◦	◦	0.5
Dothan					◦	1.0
Brenan–Schwartz	◦	◦			◦	1.0
Merton (Ho–Lee)	◦			◦		1.0
Vasiček	◦	◦		◦		1.0
Black–Karasinski		◦	◦		◦	1.0
Constantinides–Ingersoll					◦	1.5

6 Affine Single-Factor Models

A function $g : \mathbb{R}^k \to \mathbb{R}$ is called *affine* if there are constants $a \in \mathbb{R}^1$ and $b \in \mathbb{R}^k$ such that

$$g(x) = a + b.x \quad \text{for all} \quad x \in \mathbb{R}^k.$$

Theorem *Suppose that the functions $\mu(x,t)$ and $\sigma^2(x,t)$ are affine in $x \in \mathbb{R}^1$, i.e.*

$$\mu(x,t) = K_0(t) + K_1(t)x, \qquad \sigma^2(x,t) = H_0(t) + H_1(t)x. \tag{6.1}$$

Then the solution $u(x,t;\tau) = \Lambda^{\tau}(t)$ of (5.4), (5.5) is

$$u(x,t;\tau) = \mathrm{e}^{\alpha(t)+\beta(t)x} \tag{6.2}$$

where $\alpha(t)$, $\beta(t)$ solve the following ordinary differential equation problem

$$\begin{aligned} \frac{\mathrm{d}\beta(t)}{\mathrm{d}t} &= 1 - K_1(t)\beta(t) - \tfrac{1}{2}H_1(t)\beta^2(t) \\ \frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} &= -K_0(t)\beta(t) - \tfrac{1}{2}H_0(t)\beta^2(t) \\ \alpha(t)\big|_{t=\tau} = 0&, \qquad \beta(t)\big|_{t=\tau} = 0. \end{aligned} \tag{6.3}$$

Thus the yield curve $-\dfrac{\log \Lambda^\tau(t)}{\tau - t}$ *is affine.*

Proof. Let $\alpha(t)$, $\beta(t)$ solve the problem (6.3). We shall verify that the function (6.2) solves the problem (5.4)–(5.5). We have

$$\frac{\partial u}{\partial t} = u\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \frac{\mathrm{d}\beta}{\mathrm{d}t}x\right), \qquad \frac{\partial u}{\partial x} = u\beta, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x}\beta = u\beta^2 .$$

We substitute in (5.4) and obtain that (5.4) is equivalent to

$$u\left(\frac{\partial \alpha}{\partial t} + \frac{\partial \beta}{\partial t}\right) + \mu u\beta + \tfrac{1}{2}\sigma^2 u\beta^2 - xu = 0 .$$

After dividing by u we substitute $\mathrm{d}\alpha/\mathrm{d}t$, $\mathrm{d}\beta/\mathrm{d}T$ from (6.3) and μ, σ^2 from (6.1)

$$\begin{aligned} - K_0\beta - \tfrac{1}{2}H_0\beta^2 + \left(1 - K_1\beta - \tfrac{1}{2}H_1\beta^2\right)x + (K_0 + K_1 x)\beta \\ + \tfrac{1}{2}(H_0 + H_1 x)\beta^2 - x = 0 . \end{aligned}$$

Thus the function (6.2) satisfies the equation (5.4). Let us verify the boundary condition (5.5)

$$u(x,t;\tau)|_{t=\tau} = \mathrm{e}^{\alpha(t)+\beta(t)x}\big|_{t=\tau} = \exp\left[\alpha(t)|_{t=\tau} + \beta(t)|_{t=\tau}\, x\right] = \mathrm{e}^0 = 1 . \blacksquare$$

The "affine class" of therm structure models includes:

- Vasiček (Hull–White): $\mathrm{d}r = (K_0 + K_1 r)\,\mathrm{d}t + H_0\,\mathrm{d}B^Q$
- CIR: $\mathrm{d}r = (K_0 + K_1 r)\,\mathrm{d}t + \sqrt{H_1 r}\,\mathrm{d}B^Q$
- Merton (Ho–Lee): $\mathrm{d}r = K_0\,\mathrm{d}t + H_0\,\mathrm{d}B^Q$
- Pearson–Sun: $\mathrm{d}r = (K_0 + K_1 r)\,\mathrm{d}t + \sqrt{H_0 + H_1 r}\,\mathrm{d}B^Q$

Vasiček model: $\mathrm{d}r = (K_0 + K_1 r)\,\mathrm{d}t + H_0\,\mathrm{d}B^Q$

$$\Lambda^\tau(t) = \exp\left[\alpha^\tau(t) + \beta^\tau(t)x\right], \quad \beta^\tau(t) = \frac{1}{K_1}\left(1 - \mathrm{e}^{K_1(\tau - t)}\right),$$
$$\alpha^\tau(t) = \frac{H_0^2}{2}\int_t^\tau \beta^{\tau^2}(s)\,\mathrm{d}s + K_0\int_t^\tau \beta^\tau(s)\,\mathrm{d}s .$$

CIR model: $\mathrm{d}r(t) = k(\overline{x} - r(t))\,\mathrm{d}t + C\sqrt{r(t)}\,\mathrm{d}B^Q$, $r_0 > 0$

$$\mu(x,t) = k(\overline{x} - x), \quad \sigma(x,t) = C\sqrt{x}, \quad x \geq 0,\ k > 0,\ \overline{x} > 0\,.$$

Given r_0, for fixed t, $r(t)$ has a noncentral χ^2 distribution. The expectation of $r(t)$ is

$$\mathbf{E}^Q[r(t)] = \overline{x} + \mathrm{e}^{-kt}(r_0 - \overline{x})\,.$$

It tends to $\overline{x}$ when $t \to \infty$. (We call this property reversion of $r_\tau(t)$ toward $\overline{x}$. Vasiček model can have similar property for convenient parameters.)

7 Multifactor Term Structure Models. Affine Models

State variables:

$$Z(t) = (Z_1(t), \ldots, Z_n(t)), \qquad r(t) = R(Z(t), t),$$

$$\mathrm{d}Z(t) = \mu(Z(t),t)\,\mathrm{d}t + \sigma(Z(t),t)\,\mathrm{d}B^Q(t), \qquad B^Q \text{ is } d\text{-dimensional},$$

$$Z(t) \in D \subset \mathbb{R}^k, \qquad \begin{array}{l} R : D \times [0,\infty) \to \mathbb{R} \\ \mu : D \times [0,\infty) \to \mathbb{R}^k \\ \sigma : D \times [0,\infty) \to \mathbb{R}^{k \times d}. \end{array}$$

Term structure $\Lambda^\tau(t) = u(Z(t), t; \tau)$ where

$$u(z,t;\tau) = \mathbf{E}^Q_{x,t}\left[\exp\left(-\int_t^\tau R(Z(u),u)\,\mathrm{d}u\right)\right], \quad Z(u)\big|_{u=t} = z \in \mathbb{R}^k.$$

Term structure equation:

$$\text{(7.1)} \quad \frac{\partial u}{\partial t}(z,t) + \mu(z,t)\frac{\partial u}{\partial z}(z,t) + \frac{1}{2}\,\mathrm{tr}\left[\sigma(z,t)\sigma(z,t)^T \frac{\partial^2 u}{\partial x^2}\right] - R(z,t)u(z,t) = 0,$$

$$u(z,t;\tau)|_{t=\tau} = 1, \qquad (z,t) \in D \times [0,\tau).$$

Affine Multifactor Models

For these models $R(z,t)$, $\mu(z,t)$, $\sigma(z,t)$ are affine functions of z, i.e.

$$\begin{aligned}\mu(z,t) &= K_0 + K_1 z \\ \left(\sigma(z,t)\cdot\sigma(z,t)^T\right)_{ij} &= H_{0ij} + H_{1ij}\cdot z \\ R(z,t) &= \rho_0 + \rho_1 \cdot z\end{aligned}$$

where $z \in \mathbb{R}^k$, $K_0 \in \mathbb{R}^k$, $K_1 \in \mathbb{R}^{k\times k}$, $H_{0ij} \in \mathbb{R}$, $H_{1ij} \in \mathbb{R}^k$, $i,j = 1,\dots,k$, $\rho_0 \in \mathbb{R}$, $\rho_1 \in \mathbb{R}^k$.

The yield curve in this model is also an affine function of z. It is specified in the following way.

Denote H_1 the matrix whose elements H_{1ij} are vectors in $\mathbb{R}^k$, i.e. $H_{1ij} = \left(H^1_{1ij},\dots,H^n_{1ij},\dots,H^k_{1ij}\right)$. If $\beta = (\beta_1,\dots,\beta_k)$ is a vector in $\mathbb{R}^k$ then $\beta^T H_1 \beta$ is a vector with n-th component equal to $\sum\limits_{ij} \beta_i H^n_{1ij}\beta_j$. The solution $u(z,t)$ of the Cauchy problem (7.1), presenting the price of a zero coupon bond, is

$$u(z,t) = \exp\left[\alpha(t) + \beta(t)\cdot z\right]$$

where $\alpha(t) \in \mathbb{R}$, $\beta(t) \in \mathbb{R}^k$ satisfy the following ordinary differential equations and boundary conditions

$$\begin{aligned}\frac{\mathrm{d}\beta}{\mathrm{d}t} &= \rho_1 - K_1^T\beta(t) - \frac{1}{2}\beta^T(t)H_1\beta(t), \qquad \beta(\tau) = 0\,, \\ \frac{\mathrm{d}\alpha}{\mathrm{d}t} &= \rho_0 - K_0\beta(t) - \frac{1}{2}\beta^t(t)H_0\beta(t), \qquad \alpha(\tau) = 0\,.\end{aligned}$$

The price process of the zero-coupon bonds is

$$\Lambda^\tau(t) = u(Z(t),t;\tau) = \exp\left[\alpha(t) + \beta(t)\cdot Z(t)\right]$$

and the yield curve is

$$y_t(s) = -\frac{\log \Lambda^{t+s}(t)}{s} = -\frac{1}{s}\left[\alpha(t) + \beta(t)\cdot Z(t)\right].$$

8 Heath–Jarrow–Morton Model

Before discussing this model we make some preliminary considerations. Let general market model as described in Sections 2 and 4 be given. We define the *instantaneous forward rate* by

(8.1)
$$f(t,\tau) \stackrel{\text{def}}{=} -\frac{\partial}{\partial\tau}\log\Lambda^{\tau}(t) = -\lim_{s\to\tau}\frac{\log\Lambda^{s}(t)-\log\Lambda^{\tau}(t)}{s-\tau}, \qquad t<\tau<s$$

(*forward rate* $-\dfrac{\log\Lambda^{s}(t)-\log\Lambda^{\tau}(t)}{s-\tau}$ may be realized by the following trading strategy:

1. At time t we sell short a τ-bond and get $\$\Lambda^{\tau}(t)$. For that money we buy an amount $\dfrac{\Lambda^{\tau}(t)}{\Lambda^{s}(t)}$ of s-bonds.
2. At time τ the τ-bond expires and we pay \$1.
3. At time s the s-bonds expire and we obtain $\$\dfrac{\Lambda^{\tau}(t)}{\Lambda^{s}(t)}\cdot 1$.

The final result is that at time τ we have payed \$1 and at time s we obtained $\$\dfrac{\Lambda^{\tau}(t)}{\Lambda^{s}(t)}$. The return we realised is

$$\left(\frac{\Lambda^{\tau}(t)}{\Lambda^{s}(t)}\right)^{\frac{1}{s-\tau}}-1$$

at discrete time compounding or

$$-\frac{\log\Lambda^{s}(t)-\log\Lambda^{\tau}(t)}{s-\tau}$$

at continuous time compounding).

From (8.1) we obtain the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Lambda^{\tau}(t) = -f(t,\tau)\Lambda^{\tau}(t)\,. \tag{8.2}$$

Adding the boundary condition $\Lambda(t,t)=1$ we have the solution

$$\Lambda^{\tau}(t) = \exp\left[-\int_t^{\tau} f(t,u)\,\mathrm{d}u\right]. \tag{8.3}$$

Thus we can retrieve Λ and f from each other.

Proposition 3 *The relation*

$$f(t,t) = r(t)$$

holds.

Proof. By the definition of the term structure (5.1) we have

$$\Lambda^\tau(t) = \mathbf{E}_t^Q\left[\exp\left(-\int_t^\tau r(u)\,\mathrm{d}u\right)\right].$$

We differentiate w.r.t. τ and get

$$\frac{\mathrm{d}\Lambda^\tau(t)}{\mathrm{d}\tau} = \mathbf{E}_t^Q\left[-r(\tau)\exp\left(-\int_t^\tau r(u)\,\mathrm{d}u\right)\right].$$

For $\tau = t$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Lambda^\tau(t)\Big|_{\tau=t} = \mathbf{E}_t^Q\left[-r(t)\right] = -r(t).$$

On the other hand, the equation (8.2) and the boundary condition $\Lambda^t(t) = 1$ give us

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Lambda^\tau(t)\Big|_{\tau=t} = -f(t,\tau)\,\Lambda^\tau(t)\big|_{\tau=t} = -f(t,t)\Lambda^t(t) = -f(t,t).$$

The above two relations imply $f(t,t) = r(t)$. ■

Since $\Lambda^\tau(t)$ for fixed τ is an Ito process so is $f(t,\tau)$ as seen from (8.1). Suppose

(8.4)
$$f(t,\tau) = f(0,\tau) + \int_0^t \mu(s,\tau)\,\mathrm{d}s + \int_0^t \sigma(s,\tau)\,\mathrm{d}B^Q(s), \qquad 0 \le t \le \tau$$

where $\mu(s,\tau)$ and $\sigma(s,\tau)$ are adapted processes such that almost surely $\int_0^\tau |\mu(s,\tau)|\,\mathrm{d}s < \infty$ and $\int_0^\tau \sigma(s,\tau)\cdot\sigma(s,\tau)\,\mathrm{d}s < \infty$.

The martingale condition for $\Lambda^\tau(t)$ matches a condition for $f(t,\tau)$, namely:

Proposition 4 *The drift* $\mu(s,\tau)$ *and volatility* $\sigma(s,\tau)$ *of the process* $f(t,\tau)$ *satisfy the condition*

(8.5)
$$\mu(t,\tau) = \sigma(t,\tau)\cdot\int_t^\tau \sigma(t,u)\,\mathrm{d}u.$$

Proof. By the "martingale postulate" the discounted price process of the bond, for fixed τ

(8.6)
$$\begin{aligned}\overline{\Lambda}^{\tau}(t) &\overset{\text{def}}{=} \Lambda^{\tau}(t)\exp\left[-\int_0^t r(u)\,\mathrm{d}u\right]\\ &= \exp\left[-\int_0^t r(u)\,\mathrm{d}u - \int_t^{\tau} f(t,u)\,\mathrm{d}u\right] = \exp(X(t)+Y(t))\end{aligned}$$

is a martingale. Here we used (8.3) and denoted

$$X(t) = -\int_0^t r(u)\,\mathrm{d}u, \qquad Y(t) = -\int_t^{\tau} f(t,u)\,\mathrm{d}u\,. \tag{8.7}$$

Suppose that $\mu(t,u,\omega)$ and $\sigma(t,u,\omega)$ are uniformly bounded and, for each ω, continuous in t, u. From (8.4) and (8.7) we have

$$\begin{aligned}Y(t) = &-\int_t^{\tau}\left[f(0,u)+\int_0^t \mu(s,u)\,\mathrm{d}s+\int_0^t \sigma(s,u)\,\mathrm{d}B^Q(s)\right]\mathrm{d}u\\ = &-\int_t^{\tau} f(0,u)\,\mathrm{d}u-\int_t^{\tau}\left[\int_0^t \mu(s,u)\,\mathrm{d}s\right]\mathrm{d}u\\ &-\int_t^{\tau}\left[\int_0^t \sigma(s,u)\,\mathrm{d}B^Q(s)\right]\mathrm{d}u\,.\end{aligned}$$

By Fubini's theorem (both classical and for stochastic processes) we obtain

$$\begin{aligned}Y(t) = -\int_t^{\tau} f(0,u)\,\mathrm{d}u &- \int_0^t\left[\int_t^{\tau}\mu(s,u)\,\mathrm{d}u\right]\mathrm{d}s\\ &-\int_0^t\left[\int_t^{\tau}\sigma(s,u)\,\mathrm{d}u\right]\mathrm{d}B^Q(s)\,.\end{aligned}$$

Here from, differentiating

(8.8)
$$\begin{aligned}
\mathrm{d}Y(t) = f(0,t)\,\mathrm{d}t - &\left[\int_t^\tau \mu(t,u)\,\mathrm{d}u - \int_0^t \mu(s,t)\,\mathrm{d}s\right]\mathrm{d}t \\
&- \left[\int_t^\tau \sigma(t,u)\,\mathrm{d}u\right]\mathrm{d}B^Q(t) + \left[\int_0^t \sigma(s,t)\,\mathrm{d}B^Q(s)\right]\mathrm{d}t \\
= &\left[f(0,t) + \int_0^t \mu(s,t)\,\mathrm{d}s + \int_0^t \sigma(s,t)\,\mathrm{d}B^Q(s)\right. \\
&\qquad \left. - \int_t^\tau \mu(t,u)\,\mathrm{d}u\right]\mathrm{d}t - \left[\int_t^\tau \sigma(t,u)\,\mathrm{d}u\right]\mathrm{d}B^Q(t) \\
= &\left[f(t,t) - \int_t^\tau \mu(t,u)\,\mathrm{d}u\right]\mathrm{d}t + \left[-\int_t^\tau \sigma(t,u)\,\mathrm{d}u\right]\mathrm{d}B^Q(t) \\
= &\mu_Y(t)\,\mathrm{d}t + \sigma_Y(t)\,\mathrm{d}B^Q(t)\,.
\end{aligned}$$

Here we used (8.4) and denoted

(8.9) $$\mu_Y(t) = f(t,t) - \int_t^\tau \mu(t,u)\,\mathrm{d}u, \qquad \sigma_Y(t) = -\int_t^\tau \sigma(t,u)\,\mathrm{d}u.$$

Now we apply Ito's formula to the process (8.6) and get

$$\begin{aligned}
\mathrm{d}\overline{\Lambda}^\tau(t) &= \overline{\Lambda}^\tau\left[\mathrm{d}X + \mathrm{d}Y + \tfrac{1}{2}\left((\mathrm{d}X)^2 + 2\mathrm{d}X\mathrm{d}Y + (\mathrm{d}Y)^2\right)\right], \\
\mathrm{d}X &= -r(t)\,\mathrm{d}t, \qquad \mathrm{d}Y = \mu_Y\,\mathrm{d}t + \sigma_Y\,\mathrm{d}B^Q, \\
\mathrm{d}\overline{\Lambda}^\tau &= \overline{\Lambda}^\tau\left[-r\mathrm{d}t + \mu_Y\,\mathrm{d}t + \sigma_Y\,\mathrm{d}B^Q + \tfrac{1}{2}\sigma_Y^2\,\mathrm{d}t\right] \\
&= \overline{\Lambda}^\tau\left[\left(-r + \mu_Y + \tfrac{1}{2}\sigma_Y^2\right)\mathrm{d}t + \sigma_Y\,\mathrm{d}B^Q\right]\,.
\end{aligned}$$

Since $\overline{\Lambda}^\tau$ is a martingale its drift should be zero, so

(8.10) $$\mu_Y + \frac{1}{2}\sigma_Y\cdot\sigma_Y - r = 0\,.$$

Substituting (8.9) into (8.10), in view of Proposition 3, we obtain

$$\int_t^\tau \mu(t,u)\,\mathrm{d}u = \frac{1}{2}\left(\int_t^\tau \sigma(t,u)\,\mathrm{d}u\right)\cdot\left(\int_t^\tau \sigma(t,u)\,\mathrm{d}u\right)\,.$$

Taking the derivative w.r.t. τ we obtain the relation (8.5).

Now we can define (postulate) the Heath-Jarrow-Morton model. This is the general market model (of Section 2) in which the state processes are specified as the process (8.4) satisfying (8.5) and the short rate $r(t) = f(t,t)$. We see that the knowledge of the initial forward rates $\{f(0,t), t \in [0,\tau]\}$ and the forward rate volatility process σ is sufficient to determine the prices of all the securities on the market.

A Mathematical Tools

List of some mathematical tools used.

A.1 Basic Probability Notions

- σ-algebra
- Measurable space $(\Omega, \mathcal{F})$
- Measure μ; measurable function $f : \Omega \to \mathbb{R}^n$, $\int_\Omega f(\omega)\,\mathrm{d}\mu(\omega)$
- Absolute continuity: $\mu_2 \ll \mu_1 \iff \mu_1(A) = 0 \Rightarrow \mu_2(A) = 0$, $A \in \mathcal{F}$
- Radon–Nikodym theorem: If $\mu_2 \ll \mu_1$ then there exists measurable $f : \Omega \to \mathbb{R}$ such that $\mu_2(a) = \int_A f(\omega)\,\mathrm{d}\mu_1(\omega)$, $A \in \mathcal{F}$
- Probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- Random variable X
- σ-algebra $\sigma(X)$ generated by the random variable X;

$$\sigma(X) = \sigma\left(X^{-1}(B), B \subset \mathbb{R}^n \text{ open}\right)$$

- Conditional expectation $\mathbf{E}(X|\mathcal{H})$: $X : \Omega \to \mathbb{R}^n$ r.v., $\mathbf{E}(|X|) < \infty$, σ-algebra $\mathcal{H} \subset \mathcal{F}$

  1) $\mathbf{E}(X|\mathcal{H}) : \Omega \to \mathbb{R}^n$ is $\mathcal{H}$-measurable
  2) $\int_H \mathbf{E}(X|\mathcal{H})\,\mathrm{d}P = \int_H X\,\mathrm{d}P$, $H \in \mathcal{H}$

- Probability distribution of X = probability measure $P_X : \mathcal{B}(\mathbb{R}^n) \ni B \to P(X^{-1}(B))$

- Distribution function of X: $F_X(x) = \mathbf{P}\{\omega : X(\omega) \leq x\}$, $x \in \mathbb{R}$

- Distribution density of X, $p_X : F_X(x) = \int_{-\infty}^{x} p_X(y)\,\mathrm{d}y$

- Normal random variables:

  a) $X : \Omega \to \mathbb{R}$

  $$p_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right], \quad \sigma > 0,$$

  $\mathbf{E}(X) = m$, $\mathrm{var}(X) = \sigma^2$

  b) $X : \Omega \to \mathbb{R}^n$

  $$p_X(x_1, \ldots, x_n) = \frac{\sqrt{|A|}}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}\sum_{j,k}(x_j - m_j)a_{jk}(x_k - m_k)\right]$$

  $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$, $A = (a_{jk}) \in \mathbb{R}^{n\times n}$ is symmetric positive matrix, $\mathbf{E}(X) = m$, $\mathrm{cov}(X) = A^{-1}$ is the covariance matrix of X

A.2 Ito processes

- Filtration on $(\Omega, \mathcal{F})$ (Information flow):

  $$\{\mathcal{F}_t\}_{t\geq 0}, \quad \mathcal{F}_t \subset \mathcal{F} \text{ is } \sigma\text{-algebra s.t. } 0 \leq s < t \Rightarrow \mathcal{F}_s \subset \mathcal{F}_t$$

- Stochastic process $[0,T] \times \Omega \ni (t,\omega) \to X(t,\omega) \in \mathbb{R}^d$ for $[0,T] \ni t$, $X(t,\cdot)$ is a random variable; for $\Omega \ni \omega$, $X(\cdot,\omega)$ is a sample path.

- History $\{\mathcal{F}_t^X\}$ of the process X

  $$\mathcal{F}_t^X = \sigma\left(X_s, s \in [0,t]\right), \qquad \sigma(X_s) = \{X_s^{-1}(B), B \in \mathcal{B}\}$$

- $\mathcal{F}_t$-adapted process (nonanticipative): $\sigma(X_t) \subset \mathcal{F}_t$, $t \in [0,T]$ (X_t is $\mathcal{F}_t$ measurable)
- Martingale $\mathcal{M}_t$ w.r.t. $\{\mathcal{F}_t\}$
  - $\mathcal{M}_t$ is $\mathcal{F}_t$ measurable for all t
  - $\mathbf{E}[|\mathcal{M}_t|] < \infty$ for all t
  - If $s \le t$ then $\mathbf{E}[\mathcal{M}_t|\mathcal{F}_t] = \mathcal{M}_s$
- Brownian motion $B(t) = (B^1(t), \dots, B^d(t))$, $t \ge 0$
  1. $B(0) = 0$ a.s.
  2. $B(s) - B(t)$ is normally distributed in $\mathbb{R}^d$ with mean zero and covariance matrix $(s-t)I$, $s > t$
  3. $B_0(t_0)$, $B(t_1) - B(t_0)$, $\dots$, $B(t_n) - B(t_{n-1})$ are independent, $t_0 < t_1 < \cdots < t_n < \infty$
  4. For all $\omega \in \Omega$ the sample path $t \to B(\omega; t)$ is continuous

  Brownian motion is a martingale w.r.t. its history.
- Ito Integral

  Simple process $\theta(t,\omega) = \sum_j e_j(\omega)\chi_{[t_j,t_{j+1})(t)}$ where $s = t_0 < t_1 < \cdots < t_n = T$

  $$\chi_{[t_j,t_{j+1})(t)} = \begin{cases} 1 & t \in [t_j, t_{j+1}) \\ 0, & t \notin [t_j, t_{j+1}) \end{cases}, \qquad e_j(\omega) \text{ is } \mathcal{F}_{t_j}\text{-measurable}$$

  The Ito integral for a simple process θ is defined by

  $$\int_s^T \theta(t,\omega)\, \mathrm{d}B_t(\omega) \overset{\text{def}}{=} \sum_{j \ge 0} e_j(\omega) \left(B_{t_{j+1}}(\omega) - B_{t_j}(\omega) \right),$$

  where B_t is one-dimensional Brownian motion;

  The class of $\mathcal{H}^2(S,T)$ is defined by:

  - $\theta(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$

- $\theta(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable, $\mathcal{B}$ is the Borel σ-algebra on $[0,\infty)$.
- $\theta(t,\omega)$ is $\mathcal{F}_t$-adapted
- $\mathbf{E}\left[\int_S^T \theta(t,\omega)^2 \,\mathrm{d}t\right] < \infty$

The Ito integral for processes $\theta \in \mathcal{H}^2(S,T)$ is defined by

$$\int_S^T \theta(t,\omega)\,\mathrm{d}B_t(\omega) \stackrel{\text{def}}{=} \lim_{n\to\infty} \int_S^T \theta_n(t,\omega)\,\mathrm{d}B_t(\omega) \quad (\text{limit in } L^2(P))$$

where θ_n is a sequence of simple processes such that

$$\mathbf{E}\left[\int_S^T (\theta(t,\omega) - \theta_n(t,\omega))^2 \,\mathrm{d}t\right] \xrightarrow[n\to\infty]{} 0$$

- Ito process (one-dimensional)

$$X(t,\omega) = X_0(\omega) + \int_0^t \mu(s,\omega)\,\mathrm{d}s + \int_0^t \sigma(s,\omega)\,\mathrm{d}B(s,\omega),$$
$$\mu \in L^1,\ \sigma \in L^2, \quad \mathrm{d}X(t) = \mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t), \quad X(0) = X_0\,.$$

Ito formula (one-dimensional): Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$. Then $Y_t = g(t,X_t)$ is again an Ito process and

$$\mathrm{d}Y_t = \frac{\partial g}{\partial t}(t,X_t)\,\mathrm{d}t + \frac{\partial g}{\partial x}(t,X_t)\,\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)(\mathrm{d}X_t)^2$$

where $(\mathrm{d}X_t)^2$ is computed according to the rules

$$\mathrm{d}t\,\mathrm{d}t = \mathrm{d}t\,\mathrm{d}B_t = \mathrm{d}B_t\,\mathrm{d}t = 0, \qquad \mathrm{d}B_t \cdot \mathrm{d}B_t = \mathrm{d}t$$

- Martingale representation theorem
- Multi-dimensional Ito process.

Let $B(t,\omega) = \big(B^1(t,\omega),\ldots,B^d(t,\omega)\big)$ be a d-dimensional Brownian motion.

$$\mathrm{d}X_i = \mu_i\,\mathrm{d}t + \sigma_{i1}\,\mathrm{d}B^1 + \cdots + \sigma_{id}\,\mathrm{d}B^d, \qquad i = 1,\ldots,n$$

or

$$\mathrm{d}X = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$$

where

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix},\ \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix},\ \sigma = \begin{pmatrix} \sigma_{11} & \dots & \sigma_{1d} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \dots & \sigma_{nd} \end{pmatrix},\ \mathrm{d}B = \begin{pmatrix} \mathrm{d}B^1 \\ \vdots \\ \mathrm{d}B^d \end{pmatrix}.$$

Let

$$g(t,x) = (g_1(t,x),\dots,g_p(t,x))\,, \quad Y(t) = g(t,X(t)),$$

where $Y = (Y_1,\dots,Y_p)$. Then

$$\mathrm{d}Y_k = \frac{\partial g_k}{\partial t}(t,X)\,\mathrm{d}t + \sum_i \frac{\partial g_k}{\partial x_i}(t,X)\mathrm{d}X_i + \frac{1}{2}\sum_{ij}\frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X)\,\mathrm{d}X_i\,\mathrm{d}X_j,$$

$$\mathrm{d}B_i\,\mathrm{d}B_j = \delta_{ij}\,\mathrm{d}t, \qquad \mathrm{d}B_i\,\mathrm{d}t = \mathrm{d}t\,\mathrm{d}B_i = 0, \quad k = 1,\dots,p\,.$$

Example: $Y = X_1X_2$; $g(t,x_1,x_2) = x_1x_2$, $\partial g/\partial t = 0$, $\partial g/\partial x_1 = x_2$, $\partial g/\partial x_2 = x_1$, $\partial^2 g/\partial x_1 \partial x_2 = 1$

$$\mathrm{d}Y = X_2\,\mathrm{d}X_1 + X_1\,\mathrm{d}X_2 + \mathrm{d}X_1\mathrm{d}X_2\,.$$

A.3 Girsanov's Theorem

- Stochastic exponent

  Let B_t be a Brownian motion and $\{\mathcal{F}_t\}$ the accompanying filtration. Let $\eta = (\eta^1,\dots,\eta^d) \in L^2$ be $\mathcal{F}_t$ adapted and

  $$\mathbf{E}\left[\exp\left(\frac{1}{2}\int_0^T \eta_s\cdot\eta_s\,\mathrm{d}s\right)\right] < \infty \qquad \text{Novikov's condition.}$$

  Then the Ito process

  $$\xi_t = \exp\left(-\frac{1}{2}\int_0^t \eta_s\cdot\eta_s\,\mathrm{d}s - \int_0^t \eta_s\,\mathrm{d}B_s\right)$$

  is a martingale w.r.t. $\{\mathcal{F}_t\}$. It satisfies the equation

  $$\mathrm{d}\xi_t = -\xi_t\eta_t\,\mathrm{d}B_t\,.$$

- **Theorem (Girsanov)** *Let*

  (1) $$\mathrm{d}Y(t) = \beta(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t), \qquad t \le T,$$

  where $Y(t) \in \mathbb{R}^n$*,* $B(t) \in \mathbb{R}^d$*,* $\beta(t) \in \mathbb{R}^n$*,* $\sigma(t) \in \mathbb{R}^{n\times d}$*. Define the measure* Q *(equivalent to* P*) by*

  (2) $$\mathrm{d}Q(\omega) = \xi_T(\omega)\,\mathrm{d}P(\omega) \qquad \left(Q(A) = \int_A \xi_T\,\mathrm{d}P,\ A \in \mathcal{F}\right).$$

  Then the process

  $$\widetilde{B}(t) \stackrel{\text{def}}{=} \int_0^t \eta(s)\,\mathrm{d}s + B(t)$$

  is a Brownian motion w.r.t. Q *and the process* $Y(t)$ *has the representation*

  $$\mathrm{d}Y(t) = (\beta(t) - \sigma(t)\eta(t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}\widetilde{B}(t)\,.$$

- **Corollary** *If the equation*

  $$\sigma(t,\omega)\eta(t,\omega) = \beta(t,\omega)$$

  has a solution $\eta(t,\omega)$ *which is* $\mathcal{F}_t$*-adapted and satisfies the Novikov condition then the process* (1) *can be transformed by the equivalent change of measure* (2) *into a martingale.*

A.4 Feynman–Kac formula

- Stochastic differential equations
  (1)
  $$\mathrm{d}X(\theta) = \mu(X(\theta),\theta)\,\mathrm{d}\theta + \sigma(X(\theta),\theta)\,\mathrm{d}B(\theta), \qquad X(\theta_0) = x \in \mathbb{R}^n\,,$$

  $B(\theta) \in \mathbb{R}^d$, $X(\theta) \in \mathbb{R}^n$, $\mu : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^{n\times d}$.

  Other version:

  $$X(\theta) = x + \int_{\theta_0}^{\theta} \mu(X(s),s)\,\mathrm{d}s + \int_{\theta_0}^{\theta} \sigma(X(s),s)\,\mathrm{d}B(s)\,.$$

- Partial differential equation

  $$(2) \quad \mathcal{D}u(x,t) - R(x,t)u(x,t) + h(x,t) = 0, \qquad u(x,T) = g(x)$$

  where $(x,t) \in \mathbb{R}^n \times [0,T]$, $u(x,t) \in C^{2,1}(\mathbb{R}^n \times [0,T])$,

  $$\mathcal{D}u(x,t) = u_t(x,t) + \mu(x,t)u_x(x,t) + \frac{1}{2}\operatorname{tr}[\sigma(x,t)\sigma(x,t)^T u_{xx}(x,t)],$$

  $R : \mathbb{R}^n \times [0,T] \to \mathbb{R}$, $h : \mathbb{R}^n \times [0,T] \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$.

- Feynman–Kac formula for the solution $u(x,t)$ of the Cauchy problem (2)

  $$u(x,t) = \mathbf{E}_{x,t}\left\{ \int_t^T h(X(s),s) \exp\left[-\int_t^s R(X(\tau),\tau) \right] \mathrm{d}s + g(X_T) \exp\left[-\int_t^T R(X(\tau),\tau)\, \mathrm{d}\tau \right] \right\}$$

  where $X(\cdot)$ is a solution of (1) with initial condition $X(t) = x$.