Advanced Numerical Methods for Financial Problems
Pricing of Derivatives

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Outline

1. Motivation
   - Financial Problems
   - The Basic Problem That We Studied
   - Previous Work

2. Our Results/Contribution
   - Main Results
   - Basic Ideas for Proofs/Implementation

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Pricing of derivatives.
Black-Scholes Theory.

- Equity Options: American, Bermudan, etc.
- Hybrid Derivatives: Convertible Bonds (CBs).
- Fixed-Income Products: Callable Bonds, Putable Bonds and Callable/Putable Bonds
- Credit Risk Derivative: CBs - TF model.
- Gaussian underlying driven process:
  - Log-normal process
    \[ dS_t = \mu S_t dt + \sigma S_t dW_t. \]
  - Ornstein-Uhlenbeck process
    \[ dr_t = (b - ar_t) dt + \sigma dW_t. \]
- etc.
Financial Problems

The Basic Problem That We Studied

Previous Work

Pricing of derivatives.
Partial Differential Equations (PDE).

- Black-Scholes type equations - parabolic PDEs:
  - heat equations with zero right hand side
    \[ u_t = u_{xx}. \] (1)
  - heat equations with non-zero right hand side
    \[ u_t = u_{xx} + f. \] (2)

- Solve the problems when the initial data are non-smooth.
- Derivatives with embedded features (options) and constraints involve non-close form solution for its pricing:
  - early exercise - both American and Bermudan Options.
  - call-back, put and conversion features of CBs.
  - hard/soft-call provision of CBs.
Two main directions: to find (reproduce) by the natural manner the so called advanced two and three time-level FDS and explain the advantages and disadvantages of them from a point of view of the financial math.

- Finite Difference Schemes (FDS):
  - Two time-level FDS ($\theta$-method family):
    - Euler’s schemes: explicit and implicit
    - Crank-Nicholson (CN)
    - Douglas (2TLD)
  - Tree time-level FDS (Douglas).

- Truncation Error Estimation

- Numerical Methods for algebraic systems: Gauss-Seidel, SOR, PSOR.
FDS for Financial Problems.
Usage Technics and Applications.

- The Operator Approach (Mitchell and Griffiths).
- The optimal(kill)-value for 2TLD scheme is $\alpha = \frac{1}{\sqrt{20}}$ (William Shaw and may be Saulev about 1958).
- American Options (Wilmott, Hull, Shaw)
  - Over BS-equation (Wilmott and Hull).
  - Especially 3TLD over (1) (William Shaw).
- Convertible Bonds (CBs).
  - Euler’s and CN standard FDS over couple BS-equations from Tsiveriotis-Fernandes model - Lucy Xingven Li, 2005.
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The optimal(kill)-value for 2TLD scheme is $\alpha = \frac{1}{\sqrt{12}}$.

Reproduce FDS by the natural manner and explain the advantages/disadvantages in general.

Develop end implement a method for CBs evaluation with smallest "bad" effects in the following directions:

- Convergence to the conversion state.
- Description of influence of the coupons and the features: put, call-back and conversion.
- Spurious oscillations (Fig.1).
- Stability: (2TLD - Fig.2) and (Binary Tree - Fig.3).
- Provide fine mesh in the most important and difficult for description phases credit risk, investment and hybrid, and produce non-fine mesh in the phase of conversion, which is a line (Fig.4).
Advantages and disadvantages for considered FDSs:

- *In general* $\theta$-method family
  
  The obtained Pricing is continuous w.r.t. time. Theta is left-cont. and *inappropriate* for prediction. Pricing is smooth w.r.t. the underling-stock.

- *In general* 3-time level Douglas
  
  The obtained Pricing is continuous w.r.t. time. Theta is cont. and *appropriate* for prediction. Pricing is smooth w.r.t. the underling-stock.
Elimination of Oscillatory Terms.
Based on Tsiveriotis-Fernandes Math Model

**Figure:** Methods based on CN and 2TLD eliminate the oscillations.
Stability of the Method.
Based on 2-time level Douglas.

Variance of Douglas Scheme w.r.t. time-points and equity-points

Figure: Magenta for 160 equity points; Blue for 180 equity points; Red for 200 equity points; Orange for 250 equity points; Green for 300 equity points; Cyan for 350 equity points; Black for 400 equity points.
Stability of the Method.
Based on Binary Tree.

Figure: This figure shows the variance of the Binary Tree method w.r.t. the time-points (time-level).
Distribution of Spatial Points.
Based on 2-time level Douglas.

- 17 points for conversion phase (in the range from 20 to 120), and 178 points for the other 3 phases (in the range from 0 to 20)

Figure: By a grid with 20 time-points and 195 equity points.
Informally speaking, any definition of truncation error gives a measure of the extent to which an exact solution of the differential equation fails to satisfy the difference equation. Let’s an exact solution we denote with \( u : (t, x) \rightarrow u(t, x) \).

When \( u \) satisfy the difference equation of \( \theta \)-method, for its left hand side \( L \) and its right hand side \( R \) we have, respectively

\[
L = \tau \left( \partial_t u_n^m + \frac{1}{2} \tau \partial_t^2 u_n^m + \frac{1}{6} \tau^2 \partial_t^3 u_n^m + O(\tau^3) \right)
\]

\[
R = \tau \left( \partial_x^2 u_n^m + \tau \theta \partial_t \partial_x^2 u_n^m + \frac{1}{12} h^2 \partial_x^4 u_n^m + \frac{1}{2} \tau^2 \theta \partial_t \partial_x^2 u_n^m + \frac{1}{12} h^2 \tau \theta \partial_t \partial_x^4 u_n^m + \theta O(\tau^3) + \frac{1}{12} h^2 \theta O(\tau^2) + O(h^4) \right).
\]
The kill-value in 2TLD.

Now, by the choice of Douglas: \( \theta = \frac{1}{2} - \frac{1}{12\alpha} \), we obtain

\[
( \partial_t - \partial_x^2 ) u^m_n + \frac{\tau}{2} \partial_t ( \partial_t - \partial_x^2 ) u^m_n + \frac{h^2}{12} \partial_x ( \partial_t - \partial_x^2 ) u^m_n -
\]

\[
- ( \frac{\tau}{2} - \frac{h^2}{12} ) \partial_t \partial_x^2 ( \frac{\tau}{2} \partial_t + \frac{h^2}{12} \partial_x^2 ) u^m_n = \frac{1}{6} \tau^2 \partial_t^3 u^m_n + O(\tau^2) + O(h^4).
\]

Finally, by the equation \( \partial_t = \partial_x^2 \), for the truncation error \( \psi^m_n \) in the grid point \((t_m, x_n)\) we obtain the following expression:

\[
\psi^m_n = -\frac{1}{12} ( \tau^2 - \frac{h^4}{12} ) \partial_t^3 u^m_n + O(\tau^2) + O(h^4).
\]

Thus for the heat equation with zero right hand side, we obtain an error for the Douglas 2-time level scheme with order:

\[
O(\tau^2) + O(h^4).
\]
The kill-value in 2TLD.

Main Results
Basic Ideas for Proofs/Implementation

Now, firstly let we remark that in contrast to William Shaw (and maybe to Saulev about 1958), who claim that the optimal-value (kill-value) of $\alpha$ is $\alpha = \frac{1}{\sqrt{20}}$ we can propound another kill-value, namely $\alpha = \frac{1}{\sqrt{12}}$. Secondly let we remark that the value $\alpha = \frac{1}{\sqrt{12}}$ reduce the number of time levels in the FDS over 22.5 percentage (just reduction-percentage is $1 - \sqrt{\frac{3}{5}}$). For instance: instead we solve the problem with 26 time-steps (based on $\frac{1}{\sqrt{20}}$ ) we can solve that problem with 20 time-steps (based on $\frac{1}{\sqrt{12}}$ ) via non-bad truncation error.
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The computations, we did for evaluation date 31. Aug. 2005, and the definition of CBs which we used as example is as follows:

- Redemption Price $1000.00
- Coupon (semi-annual) 4.75%
- Conversion ratio 42.7716
- Exchange rate 1.00
- Risk-free Yield 4.2232%
- Stock volatility 27.2030%
The Feature schedules. The date-format is yyyy-m-d.

- **Maturity Date**: 2022-2-1
- **Conversion Schedule**: from 2002-8-7 to 2022-1-31
- **Call Schedule**
  - from 2006-2-6 to 2007-2-5 by $1015.83
  - from 2007-2-5 to 2008-2-5 by $1007.92
  - from 2008-2-5 to 2022-2-1 by $1000.00
- **Put Schedule**
  - from 2009-2-2 to 2009-2-2 by $1000
  - from 2012-2-1 to 2012-2-1 by $1000
  - from 2017-2-1 to 2017-2-1 by $1000
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Reference
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