Advanced Numerical Methods for Financial Problems

Pricing of Derivatives

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Outline

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Pricing of derivatives.
Black-Scholes Theory.

- Equity Options: American, Bermudan, etc.
- Hybrid Derivatives: Convertible Bonds (CBs).
- Fixed-Income Products: Callable Bonds, Putable Bonds and Callable/Putable Bonds
- Credit Risk Derivative: CBs - TF model.
- Gaussian underling driven process:
  - Log-normal process
    $$dS_t = \mu S_t dt + \sigma S_t dW_t.$$
  - Ornstein-Uhlenbeck process
    $$dr_t = (b - ar_t)dt + \sigma dW_t.$$
  - etc.

Pricing of derivatives.
Partial Differential Equations (PDE).

- Black-Scholes type equations - parabolic PDEs:
  - heat equations with zero right hand side

$$u_t = u_{xx}. \tag{1}$$

  - heat equations with non-zero right hand side

$$u_t = u_{xx} + f. \tag{2}$$

- Solve the problems when the initial data are non-smooth.
- Derivatives with embedded features (options) and constraints involve non-close form solution for its pricing:
  - early exercise - both American and Bermudan Options.
  - call-back, put and conversion features of CBs.
  - hard/soft-call provision of CBs.

Need for Numerical Methods

Two main directions: to find (reproduce) by the natural manner the so called advanced two and three time-level FDS and explain the advantages and disadvantages of them from a point of view of the financial math.

- Finite Difference Schemes (FDS):
  - Two time-level FDS (θ-method family):
    - Euler's schemes: explicit and implicit
    - Crank-Nicholson (CN)
    - Douglas (2TLD)
  - Tree time-level FDS (Douglas).
- Truncation Error Estimation
- Numerical Methods for algebraic systems: Gauss-Seidel, SOR, PSOR.

FDS for Financial Problems.

Usage Technics and Applications.

- The Operator Approach (Mitchell and Griffiths).
- The optimal(kill)-value for 2TLD scheme is $\alpha = \frac{1}{\sqrt{20}}$ (Wiliam Shaw and may be Saulev about 1958).
- American Options (Wilmott, Hull, Shaw)
  - Over BS-equation (Wilmott and Hull).
  - Especially 3TLD over (1) (William Shaw).
- Convertible Bonds (CBs).
  - Binomial Model - J. Hull, 2000.
  - Over BS-equation - P. Wilmott, 2000.
  - Euler's and CN standard FDS over couple BS-equations from Tsiveriotis-Fernandes model - Lucy Xingven Li, 2005.

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Outline

List of results.
Page One.

- The optimal(kill)-value for 2TLD scheme is $\alpha = \frac{1}{\sqrt{12}}$.
- Reproduce FDS by the natural manner and explain the advantages/disadvantages in general.
- Develop end implement a method for CBs evaluation with smallest "bad" effects in the following directions:
  - Convergence to the conversion state.
  - Description of influence of the coupons and the features: put, call-back and conversion.
  - Spurious oscillations (Fig.1).
  - Stability: (2TLD - Fig.2) and (Binary Tree - Fig.3).
  - Provide fine mesh in the most important and difficult for description phases credit risk, investment and hybrid, and produce non-fine mesh in the phase of conversion, which is a line (Fig.4).

List of results.
Page Two.

Advantages and disadvantages for considered FDSs:

- *In general* θ-method family

  The obtained Pricing is continuous w.r.t. time.
  Theta is left-cont. and inappropriate for prediction.
  Pricing is smooth w.r.t. the underling-stock.

- *In general* 3-time level Douglas

  The obtained Pricing is continuous w.r.t. time.
  Theta is cont. and appropriate for prediction.
  Pricing is smooth w.r.t. the underling-stock.

Elimination of Oscillatory Terms.

Based on Tsiveriotis-Fernandes Math Model

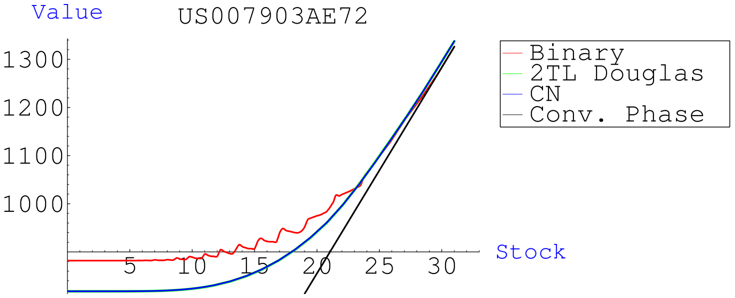

Figure: Methods based on CN and 2TLD eliminate the oscillations.

Stability of the Method.

Based on 2-time level Douglas.

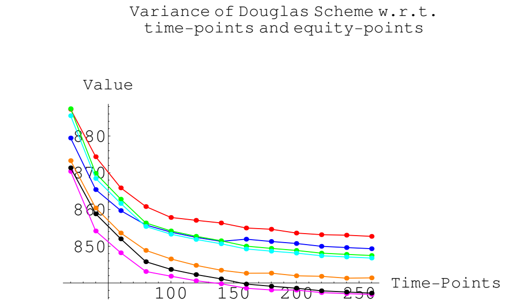

Figure: **Magenta** for 160 equity points; **Blue** for 180 equity points; **Red** for 200 equity points; **Orange** for 250 equity points; **Green** for 300 equity points; **Cyan** for 350 equity points; **Black** for 400 equity points.

Stability of the Method.
Based on Binary Tree.

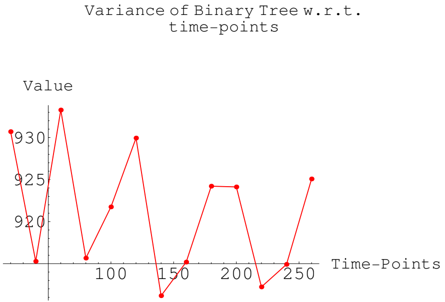

Figure: This figure shows the variance of the Binary Tree method w.r.t. the time-points (time-level).

Distribution of Spatial Points.

Based on 2-time level Douglas.

- 17 points for conversion phase (in the range from 20 to 120), and 178 points for the other 3 phases (in the range from 0 to 20)

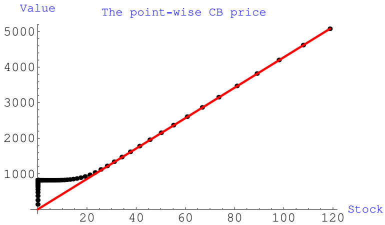

Figure: By a grid with 20 time-points and 195 equity points.

The kill-value in 2TLD.
Step One

INFORMALLY speaking, any definition of truncation error gives a measure of the extant to which an exact solution of the differential equation fails to satisfy the difference equation. Let's an exact solution we denote with $u : (t, x) \to u(t, x)$.
When u satisfy the difference equation of θ-method, for its left hand side L and its right hand side R we have, respectively

$$L = \tau\left(\partial_t u_n^m + \frac{1}{2}\tau\partial_t^2 u_n^m + \frac{1}{6}\tau^2\partial_t^3 u_n^m + O(\tau^3)\right)$$

$$\begin{aligned} R = \tau\Big(&\partial_x^2 u_n^m + \tau\theta\partial_t\partial_x^2 u_n^m + \frac{1}{12}h^2\partial_x^4 u_n^m + \\ &+\frac{1}{2}\tau^2\theta\partial_t^2\partial_x^2 u_n^m + \frac{1}{12}h^2\tau\theta\partial_t\partial_x^4 u_n^m \\ &+\theta O(\tau^3) + \frac{1}{12}h^2\theta O(\tau^2) + O(h^4)\Big). \end{aligned}$$

The kill-value in 2TLD.

Finally

Now, by the choice of Douglas: $\theta = \frac{1}{2} - \frac{1}{12\alpha}$, we obtain

$$(\partial_t - \partial_x^2)u_n^m + \tfrac{\tau}{2}\partial_t(\partial_t - \partial_x^2)u_n^m + \tfrac{h^2}{12}\partial_x^2(\partial_t - \partial_x^2)u_n^m -$$

$$-\left(\tfrac{\tau}{2} - \tfrac{h^2}{12}\right)\partial_t\partial_x^2\left(\tfrac{\tau}{2}\partial_t + \tfrac{h^2}{12}\partial_x^2\right)u_n^m = \tfrac{1}{6}\tau^2\partial_t^3 u_n^m + O(\tau^2) + O(h^4).$$

Finally, by the equation $\partial_t = \partial_x^2$, for the truncation error Ψ_n^m in the grid point (t_m, x_n) we obtain the following expression:

$$\Psi_n^m = -\frac{1}{12}\left(\tau^2 - \frac{h^4}{12}\right)\partial_t^3 u_n^m + O(\tau^2) + O(h^4).$$

Thus for the heat equation with zero right hand side, we obtain an error for the Douglas 2-time level scheme with order:

$$O(\tau^2) + O(h^4).$$

The kill-value in 2TLD.

Relevant Effect.

Now, firstly let we remark that in contrast to William Shaw (and maybe to Saulev about 1958), who claim that the optimal-value (kill-value) of α is $\alpha = \frac{1}{\sqrt{20}}$ we can propound another kill-value, namely $\alpha = \frac{1}{\sqrt{12}}$. Secondly let we remark that the value $\alpha = \frac{1}{\sqrt{12}}$ reduce the number of time levels in the FDS over 22.5 percentage (just reduction-percentage is $1 - \sqrt{\frac{3}{5}}$). For instance: instead we solve the problem with 26 time-steps (based on $\frac{1}{\sqrt{20}}$) we can solve that problem with 20 time-steps (based on $\frac{1}{\sqrt{12}}$) via non-bad truncation error.

Outline

Example Definition
Part One

THE computations, we did for evaluation date 31.Aug.2005, and the definition of CBs which we used as example is as follows:

- Redemption Price $1000.00
- Coupon (semi-annual) 4.75 %
- Conversion ratio 42.7716
- Exchange rate 1.00
- Risk-free Yield 4.2232 %
- Stock volatility 27.2030 %

Example Definition
Part Two

The Feature schedules. The date-format is yyyy-m-d.

- Maturity Date: 2022-2-1
- Conversion Schedule: from 2002-8-7 to 2022-1-31
- Call Schedule
    - from 2006-2-6 to 2007-2-5 by $1015.83
    - from 2007-2-5 to 2008-2-5 by $1007.92
    - from 2008-2-5 to 2022-2-1 by $1000.00
- Put Schedule
    - from 2009-2-2 to 2009-2-2 by $1000
    - from 2012-2-1 to 2012-2-1 by $1000
    - from 2017-2-1 to 2017-2-1 by $1000

Outline

1. Motivation
   - Financial Problems
   - The Basic Problem That We Studied
   - Previous Work
2. Our Results/Contribution
   - Main Results
   - Basic Ideas for Proofs/Implementation
3. Appendix
   - Example Definition
4. Reference

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