Higher order approximations of affinely controlled nonlinear systems

Nikolay Kirov and Mikhail Krastanov

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
Acad. G. Bonchev str. bl.8, 1113 Sofia
nkirov@math.bas.bg, krast@math.bas.bg

Abstract. An numerical approach for numerical approximation of trajectories of a smooth affine control system is proposed under suitable assumptions. This approach is based on expansion of solutions of systems of ordinary differential equations (ODE) by Volterra series and allows to estimate the distance between the obtained approximation and the true trajectory.

1 Introduction

To apply the traditional numerical schemes (such as Runge-Kutta schemes) of higher order for approximation trajectories of nonlinear control systems is a nontrivial task (cf. for example [4], [11] etc.) due to nonsmoothness of the control functions. For the case of linear differential inclusions, this problem is studied in [9]. There is proposed a numerical procedure based on a suitable approximation of integrals of multivalued mappings (cf. for example [10]) and on the ideology of algorithms with result verification (cf. for example [2]). Another approach for approximating trajectories of affinelly controlled systems is proposed in [5] and [6]. This approach is based on the expansion of the solution of the systems of ordinary differential equations by Volterra series (cf. for example [7]). In this note, combining this approach with the ideas developed in [9], we propose a method for approximation of trajectories of analytic control systems with guaranteed accuracy. We would like to point out that our approach can be applied for general control systems that are smooth with respect to the phase variables and continuous with respect to the control.

2 Systems of ODE and Volterra series.

First, we introduce briefly some notations and notions: For every point $y = (y^1, \ldots, y^n)^T$ from R^n we set $\|y\| := \sum_{i=1}^n |y^i|$ and let B be the unit ball in R^n (according to this norm) centered at the origin. Let C denote the set of all complex numbers, $x_0 \in R^n$ and Ω be a convex compact neighbourhood of the

[1] This research was partially supported by Swiss NSF No. 7 IP 65642 (Program SCOPES) and by the Ministry of Science and Higher Education - National Fund for Science Research under contracts MM-807/98 and MM-1104/01.

point x_0. If $z \in C$, then by $|z|$, Re z and Im z we denote its norm, its real and imaginary parts, respectively. Let $\sigma > 0$. We set $\Omega^\sigma :=$

$$\{z = (z_1, \ldots, z_n) \in C^n : (\mathrm{Re}\, z_1, \ldots, \mathrm{Re}\, z_n) \in \Omega + \sigma B,\ |\mathrm{Im}\, z_i| < \sigma,\ i = 1, \ldots, n\}.$$

By $\mathcal{F}_\Omega^\sigma$ we denote the set of all real analytic functions defined on Ω, such that every $\phi \in \mathcal{F}_\Omega^\sigma$ has a bounded analytic extension $\bar{\phi}$ on Ω^σ. We define a norm in the set $\mathcal{F}_\Omega^\sigma$ as follows:

$$||\phi||_\Omega^\sigma = \sup\ \{|\bar{\phi}(z)|\ : z \in \Omega^\sigma\}.$$

Let $h(x) = (h_1(x), h_2(x), \ldots, h_n(x))^T, x \in \Omega$, be a vector field defined on Ω. As usually, we identify h with the corresponding differential operator

$$\sum_{i=1}^{n} h_i(x)\ \frac{\partial}{\partial x_i},\ x \in \Omega.$$

Let $\mathcal{V}_\Omega^\sigma$ be the set of all real analytic vector fields h defined on Ω such that every $h_i, i = 1, \ldots, n$, belongs to $\mathcal{F}_\Omega^\sigma$. We define the following norm in $\mathcal{V}_\Omega^\sigma$:

$$||h||_\Omega^\sigma = \max\ (||h_i||_\Omega^\sigma,\ i = 1, \ldots, n).$$

An integrable analytic vector field $X_t(x) = (X_1(t,x), X_2(t,x), \ldots, X_n(t,x)), x \in \Omega,\ t \in R$ (parameterized by t), is a map $t \to X_t \in \mathcal{V}_\Omega^\sigma$ such that:
i) for every $x \in \Omega$, the functions $X_1(.,x), X_2(\cdot,x), \ldots, X_n(\cdot,x)$ are measurable;
ii) for every $t \in R$ and $x \in \Omega$, $|X_i(t,x)| \leq m(t),\ i = 1, 2, \ldots, n$, where m is an integrable (on every compact interval) function.

Further we shall consider only *uniformly integrable* vector fields X_t, i.e.

$$\int_{t_1}^{t_2} ||X_\tau||_\Omega^\sigma \to 0 \ \text{ whenever } \ |t_2 - t_1| \to 0.$$

Let M be a compact set contained in the interior of Ω and containing the point x_0, and let X_t be an integrable analytic vector field defined on Ω. Then there exists a real number $T(M, X_t) > t_0$ such that for every point x of M the solution $y(.,x)$ of the differential equation

$$\dot{y}(t,x) = X_t(y(t,x)),\quad y(t_0,x) = x, \tag{1}$$

is defined on the interval $[t_0, T(M, X_t)]$ and $y(T,x) \in \Omega$ for every T from $[t_0, T(M)]$. In this case we denote by $\exp \int_{t_0}^{T} X_t\ dt\ :\ M \to \Omega$, the diffeomorphism defined by

$$\exp \int_{t_0}^{T} X_t\ dt\ (x) := y(T,x).$$

According to proposition 2.1 from [1], $T(M, X_t) > t_0$ can be chosen in such a way that for every $T,\ 0 < T < T(M, X_t)$, for every point x from M and for every function ϕ from $\mathcal{F}^{\sigma}_{\Omega}$, the following expansion of $\phi\left(\exp\int_{t_0}^{T} X_t\ dt\ (x)\right)$ holds true:

$$\phi\left(\exp\int_{t_0}^{T} X_t\ dt\ (x)\right) = \widetilde{\exp}\int_{t_0}^{T} X_t\ dt\ \phi(x), \tag{2}$$

where $\widetilde{\exp}\ \int_{t_0}^{T} X_t\ dt\ \phi(x) =$

$$= \phi(x) + \sum_{N=1}^{\infty}\int_{t_0}^{T}\int_{t_0}^{\tau_1}\int_{t_0}^{\tau_2}\dots\int_{t_0}^{\tau_{N-1}} X_{\tau_N}X_{\tau_{N-1}}\dots X_{\tau_2}X_{\tau_1}\phi(x)d\tau_N d\tau_{N-1}\dots d\tau_1.$$

and the series is absolutely convergent. The proof of these identities is based on the following estimate: for every positive real numbers $\sigma_1 < \sigma$, for every point x of M, for every function ϕ from $\mathcal{F}^{\sigma}_{\Omega}$ and for every points $\tau_N, \tau_{N-1}, \dots, \tau_2, \tau_1$, from $[t_0, T]$, the following inequality holds true $\|X_{\tau_N}\dots X_{\tau_2}X_{\tau_1}\phi\|^{\sigma_1}_{\Omega} \leq$

$$\leq N!\left(\frac{2n}{\sigma-\sigma_1}\right)^{N}\|X_{\tau_N}\|^{\sigma}_{\Omega}\cdots\|X_{\tau_2}\|^{\sigma}_{\Omega}\cdot\|X_{\tau_1}\|^{\sigma}_{\Omega}\cdot\|\phi\|^{\sigma}_{\Omega}. \tag{3}$$

This estimate implies the following technical lemma:

Lemma 1. *Let M be a convex compact subset of Ω, $\psi \in \mathcal{F}^{\sigma}_{\Omega}$, $X_t \in \mathcal{V}^{\sigma}_{\Omega}$ and $t_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_\mu \leq T$. We set*

$$\Psi_{\tau_1,\tau_2,\dots,\tau_\mu} := X_{\tau_\mu}\dots X_{\tau_2}X_{\tau_1}\psi$$

Then for every $0 < \sigma_1 < \sigma$ and for every two points y_1, y_2 of M the following inequality holds true:

$$|\Psi_{\tau_1,\tau_2,\dots,\tau_\mu}(y_2) - \Psi_{\tau_1,\tau_2,\dots,\tau_\mu}(y_1)| \leq$$

$$\leq (\mu+1)!\left(\frac{2n}{\sigma-\sigma_1}\right)^{\mu+1}\|X_{\tau_\mu}\|^{\sigma}_{M}\cdots\|X_{\tau_2}\|^{\sigma}_{M}\cdot\|X_{\tau_1}\|^{\sigma}_{M}\cdot\|\psi\|^{\sigma}_{M}\cdot\|y_2-y_1\|.$$

Let the functions $E_i : \Omega_\sigma \to R$, $i = 1, \dots, n$, be defined as follows: $E_i(z_1, \dots, z_n) = z_i$. We set $E := (E_1, \dots, E_n)^T$, $X_t\ E\ := (X_t\ E_1, \dots, X_t E_n)^T$. Applying Lemma 1 we obtain that for every two points y_1 and y_2 from M the following estimate holds true

$$\|X_{\tau_\mu}\dots X_{\tau_2}X_{\tau_1}E(y_2) - X_{\tau_\mu}\dots X_{\tau_2}X_{\tau_1}E(y_1)\| \leq$$

$$\leq n\mu!\left(\frac{2n}{\sigma-\sigma_1}\right)^{\mu}\|X_{\tau_\mu}\|^{\sigma}_{M}\cdots\|X_{\tau_2}\|^{\sigma}_{M}\cdot\|X_{\tau_1}\|^{\sigma}_{M}\|y_2-y_1\|. \tag{4}$$

We use the estimate (4) to prove *an existence criterion* for $\exp\int_{t_0}^{\tau} X_t\ dt\ (x)$ where x belongs to some compact set M:

Proposition 1. *Let M and M_1 be convex compact subsets of Ω, $\sigma > \sigma_1 > 0$, $T > t_0$ and ω be a positive integer such that for every point $x \in M$ and for every $t \in [t_0, T]$ the following relations hold true:*

$$0 < n\left(\frac{2n}{\sigma-\sigma_1}\int_{t_0}^{T} \|X_s\|_{M_1}^{\sigma}\, ds\right)^{\omega} < 1 \tag{5}$$

$$x + \sum_{N=1}^{\omega-1}\int_{t_0}^{t}\int_{t_0}^{\tau_1}\int_{t_0}^{\tau_2}\dots\int_{t_0}^{\tau_{N-1}} X_{\tau_N}X_{\tau_{N-1}}\dots X_{\tau_2}X_{\tau_1}E(x)d\tau_N d\tau_{N-1}\dots d\tau_1$$
$$+\frac{n}{\omega}\left(\frac{2n}{\sigma-\sigma_1}\right)^{\omega-1}\left(\int_{t_0}^{t}\|X_s\|_{M_1}^{\sigma}\, ds\right)^{\omega}\cdot B \;\in\; M_1. \tag{6}$$

Then $\exp\int_{t_0}^{T} X_t\; dt\; (x)$ *is well defined for every point $x \in M$. Moreover,* $\exp\int_{t_0}^{\tau} X_t\; dt\; (x) \in M_1$ *for every point $x \in M$ and for every τ from $[t_0, T]$.*

Proof. Let x be an arbitrary point from M. By $C(M_1; [t_0, T])$ we denote the set of all continuous functions defined on $[t_0, T]$ with values from the set M_1. We can define the following operator $F : C(M_1; [t_0, T]) \to C(M_1; [t_0, T])$ as follows: $F(y)(\tau) = x+$

$$\sum_{N=1}^{\omega-1}\int_{t_0}^{\tau}\int_{t_0}^{\tau_1}\int_{t_0}^{\tau_2}\dots\int_{t_0}^{\tau_{N-1}} X_{\tau_N}X_{\tau_{N-1}}\dots X_{\tau_2}X_{\tau_1}E(x)d\tau_N d\tau_{N-1}\dots d\tau_1+$$
$$\int_{t_0}^{\tau}\int_{t_0}^{\tau_1}\int_{t_0}^{\tau_2}\dots\int_{t_0}^{\tau_{\omega-1}} X_{\tau_\omega}X_{\tau_{\omega-1}}\dots X_{\tau_2}X_{\tau_1}E(y(\tau_\omega))d\tau_\omega d\tau_{\omega-1}\dots d\tau_1,$$

Let $\|\cdot\|_C$ denote the usual uniform norm in $C(M_1; [t_0, T])$, i.e.

$$\|y\|_C = \max\{\|y(t)\| : t \in [t_0, T]\},$$

where $y(t) = (y_1(t), \dots, y_n(t))^T$, $t \in [t_0, T]$. This operator is well defined on $C(M_1; [t_0, T])$. Let y_1 and y_2 be arbitrary elements of $C(M_1; [t_0, \tau])$. Applying the estimate (4), we obtain that

$$\|F(y_2) - F(y_1)\|_C = \max\{\|F(y_2)(\tau) - F(y_1)(\tau)\| :\ \tau \in [t_0, T]\} \le$$
$$\le \max\left\{\int_{t_0}^{\tau}\int_{t_0}^{\tau_1}\int_{t_0}^{\tau_2}\dots\int_{t_0}^{\tau_{\omega-1}} \left\|X_{\tau_\omega}X_{\tau_{\omega-1}}\dots X_{\tau_2}X_{\tau_1}E(y_2(\tau_\omega)) -\right.\right.$$
$$\left.\left. X_{\tau_\omega}X_{\tau_{\omega-1}}\dots X_{\tau_2}X_{\tau_1}E(y_1(\tau_\omega))\right\| d\tau_\omega d\tau_{\omega-1}\dots d\tau_1 :\ \tau \in [t_0, T]\right\}$$
$$\le n\omega!\left(\frac{2n}{\sigma-\sigma_1}\right)^{\omega}\int_{t_0}^{T}\int_{t_0}^{\tau_1}\int_{t_0}^{\tau_2}\dots\int_{t_0}^{\tau_{\omega-1}}$$
$$\|X_{\tau_\omega}\|_{M_1}^{\sigma}\cdots\|X_{\tau_2}\|_{M_1}^{\sigma}\cdot\|X_{\tau_1}\|_{M_1}^{\sigma}\cdot d\tau_\omega d\tau_{\omega-1}\dots d\tau_1\|y_2 - y_1\|_C =$$
$$= n\left(\frac{2n}{\sigma-\sigma_1}\int_{t_0}^{T}\|X_s\|_{M_1}^{\sigma}\, ds\right)^{\omega}\|y_2 - y_1\|_C = L\|y_2 - y_1\|_C,$$

where

$$L := n\left(\frac{2n}{\sigma-\sigma_1}\int_{t_0}^{T}\|X_s\|_{M_1}^{\sigma}\,ds\right)^{\omega}.$$

Since $0 < L < 1$, the differential operator F is contractive. Applying the Banach fixed point theorem we obtain that there exists an unique function $\overline{y}$ from $C(M_1;[t_0,\tau])$ such that $F(\overline{y}) = \overline{y}$. The last relation means that $\overline{y}(t) = \exp\int_{t_0}^{\tau} X_t\ dt\ (x)$. This completes the proof.

The next proposition helps us to estimate the global approximation error using the local approximation error and motivates our approximation procedure.

Proposition 2. *Let* $\varepsilon > 0$, $\sigma > \sigma_1 > 0$, $\exp\int_{t_0}^{T} X_t\ dt\ (x_0)$ *is well defined and* $\exp\int_{t_0}^{t} X_t\ dt\ (x_0) \in \Omega$ *for every* $t \in [t_0,T]$. *Let* $t_0 < t_1 < \ldots < t_k = T$,

$$\frac{2n}{\sigma-\sigma_1}\int_{t_i}^{t_{i+1}}\|X_s\|_{\Omega}^{\sigma}\,ds < \frac{1}{2}\ \textit{and}\ \frac{n}{\omega}\left(\frac{2n}{\sigma-\sigma_1}\right)^{\omega-1}\left(\int_{t_i}^{t_{i+1}}\|X_s\|_{\Omega}^{\sigma}\,ds\right)^{\omega} < \frac{K}{3k^{\alpha+1}},$$

Then there exist compact subsets M_i *of* Ω *such that* $\exp\int_{t_0}^{t_i} X_t\ dt\ (x_0)$ *belongs to* M_i, $i = 0,1,\ldots,k$, *and*

$$\textit{diam}\ M_i < \frac{K}{k^{\alpha}}\ \textit{exp}\ \left(\frac{4n^2}{\sigma-\sigma_1}\int_{t_0}^{t_i}\|X_s\|_{\Omega}^{\sigma}\,ds\right)$$

where $\textit{diam}\ M := \max\ \left\{\|y_2-y_1\| : y_j = (y_j^1,\ldots,y_j^n)^T \in M,\ j=1,2\right\}$.

Proof. We set $M_0 := \{x_0\}$ and define the sets M_i, $i = 1,2,\ldots,k-1$, as follows: If M_i is already defined, we set $M_{i+1} := \{F(x) : x \in M_i\} \cap \Omega$, where $F(x) :=$

$$x + \sum_{N=1}^{\omega-1}\int_{t_i}^{t_{i+1}}\int_{t_i}^{\tau_1}\int_{t_i}^{\tau_2}\cdots\int_{t_i}^{\tau_{N-1}} X_{\tau_N}X_{\tau_{N-1}}\ldots X_{\tau_2}X_{\tau_1}E(x)d\tau_N d\tau_{N-1}\ldots d\tau_1$$

$$+\frac{n}{\omega}\left(\frac{2n}{\sigma-\sigma_1}\right)^{\omega-1}\left(\int_{t_i}^{t_{i+1}}\|X_s\|_{\Omega}^{\sigma}\,ds\right)^{\omega}\cdot B.$$

We set $y_0 = x_0$, $y_{i+1} = \exp\int_{t_i}^{t_{i+1}} X_t\ dt\ (y_i)$, $i = 1,2,\ldots,k$. It can be directly verified that $y_i \in M_i$ and

$$y_k = \exp\int_{t_0}^{t_k} X_t\ dt\ (x_0) \in M_k.$$

Let us denote diam M_i by d_i. Clearly, $d_0 =$ diam $M_0 = 0$. Let us assume that $d_{i+1} \le \|F(y_2) - F(y_1)\| + K/3k^{\alpha}$, where $y_1,\ y_2\ \in M_i$. Then

$$d_{i+1} \le \sum_{i=1}^{n}\left|y_2^i - y_1^i\right|\ +\ \frac{K}{k^{\alpha+1}}+$$

$$+\sum_{N=1}^{\omega-1}\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{\tau_1}\int_{t_i}^{\tau_2}\dots\int_{t_i}^{\tau_{N-1}} X_{\tau_N}X_{\tau_{N-1}}\dots X_{\tau_2}X_{\tau_1}E(y_2)d\tau_N d\tau_{N-1}\dots d\tau_1\right.$$
$$\left.-\int_{t_i}^{t_{i+1}}\int_{t_i}^{\tau_1}\int_{t_i}^{\tau_2}\dots\int_{t_i}^{\tau_{N-1}} X_{\tau_N}X_{\tau_{N-1}}\dots X_{\tau_2}X_{\tau_1}E(y_1)d\tau_N d\tau_{N-1}\dots d\tau_1\right| \leq$$
$$\leq \|y_2-y_1\| + \frac{K}{k^{\alpha+1}} + n\sum_{N=1}^{\omega-1} N!\left(\frac{2n}{\sigma-\sigma_1}\right)^N \frac{1}{N!}\left(\int_{t_i}^{t_{i+1}}\|X_s\|_\Omega^\sigma\, ds\right)^N \|y_2-y_1\| \leq$$
$$\|y_2-y_1\|\left[1+\frac{4n}{\sigma-\sigma_1}\int_{t_i}^{t_{i+1}}\|X_s\|_\Omega^\sigma\, ds\right] + \frac{K}{k^{\alpha+1}}.$$

Hence, we have proved that

$$d_{i+1} \leq d_i(1+\delta_i) + \Delta,\ i=0,1,\dots,k-1,$$

where $\Delta = K/k^{\alpha+1}$ and

$$\delta_i = \frac{4n^2}{\sigma-\sigma_1}\int_{t_i}^{t_{i+1}}\|X_s\|_{M_i}^\sigma\, ds.$$

Applying theorem 1.7 (p. 182, [3]), we obtain that for every

$$d_i \leq \ \exp\left(\sum_{j=0}^{i-1}\delta_j\right) i\Delta,\ i=1,2,\dots,k.$$

Substituting δ_j and Δ we complete the proof.

3 A computational procedure

Let us consider the following control system:

$$\frac{d}{dt}x(t) = f_0(x(t)) + \sum_{i=1}^m u_i f_i(x(t)),\ x(0)=x_0, \tag{7}$$

where the state variable x belongs to R^n, $f_0, f_1, \dots, f_m$ are real analytic vector functions and the admissible controls $u=(u_1,u_2,\dots,u_m)$ are the Lebesgue integrable functions.

Let $\varepsilon>0$, $\hat{T}>0$ and $u:[t_0,\hat{T}]\to U$ be an admissible control. First, using Proposition 1 we can find a positive real T (not greater than $\hat{T}$) and a compact subset Ω of R^n containing the point x_0 such that the corresponding trajectory $x:[t_0,T]\to R^n$ is well defined on $[t_0,T]$ and $x(t)\in\Omega$ for every t from $[t_0,T]$. Next, we show how we can calculate a point $y\in\Omega$ such that $|x(T)-y|<\varepsilon$: We assume that $f_i\in\mathcal{V}_\Omega^\sigma$ for some $\sigma>0$, $|u_j|\leq\nu_j$, $j=1,2,\dots,m$, and $\|f_i\|_\Omega^\sigma\leq C_i$ for $i=0,1,\dots,m$. We set $\nu_0:=1$ and

$$C := \sum_{i=1}^m \nu_i C_i.$$

Let $0 < \sigma_1 < \sigma$. For every positive integer k we set $h := (T - t_0)/k$ and define the points $0 = t_0 < t_1 < \cdots < t_k$, where $t_i = t_0 + ih$. We choose k to be sufficiently large, so that the following relation holds true:

$$\frac{4nhC}{\sigma - \sigma_1} < 1.$$

Next we choose a value for $\alpha > 1$ such that the following inequality holds true:

$$k^\alpha > \frac{1}{\varepsilon} \exp\left(\frac{4n^2(T - t_0)C}{(\sigma - \sigma_1)}\right).$$

At the end we determine the accuracy of the desirable local approximation by choosing the positive integer ω to be so large that

$$n\left(\frac{2n(T - t_0)C}{(\sigma - \sigma_1)}\right)^\omega < 1 \text{ and } \frac{nhC}{\omega}\left(\frac{2nhC}{\sigma - \sigma_1}\right)^{\omega - 1} 3k^{\alpha+1} < 1.$$

Since the trajectory $x : [t_0, T] \to R^n$ is well defined on $[0, T]$ and belongs to Ω, we obtain according to Proposition 2 that $|x(T) - z| < \varepsilon$, where $z := z_k$ and for every $i = 0, 1, \ldots, k - 1$, $z_{i+1} :=$

$$= z_i + \sum_{N=1}^{\omega-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{\tau_1} \int_{t_i}^{\tau_2} \cdots \int_{t_i}^{\tau_{N-1}} X_{\tau_N} X_{\tau_{N-1}} \ldots X_{\tau_2} X_{\tau_1} E(z_i) d\tau_N d\tau_{N-1} \ldots d\tau_1.$$

In [8] is studied the following control systems:

$$\left| \begin{array}{ll} \dot{x}_1 = u & |u| \le 1 \\ \dot{x}_2 = 3x_1^2 & x(0) = (0, 0) \end{array} \right.$$

All time-optimal controls are piecewise constant with at most two pieces. The constant controls $u \equiv 1$ and $u \equiv -1$ generate the lower boundaries of the reachable sets $s \to (s, |s^3|)$, and $s \to (s, |s^3| - s^2)$, $0 \le s \le T$, respectively. The controls with $u_s(t) = \pm 1$ for $0 \le t \le s$ and $u_s(t) = \mp 1$ for $s \le t \le T$ steer to the curves of endpoints $s \to (2s - T, 2s^3 + (T - 2s)^3)$ and $s \to (2s - T, 2s^3 + (T - 2s)^3 - (T - 2s)^2)$, respectively.

Figure 1 shows the approximate reachable set for the system at the moment $t = 1$ using first and second order terms in Volterra series. The timestep is $h = \dfrac{\pi}{30}$.

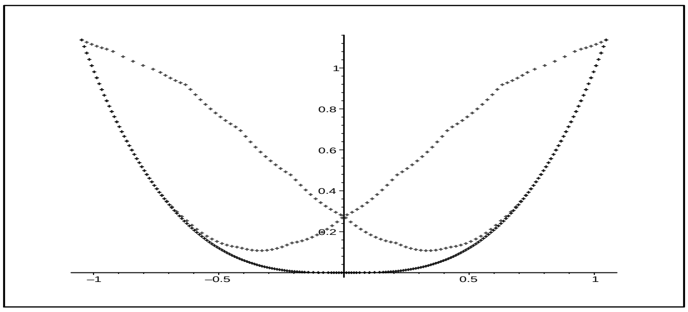

Fig. 1 First and second order terms

The exact reachable set is presented on Fig. 2, using also third order terms in Volterra series. It is enough to take timestep equal to 1 in this case.

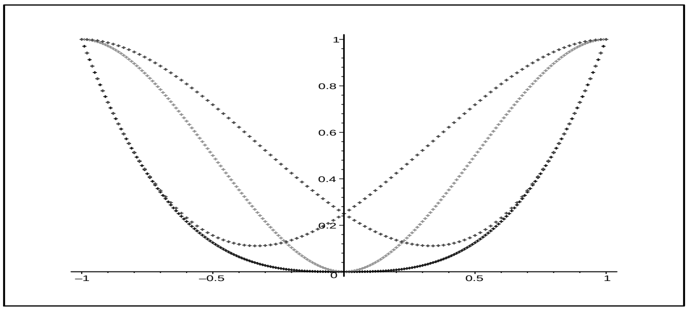

Fig. 2 The exact reachable set

References

1. A. Agrachev, R. Gamkrelidze, The exponential representation of flows and the chronological calculus, Math. USSR Sbornik **107** (1978), 467-532.
2. P. Bochev, S. Markov, A self-validating method for matrix exponential, Computing **43** (1989), 59-72.
3. F. Clarke, Yu. Ledyaev, R. Stern, P. Wolenski: Nonsmooth analysis and control theory, Graduate text ion Mathematics **178**, Springer, 1998.
4. A. Dontchev, F. Lempio, Difference methods for differential inclusions, SIAM J. Review **34** (1992), 263-294.
5. L. Grüne, P. Kloeden, Higher order numerical schemes for affinely controlled nonlinear systems, Numer. Math. **89** (2001), 669-690.
6. L. Grüne und P. E. Kloeden, Numerical schemes of higher order for a class of nonlinear control systems, In: *Numerical Methods and Applications,* Borovets, Bulgaria (2002), I. Dimov, I. Lirkov, S. Margenov, Z. Zlatev, eds, Springer Lecture Notes in Computer Science 2542(2003), 213-220.
7. A. Isidori, Nonlinear control systems. An introduction, Springer, 1995.
8. M. Kawski, High-order Maximal Principle, preprint, 2003.
9. M. Krastanov, N. Kirov, Dynamic Interactive System for Analysis of Linear Differential Inclusions, "Modeling techniques for uncertain systems", A. Kurzhanski, V. Veliov (Eds.), Birkhauser, 123-130, 1994.
10. V. Veliov, Discrete approximations to integrals of multivalued inclusions, Compt. Rend. Acad. Bulg. Sci. **42** (1989), 51-54.
11. V. Veliov, On the time discretization of control systems, SIAM J. Control Optim. **35** (1997), 1470-1486.