Perturbation Solution for the 2D Shallow-Water Waves

Michail D. Todorov

Faculty of Applied Mathematics and Computer Science
Technical University of Sofia, Bulgaria

(Work done in collaboration with and supervision of Prof. C. I. Christov from ULL)

3rd International Conference AMiTaNS'11, 20-25 June 2011, Albena, Bulgaria

Outline

- Problem Formulation
- Boussinesq Paradigm Equation
- Perturbation Method
- Governing System
- Difference Schemes
- Results and Discussion
- References

Outline of the Problem

- Boussinesq's equation (BE) was the first model for the propagation of surface waves over shallow inviscid fluid layer. He found an anlytical solution of his equation and thus proved that the balance between the steepening effect of the nonlinearity and the flattening effect of the dispersion maintains the shape of the wave. This discovery can be properly termed 'Boussinesq Paradigm.'

Outline of the Problem

- Apart from the significance for the shallow water flows, this paradigm is very important for understanding the particle-like behavior of nonlinear localized waves. As it should have been expected, most of the physical systems are not fully integrable (even in one spatial dimension) and only a numerical approach can lead to unearthing the pertinent physical mechanisms of the interactions.

Outline of the Problem

- The overwhelming majority of the analytical and numerical results obtained so far are for one spatial dimension, while in multidimension, much less is possible to achieve analytically, and almost nothing is known about the unsteady solutions that involve interactions, especially when the full-fledged Boussinesq equations are involved.

Outline of the Problem

- In the present work, we undertake an asymptotic semi-analytical solution for moderate phase speeds and compare the results with the above mentioned numerical works.

Boussinesq Paradigm Equation (BPE)

As shown in Christov(2001), the consistent implementation of the Boussinesq method yields the following Generalized Wave Equation (GWE) for $f = \phi(x, y, 0; t)$:

$$f_{tt} + 2\beta\nabla f \cdot \nabla f_t + \beta f_t \Delta f + \frac{3\beta^2}{2}(\nabla f)^2 \Delta f - \Delta f + \frac{\beta}{6}\Delta^2 f - \frac{\beta}{2}\Delta f\, t = 0. \tag{1}$$

Eq. (1) is the most rigorous amplitude equation that can be derived for the surface waves over an inviscid shallow layer, when the length of the wave is considered large in comparison with the depth of the layer. It was derived only in 2001. Besides it a plethora of different inconsistent Boussinesq equations are still vigorously investigated. The most popular are the versions that contain a quadratic nonlinearity which are useful from the paradigmatic point of view.

Unfortunately, Boussinesq did some additional unnecessary assumptions, which rendered his equation incorrect in the sense of Hadamard. We term the original model the 'Boussinesq's Boussinesq Equation' (BBE). During the years, it was 'improved' in a number of works. The mere change of the incorrect sign of the fourth derivative in BBE yields the so-called 'good' or 'proper' Boussinesq equation (BE). A different approach to removing the incorrectness is by changing the spatial fourth derivative to a mixed fourth derivative, which resulted into an equation know nowadays as the Regularized Long Wave Equation (RLWE) or Benjamin–Bona–Mahony equation (BBME).

Boussinesq Paradigm Equation (BPE)

In fact, the mixed derivative occurs naturally in Boussinesq derivation (see Eq. (1)), and was changed by Boussinesq to a fourth spatial derivative under an assumption that $\partial_t \approx c\partial_x$, which is currently known as the 'Linear Impedance Relation' (or LIA). The LIA has produced innumerable instances of unphysical results.

Boussinesq Paradigm Equation (BPE)

Boussinesq applied the LIA also to the nonlinear terms, and neglected the cubic nonlinearity. This simplified the nonlinear terms of Eq. (1) to a point where Boussinesq was able to find the first *sech* solution for the permanent localized wave. The actual nonlinearity is important because it provides for the Galilean invariance of the model. We focus here on the following two-dimensional amplitude equation:

$$w_{tt} = \Delta \left[w - \alpha(w^3 - \sigma w^5) + \beta_1 w_{tt} - \beta_2 \Delta w \right], \qquad (2)$$

where w is the surface elevation, $\beta_1, \beta_2 > 0$ are two dispersion coefficients. The parameter σ accounts for the relative importance of the quintic nonlinearity term. We term this equation the Qubic-Quintic Boussinesq Paradigm Equation (QQBPE).

An important advantage of the QQBPE is that its energy is the following functional, namely,

$$\frac{dE}{dt} = 0, \quad E = \frac{1}{2}\int_{-\infty}^{\infty} \Big[(\frac{\partial w}{\partial t})^2 + (\frac{\partial w}{\partial x})^2 - \frac{2}{3} w^4 + \frac{1}{3}\sigma w^6 + \beta_1 (\frac{\partial^2 w}{\partial t \partial x})^2 + (\frac{\partial^2 w}{\partial x^2})^2 \Big] \mathrm{d}x. \tag{3}$$

The blow up occurs when the negative term can start increasing in time. This happens when the amplitude of the function w increases. Unlike the BPE with quadratic nonlinearity, when the amplitude increases, the quintic term in QQBPE will dominate and will make the energy functional positive, which limits the increase of the amplitude. All this means that no blow-up can be expected for QQBPE. For smaller w, the energy can become even negative. because the fourth order term dominated the sixth-order on.

Boussinesq Paradigm Equation (BPE)

In one spatial dimension, an analytical solution was found in Maugin&Cadet (1991) for a system involving the QQBPE. This analytical solution was used in Christov& Maugin (1995) to investigate the collision dynamics in 1D. As usual, an analytical solution in 2D is not available. To find an approximation to the 2D solution is the objective of the present work. We follow Christov& Choudury (2011) and create an asymptotic solution valid for small phase speeds of the soliton.

Boussinesq Paradigm Equation (BPE)

For the numerical interaction of 2D Boussinesq solitons, one needs the shape of a stationary moving solitary wave in order to construct an initial condition. To this end, introduce relative coordinates $\hat{x} = x - c_1 t$, $\hat{y} = y - c_2 t$, in a frame moving with velocity (c_1, c_2). Since there is no evolution in the moving frame $v(x, y, t) = u(\hat{x}, \hat{y})$, and the following equation holds for u:

$$\begin{aligned}(c_1^2 u_{\hat{x}\hat{x}} + 2c_1 c_2 u_{\hat{x}\hat{y}} + c_2^2 u_{\hat{y}\hat{y}}) = (u_{\hat{x}\hat{x}} + u_{\hat{y}\hat{y}}) - [(u^3 - \sigma u^5)_{\hat{x}\hat{x}} \\ + (u^3 - \sigma u^5)_{\hat{y}\hat{y}})] - (u_{\hat{x}\hat{x}\hat{x}\hat{x}} + 2u_{\hat{x}\hat{x}\hat{y}\hat{y}} + u_{\hat{y}\hat{y}\hat{y}\hat{y}}) \\ + \beta_1[c_1^2(u_{\hat{x}\hat{x}\hat{x}\hat{x}} + u_{\hat{x}\hat{x}\hat{y}\hat{y}}) \\ + 2c_1 c_2(u_{\hat{x}\hat{x}\hat{x}\hat{y}} + u_{\hat{x}\hat{y}\hat{y}\hat{y}}) + c_2^2(u_{\hat{x}\hat{x}\hat{y}\hat{y}} + u_{\hat{y}\hat{y}\hat{y}\hat{y}})].\end{aligned}$$

The so-called asymptotic boundary conditions (a.b.c.) read $u \to 0$, for $\hat{x} \to \pm\infty$, $\hat{y} \to \pm\infty$. The a.b.c.'s are invariant under rotation of the coordinate system, hence it is enough to consider solitary propagating along one of the coordinate axes, only. We chose $c_1 = 0$, $c_2 = c \neq 0$. Without fear of confusion we will 'reset' the names of the independent variables to x, y and omit in what follows the hat over the function u.

Perturbation Method

The small parameter does not multiply the highest derivative, hence the expansion is regular. When $c = 0$, the solution possesses a radial symmetry, and we consider the expansion

$$u(x,y) = u_0(r) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + O(\varepsilon^3), \quad r = \sqrt{x^2+y^2}. \tag{4}$$

Here we note that

$$(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)^3 \approx u_0^3 + 3\varepsilon u_0 u_1^2 + 3\varepsilon^2(u_0 u_1^2 + u_0^2 u_2) + O(\varepsilon^3), \tag{5}$$

$$(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)^5 \approx u_0^5 + 5\varepsilon u_0^4 u_1 + 10\varepsilon^2 u_0^3 u_1^2 + 5\varepsilon^2 u_0^4 u_2 + O(\varepsilon^3), \tag{6}$$

Now, neglecting the terms of order $O(\varepsilon^3)$, we get for the three lowest orders in ε the following system

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\left[u_0(r) - u_0^3(r) + \sigma u_0^5(r) - \frac{1}{r}\frac{d}{dr}r\frac{du_0}{dr}\right] = 0, \tag{7a}$$

$$\varepsilon\left[-\frac{d^2}{dy^2}u_0 + \beta_1\frac{d^4}{dy^4}u_0 + \Delta u_1 - 3\Delta(u_0^2 u_1) + 5\sigma\Delta(u_0^4 u_1) - \Delta^2 u_1\right] = 0. \tag{7b}$$

$$\varepsilon^2\left[-\frac{d^2}{dy^2}u_1 + \beta_1\frac{d^4}{dy^4}u_1 + \Delta u_2 - 3\Delta(u_0 u_1^2 + u_0^2 u_2) + 10\sigma\Delta(u_0^3 u_1^2) + 5\sigma\Delta(u_0^4 u_2) - 2\Delta(u_0 u_2) - \Delta^2 u_2\right] = 0. \tag{7c}$$

Perturbation Method

We prefer to treat the above system in polar coordinates because then the region is unbounded only with respect to one of the variables (the polar radius r).

In this work we limit ourselves to the $O(\epsilon)$ approximation.

The governing system

We get the following equations

1. for $u_0(r) = F(r)$:

$$F(r) - F^3(r) + \sigma F^5(r) - \frac{1}{r}\frac{d}{dr}r\frac{dF}{dr} = 0. \tag{8a}$$

2.

$$-G(r) + 3F^2(r)G(r) - 5\sigma F^4(r)G(r) + \frac{d^2G}{dr^2} + \frac{1}{r}\frac{dG}{dr} = -\frac{1}{2}F(r). \tag{8b}$$

3.

$$r\frac{d}{dr}\frac{1}{r^3}\frac{d}{dr}r^2\left[-H(r) + 3F^2(r)H(r) - 5\sigma F^4(r)H(r) + r\frac{d}{dr}\frac{1}{r^3}\frac{d}{dr}r^2H(r)\right] = \frac{1}{2}\left[\frac{d^2}{dr^2}F(r) - \frac{1}{r}\frac{d}{dr}F(r)\right], \tag{8c}$$

The Governing System

When one is faced with singularities that arise from the use of specific coordinates (e.g., polar coordinates), one has to ensure the proper behavior of the functions in the point of singularity by imposing additional (purely mathematical) conditions in the geometric singularity called 'behavioral'. The behavioral conditions at the origin arise from the fact that there is a singularity in the operator:

$$H'(0) = H'''(0) = G'(0) = G'''(0) = 0, \tag{9a}$$

while the behavioral conditions at infinity are the asymptotic boundary conditions (a.b.c.):

$$G(r), H(r) \to 0 \quad \text{for} \quad r \to \infty. \tag{9b}$$

The equations possess non-trivial solutions provided a nontrivial solution is found $F(r) \not\equiv 0$. Thus, one can tackle the bifurcation problem while finding the function $F(r)$.

The boundary value problem Eqs. (8),(9) is to be solved numerically. We use a grid which is staggered by $\frac{1}{2}h$ from the origin $r = 0$, while it coincides with the "numerical infinity", $r = r_\infty$. Thus

$$r_i = \left(i - \tfrac{1}{2}\right)h, \quad r_{i\pm\frac{1}{2}} = r_i \pm \tfrac{1}{2}h, \qquad h = r_\infty/(N - 0.5),$$

where N is the total number of points. The staggered grid gives a unique opportunity to create difference approximations for the Bessel operators involved in our model that take care of the singularities of the respective Bessel operator automatically, without the need to impose explicit behavioral boundary conditions in the origin. This is made possible by the fact that $r_{-\frac{1}{2}} = 0$.

To present the results, we fist find the appropriate best-fit analytic expressions for the functions F, G, H, and then construct an analytic expression. We found the following best fit approximation for the shape of the stationary propagating soliton for $\beta_2 = 1$, namely:

$$w^s(x,y,t;c) = f(x,y) + c^2[g(x,y) + h(x,y)\cos(2\theta)], \qquad (10)$$

$$f(x,y) = \frac{1.0032}{\cosh r}\frac{1+0.3r^2}{(1+0.2r+0.65r^2)^{1.25}},$$

$$g(x,y) == 0.203\left(\frac{1.2}{\cosh r} - \frac{0.3}{\cosh 2r}\right)(1+0.1r^2)^{0.25},$$

$$h(x,y) = 0.6\frac{0.5r^2+0.4r^4}{5.2+1.3\sqrt{r}+2.7r+7r^2+3.2r^3+1.4r^4+0.095r^6}$$

where $r(x,y) = \sqrt{x^2+y^2}$, and $\theta(x,y) = \arctan(y/x)$.

Function $F(r)$

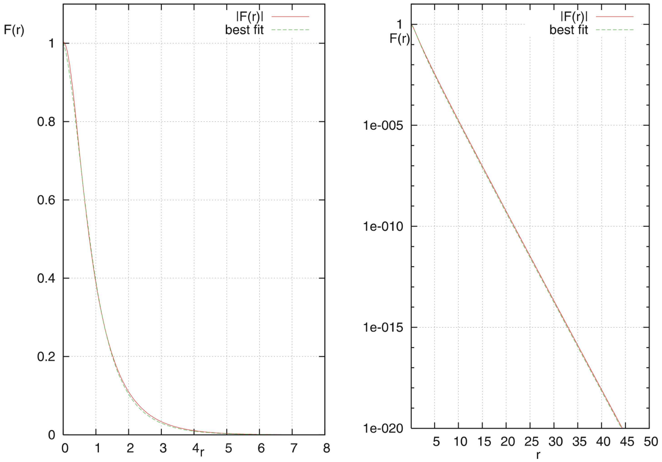

Figure: 1. Function $F(r)$

Function $G(r)$

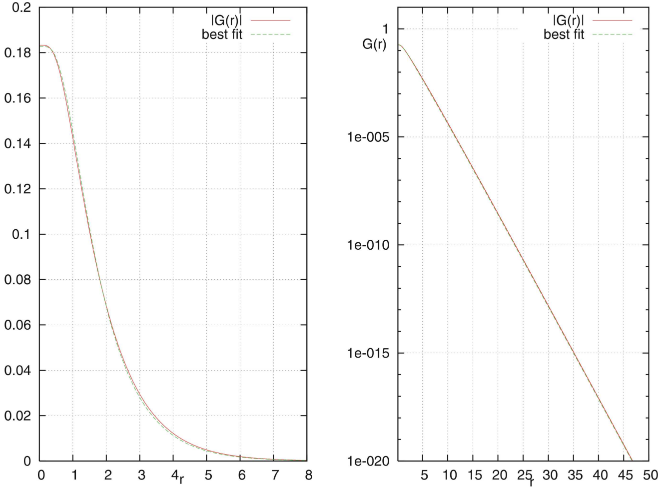

Figure: 2. Function $G(r)$

Function $H(r)$

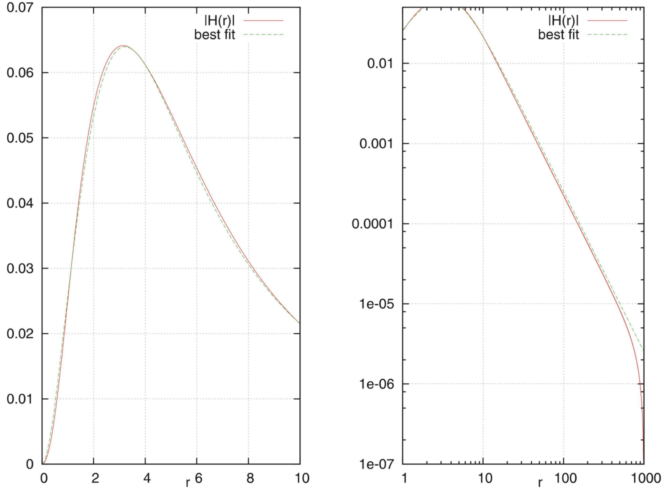

Figure: 3. Function $H(r)$

The shapes of the 2D Boussinesq solitons, $c_1 = 0$, $c_2 = 0.3$

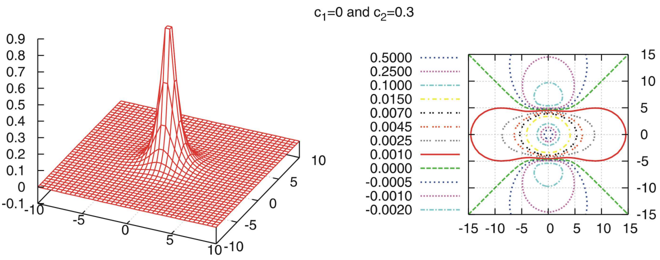

Figure: 4.

The shapes of the 2D Boussinesq solitons, $c_1 = 0$, $c_2 = 0.6$

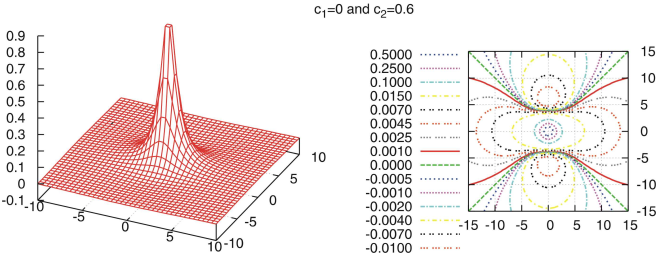

Figure: 5.

Thanks

Thank you for your kind attention !