Numerical Study of Traveling Wave Solutions to 2D Boussinesq Equation

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Posing The Problem

Hyperbolic Equation

Boussinesq Paradigm Equation (BPE):

\[ u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \beta_2 \Delta^2 u + \Delta F(u) = 0, \quad F(u) := \alpha u^2 \]

\[ u : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R} \]
\[ (x, y, t) \rightarrow u(x, y, t) \]

- Origins\(^1,^2\)
- Model
- Properties
  - soliton solution
  - behavior of the soliton

Hyperbolic Equation
Elliptic (Stationary) Equation

- Variable change
- Stationary BPE (S BPE)
  - solutions of type \( u(x,y,t) = v(x,y-ct) \):
    \[
    \beta c^2 (E - \Delta)v_{yyyy} - \beta \Delta v + \Delta^2 v + \alpha \beta \Delta (v^2) = 0, \quad (S \ BPE)
    \]
    with \( \beta = \beta_1 / \beta_2 \) and \( \alpha, \beta > 0 \).

- Equation (S BPE) as a second order system (SYS):
  \[
  -(1 - c^2 \beta) \Delta v + \beta (1 - c^2)v + \alpha \beta v^2 = w
  \]
  \[
  -\Delta w = c^2 \beta (E - \Delta)v_{xx}
  \]

Elliptic Equation
Solver Algorithm

Simple Iteration Method

- Add artificial time
- Add false time derivatives
- Solve the new pertinent transient equation system
  - Wait for $\hat{v}$ and $\hat{w}$ to converge

$$\frac{\partial \hat{v}}{\partial t} - (1 - c^2 \beta) \Delta \hat{v} + \beta (1 - c^2) \hat{v} + \alpha \beta \theta \hat{v}^2 = \hat{w} \quad (SIM.1)$$

$$\frac{\partial \hat{w}}{\partial t} - \Delta \hat{w} = c^2 \beta (E - \Delta) \hat{v}_{xx} \quad (SIM.2)$$
## Finite Differences

<table>
<thead>
<tr>
<th>order</th>
<th>finite difference</th>
<th>second derivative approx</th>
</tr>
</thead>
</table>
| $p = 2$: | $[1 \quad -2 \quad 1]$ | \[
\frac{\partial^2}{\partial x^2} v = \frac{1}{h^2} \left[ (x-h) - 2v(x) + v(x-h) \right]
\]

| $p = 4$: | $\left[ -\frac{1}{12}, 3', -\frac{5}{2}, 3', -\frac{1}{12} \right]$ | \[
\frac{\partial^2}{\partial x^2} v = \frac{1}{h^2} \left[ \frac{1}{12} v(x-2h) + \frac{4}{3} v(x-h) - \frac{5}{2} v(x) + \frac{4}{3} v(x+h) - \frac{1}{12} v(x+2h) \right]
\]

| $p = 6$: | $\left[ \frac{1}{90}, -\frac{3}{20}, 3', -\frac{49}{18}, 3', -\frac{3}{20}, \frac{1}{90} \right]$ | \[
\frac{\partial^2}{\partial x^2} v = \frac{1}{h^2} \left[ \frac{1}{90} v(x-3h) - \frac{3}{20} v(x-2h) + \frac{3}{2} v(x-h) - \frac{49}{18} v(x) + \frac{3}{2} v(x+h) - \frac{3}{20} v(x+2h) + \frac{3}{90} v(x+3h) \right]
\]
Additional Tools

➤ The trivial solution must be avoided
➤ Fix the value of the function in point (0, 0)\(^4\)
  • \(\nu(0,0) = \theta\)
  • \(\tilde{\nu} = \theta \nu\) and \(\tilde{w} = \theta w\)

\[-(1-c^2 \beta) \Delta \hat{v} + \beta(1-c^2)\hat{v} + \alpha \beta \hat{v}^2 = \hat{w} \quad (SYS.1)\]
\[-\Delta \hat{w} = c^2 \beta (E - \Delta) \hat{v}_{xx} \quad (SYS.2)\]

➤ The value of \(\theta\) is found from the equation (S BPE)\(^4\)

\[
\theta = \left. \frac{(1-c^2 \beta) \Delta \hat{v} - \beta(1-c^2)\hat{v} + \hat{w}}{\alpha \beta} \right|_{x=0, y=0} \quad (TH)\]

Boundary Condition

- **Solution Asymptotics**
  - $1/r^2$ asymptotics decay at infinity\(^4\)

\[
\beta c^2 v_{yy} - \beta \Delta v - \beta c^2 \Delta v_{yy} + \Delta^2 v + \alpha \beta \Delta(v^2) = 0, \quad (S\ BPE)
\]

- Assume that $(\partial^n/\partial r^n)v$ has $(1/r^{n+2})$ asymptotics decay at infinity

\[
\frac{\partial^2}{\partial y^2} \equiv \sin^2 \theta \frac{\partial^2}{\partial r^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\sin 2\theta}{r} \frac{\partial^2}{\partial r \partial \theta} - \frac{\sin 2\theta}{r^2} \frac{\partial}{\partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial}{\partial r}.
\]

\[
c^2 v_{yy} = \Delta v, \quad (\text{inf})
\]

\[v(x, y), \Delta v(x, y) \rightarrow \infty \quad \text{as} \quad r = \sqrt{x^2 + y^2} \rightarrow \infty\]

Using the given properties of the equation:

- $1/r^2$ asymptotics decay at infinity
- the symmetry of the solution
- positive/negative domains

the following formula is obtained for the boundary condition:

$$
\bar{v}(x, y) = \mu \frac{(1-c^2)x^2 - y^2}{(1-c^2)x^2 + y^2} \quad (vB)
$$

$$
\bar{w}(x, y) = \bar{\mu} \frac{(1-c^2)x^2 - y^2}{(1-c^2)x^2 + y^2} \quad (wB)
$$
Validation

New Stop Criterion

- Choose neutral condition - the $1/r^2$ profile of the solution
- better convergence results for all finite difference schemes
- legit results
  - solution
  - boundary function
**x-y cross-sections of the solution**

Upper panels:
- The absolute value of the function on log-log plots.
- Black line describes \((vB)\) function with the respective \(\mu\) parameter.

Lower panels show:
- Plots display \(vr^2\) values along the vertical \(z\)-axis.

The solution settles down as the number of points \(N_x, N_y\) per simulation increases!

The effect of the mesh size. Lower panels: the function scaled by \(r^2\). \(N_x, N_y\) – number of mesh-points along \(x, y\) axis.
Validation - Algorithm's Convergence

<table>
<thead>
<tr>
<th>FDS</th>
<th>h</th>
<th>Errors $E_i$ in $L_2$</th>
<th>Conv Rate</th>
<th>Errors $E_i$ in $L_\infty$</th>
<th>Conv Rate</th>
<th>Diff $D_i$ in $L_2$</th>
<th>Diff $D_i$ in $L_\infty$</th>
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</table>

1. Runge's formula for convergence rate

$(\log E_1 - \log E_2) / \log 2$

$$E_1 = \left\| \hat{v}_{[h]} - \hat{v}_{[h/2]} \right\|, E_2 = \left\| \hat{v}_{[h/2]} - \hat{v}_{[h/4]} \right\|,$$

2. Diff between Chr and Nat solutions

$$D_1 = \left\| \hat{v}.Chr_{[h]} - \hat{v}.Nat_{[h]} \right\|, D_2 = \left\| \hat{v}.Chr_{[h/2]} - \hat{v}.Nat_{[h/2]} \right\|, D_3 = \left\| \hat{v}.Chr_{[h/4]} - \hat{v}.Nat_{[h/4]} \right\|$$
Derivative Convergence

<table>
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<tr>
<th></th>
<th>FDS</th>
<th>h</th>
<th>errors in $L_2$</th>
<th>Conv. Rate</th>
<th>errors in $L_\infty$</th>
<th>Conv. Rate</th>
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<td>0.2</td>
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</table>

Errors in $L_2$ and $L_\infty$ norms and convergence rate for fourth order x-derivative evaluated by the FDS with $O(h^2)$ and $O(h^6)$ approximation order

Runge's test, evaluating the fourth x-derivative of the solution, show that it converges numerically. Tests for other fourth order derivatives are similar and we do not present them here.
Best-Fitt Approximation formulae

\[ w^s(x, y, t; c) = f(x, y) + c^2 [(1 - \beta_1)g_a(x, y) + \beta_1 g_b(x, y)] \\
+ c^2 [(1 - \beta_1)h_1(x, y) + \beta_1 h_2(x, y) \cos(2\theta)], \]

where

\[ f(x, y) = \frac{2.4(1 + 0.24r^2)}{\cosh(r)(1 + 0.095r^2)^{1.5}}, \]

\[ g_a(x, y) = -\frac{1.2(1 - 0.177r^{2.4})}{\cosh(r)(1 + 0.11r^{2.1})}, \quad g_b(x, y) = -\frac{1.2(1 + 0.22r^2)}{\cosh(r)(1 + 0.11r^{2.4})}, \]

\[ h_1(x, y) = \frac{a_i r^2 + b_ir^3 + c_i r^4 + v_i r^6}{1 + d_i r + e_i r^2 + f_i r^3 + g_i r^4 + h_i r^5 + q_i r^6 + w_i r^8}. \]

Comparison between the numerical solution $\tilde{\nu}$ and the best fit formulae $c = \beta$:

- $c = 0.3$, $\beta = 1$
- $c = 0.3$, $\beta = 3$
- $c = 0.3$, $\beta = 5$
- $c = 0.1$, $\beta = 1$
- $c = 0.5$, $\beta = 1$
- $c = 0.9$, $\beta = 1$
Thank you for your attention