# Global existence of the solutions to Boussinesq paradigm equation with supercritical initial energy

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**In memory of our colleague and friend Christo Christov**

# Introduction

We consider the Cauchy problem for Boussinesq Paradigm Equation (BPE)

$$U_{tt} - \Delta U - \beta_1 \Delta U_{tt} + \beta_2 \Delta^2 U = \Delta f(U) \text{ for } X \in \mathbb{R}^n,\ t \in [0,T), T \le \infty \quad (1)$$
$$U(X,0) = U_0(X), \quad U_t(X,0) = U_1(X) \quad \text{for } X \in \mathbb{R}^n,$$

with nonlinear term $f(U) = \alpha |U|^p$, where $\alpha > 0$, $\beta_1 \ge 0$, $\beta_2 > 0$ are real constants, $1 < p < \infty$ for $n = 1,2$ and $(n+2)/n \le p \le (n+2)/(n-2)$ for $n \ge 3$.

This problem arises in number of mathematical models of physical processes, for example in the modeling of surface waves in shallow waters. The derivation of equation (1) from the full Boussinesq model can be found e. g. in

- Christov, C.I., *Wave motion* 34, (2001) 161 – 174.

**Problem**: When the weak solution of (1) is globally defined or blow up for a finite time?

# Some of the scientific contributions of C. Christov to this problem

- Christov, C.I., Velarde, M., *Int. J Bifurcation Chaos*, 4 (1994) 1095–1112.
- C.I. Cristov, *Proc. ICFDS*, Oxford, (1996), 343–349.
- Christov, C.I., *Wave motion* 34 (2001) 161–174; 34 (2001), 161-174; 45 (2008), 154–161.
- Christov C.I., *Int. J. Bifurcation Chaos*, 12 (2002), 2435-2444.
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- Christov C.I., Maugin G.A. and Porubov A., *C. R. Mechanique*, 335 (2007) 521-535.
- Christou, M., Christov, C. I., *AIP* 1186, (2009) 217–224.
- Chertock, A., Christov, C. I., Kurganov, A., *Computational Sci., & High performance Computing IV, NNFM*, 113, (2011), 267–281.
- Christov, C.I., Choudhury, J.: *Mech. Res. Commun.* 38 (2011), 274–281.
- Christov, C.I., Todorov, M.T., Christou, M.A.: *AIP Conference Proceedings*, 1404 (2011), 49–56.

# Theoretical results for global existence or finite time blow up of the solutions

- J. Bona, R. Sachs, *Commun. Math. Phys.* 118 (1988) 15–29.
  $n = 1,\ \beta_1 = 0,\ \beta_2 = 1$

- S. K. Turitsyn, *Phys. Rev. E* 47 (1993) R13–R16, *Phys. Rev. E* 47 (1993) R796–R799.
  $n = 1,\ f(U) = U^2$ and $\beta_1 = 0,\ \beta_2 = 1$ or $\beta_1 = 1,\ \beta_2 = 0$

- Y. Liu and R. Xu , *Physica D.* 237 (2008) 721–731.
  $n = 1,\ \beta_1 = 0,\ \beta_2 = 1,\ f(U) = \alpha|U|^{p-1}U,\ \alpha > 0$

- S. Wang and G. Chen, *Nonlinear Analysis* 64 (2006) 159–173.
  $n = 1,\ \beta_1 = 1,\ \beta_2 = 1$

- Q. Lin, Y. Wu, R. Loxton, *J. Math. Anal. and Appl.* 353 (2009) 186–195.
  $n = 1,\ \beta_1 = 0,\ \beta_2 = 1$

- F. Linares, *J. Diff. Eq.* 106 (1993) 257–293.
  $n \geq 1$, $f(U) = \alpha|U|^{p-1}U$, $\alpha < 0$

- R. Xue, *J. Math. Anal. Appl.* 316 (2006) 307–327.
  $n \geq 1$, $f(U) = \alpha|U|^{p-1}U$, $\alpha < 0$

- N. Polat and A. Ertas, *J. Math. Anal. Appl.* 349 (2009) 10–20.
  $n \geq 1$, $\beta_1 = 1$, $\beta_2 = 1$

- R. Xu Y. Liu, *J. Math. Anal. Appl.* 359 (2009) 739–751.
  $n \geq 1$, $\beta_1 = 1$, $\beta_2 = 1$, $f(U) = \alpha|U|^p$

- R. Xu Y. Liu, Y. Tao, *Nonlinear Anal.* 71 (2009) 4977–4983.
  $n \geq 1$, $\beta_1 = 1$, $\beta_2 = 1$

# Preliminaries

Change of the variables: $x = X/\sqrt{\beta_2}$

BPE can be rewritten in the following form with $u(x,t) = U(X,t)$, $u_0(x) = U_0(X)$ and $u_1(x) = U_1(X)$:

$$\begin{aligned} &\beta_2 u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \Delta^2 u = \Delta f(u), \quad \text{for } x \in \mathbb{R}^n,\ t \in [0,T),\ T \leq \infty, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad \text{for } x \in \mathbb{R}^n. \end{aligned} \tag{2}$$

We consider problem (2) with initial data

$u_0 \in \mathrm{H}^1(\mathbb{R}^n), \quad u_1 \in \mathrm{L}^2(\mathbb{R}^n), \quad (-\Delta)^{-1/2} u_1 \in \mathrm{L}^2(\mathbb{R}^n).$

$(-\Delta)^{-s} u = F^{-1}\left(|\xi|^{-2s} F(u)\right)$, $s > 0$,

$F(u)$, $F^{-1}(u)$ – Fourier transformation and the inverse Fourier transformation, respectively.

Conservation low: $E(t) = E(0)$ for every $t \in [0, T)$,

$$E(t) = E(u(\cdot,t), u_t(\cdot,t)) = \frac{1}{2}\left[\beta_2 \left\|(-\Delta)^{-1/2} u_t\right\|^2 + \beta_1 \|u_t\|^2 + \|u\|^2_{\mathrm{H}^1}\right] + \frac{\alpha}{p+1}\int_{\mathbb{R}^n} |u|^p u\, dx$$

Let us consider the functionals $J(u)$ and $I(u)$:

$$J(u) = \frac{1}{2}\|u\|^2_{\mathrm{H}^1} + \frac{\alpha}{p+1}\int_{\mathbb{R}^n} |u|^p u\, dx, \qquad I(u) = \|u\|^2_{\mathrm{H}^1} + \alpha \int_{\mathbb{R}^n} |u|^p u\, dx.$$

By means of these functionals the critical energy constant $d$ is defined as

$$d = \inf_{u \in \mathrm{N}} J(u), \quad \mathrm{N} = \{u \in \mathrm{H}^1;\ I(u) = 0,\ \|u\|_{\mathrm{H}^1} \neq 0\},$$

or equivalently $d = \inf_{u \in \mathrm{H}^1 \setminus \{0\}} \sup_{\lambda} J(\lambda u)$.

- R. Xu Y. Liu, *J. Math. Anal. Appl.* 359 (2009) 739–751

**Theorem 1:**(R. Xu, Y. Lin)

(i) If $E(0) < 0$ and $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, then every weak solution of BPE blows up for an appropriate finite time.

(ii) If $E(0) = 0$ and $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, then every weak solution of BPE, except the trivial one, blows up for an appropriate finite time.

(iii) Let $0 < E(0) < d$.

If $I(u_0) < 0$ and $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, then the weak solution of BPE blows up for an appropriate finite time.

If $I(u_0) > 0$ then the weak solution of BPE is globally defined for every $t \in [0, \infty)$.

• N. Kutev, N. Kolkovska, M. Dimova, Christov C.I., *AIP Conf. Proc.* 1404 (2011) 68–76.

We have proved that

$$d = \frac{p-1}{2(p+1)} \left(\alpha C_\star^{p+1}\right)^{-2/(p-1)}, \qquad C_\star = \sup_{\substack{u \in \mathrm{H}^1 \\ u \neq 0}} \frac{\|u\|_{\mathrm{L}^{p+1}}}{\|u\|_{\mathrm{H}^1}}.$$

$C_\star$ - the constant of the embedding of $\mathrm{H}^1$ into $\mathrm{L}^{p+1}$, $p > 1$

Moreover, we get explicitly the value of $d$ in the one dimensional case

$$d\big|_{n=1} = \frac{1}{p+3} \left(\frac{2(p+1)}{\alpha}\right)^{\frac{2}{p-1}} \left[\Gamma\left(\frac{2}{p-1}\right)\right]^2 \Big/ \Gamma\left(\frac{4}{p-1}\right).$$

# Numerical experiments

**Aim**: To find a hint for answering the question whether the constant $d$ in Theorem 1 is exact for the validity of its statements, or there exists a constant $\tilde{d} > d$ such that for $E(0) < \tilde{d}$ the statement is still true.

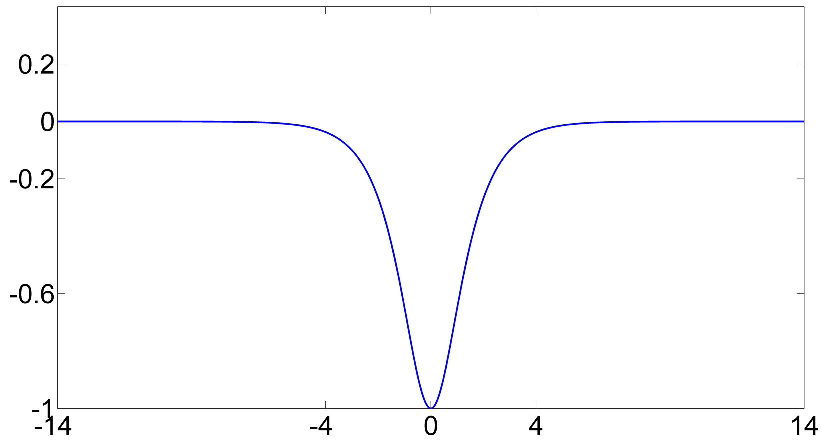
$$\beta_2 u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \Delta^2 u = \alpha \Delta |u|^3, \quad \text{for } x \in \mathbb{R},\ t \in [0, T),\ T \leq \infty,$$
$$u_0(x) = -A\ \mathrm{sech(x)}, \quad u_t(x,0) = u_1(x) = \varepsilon \tilde{u}_0(x), \qquad \text{for } x \in \mathbb{R}$$

$A = const > 0$, $\varepsilon = const$, $\tilde{u}_0(x) \in \mathrm{C}(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R})$ and $\tilde{u}_0(x)$ is defined as:

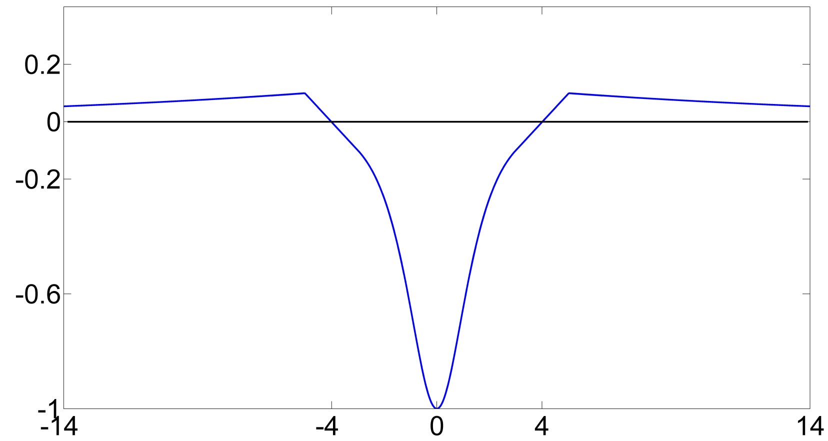
$$\tilde{u}_0(x) = \begin{cases} u_0(x) & \text{for} \qquad\qquad |x| \leq L-1, \\ A\ (|x| - L)\ \mathrm{sech}(L-1) & \text{for} \quad L-1 \leq |x| \leq L+1, \\ A\ e^{-|x|\delta} e^{\delta(L+1)}\ \mathrm{sech}(L-1) & \text{for} \qquad\qquad |x| \geq L+1. \end{cases}$$

$\delta = const > 0$: $\displaystyle\int_{-\infty}^{\infty} \tilde{u}_0(x)\, dx = 0$; $\qquad L = const > 0$: $|u_0(L)| \leq 1 \times 10^{-3}$.

The choice of $\tilde{u}_0(x)$ ensures that $u_1 \in \mathrm{L}^2$ and $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$.




$u_0(x) = -\mathrm{sech}(\mathrm{x})$




$\tilde{u}_0(x)$, $L = 4$

- C. Cristov, *Proc. ICFDS Oxford* (1996) 343–349. – first conservative scheme for BPE

To solve BPE we use conservative, implicit with respect to the nonlinearities, finite difference schemes studied in

- N. Kolkovska, M. Dimova, *Cent. Eur. J. Math.* 10(3) (2012) 1159–1171.

- M. Dimova, N. Kolkovska, *LNCS* 7125 (2012) 215–220.

A second order of convergence of the discrete solution to the exact one is proved in the uniform and in $W_2^1$ mesh norms. Moreover, during the time evolution the finite difference schemes preserve the discrete energy functional.

The used finite difference method is thoroughly tested on the models of propagation of one or two solitary waves and for different nonlinearities.

A regular mesh with space step $h = 0.025$ and time step $\tau = 0.025$ is used for the numerical tests. In addition mesh refinement analysis is performed.

For all numerical tests we set: $\beta_1 = 1.5$, $\beta_2 = 0.5$ and $\alpha = 2$.

**Numerical test 1. Aim: To clarify the question about the exactness of the value of the critical energy $d$.**

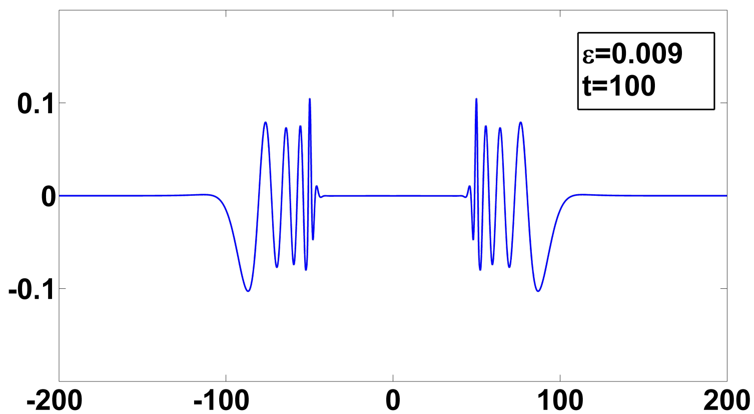
For $f(u) = 2|u|^3$ we have $d = 4/(3\alpha) = 2/3 \approx 0.66666666\ldots$.
$A = 1$: $u_0(x) = -\mathrm{sech}(x)$ is a ground state solution of BPE

If $A = 1$ and $\varepsilon = 0$ (i.e. $u_1(x) = 0$), then $E(0) = d = 2/3$ and $I(u_0) = 0$.
If $A \approx 1$, $A < 1$ and $\varepsilon = 0$ (i.e. $u_1(x) = 0$), then $E(0) < d = 2/3$; $I(u_0) > 0$ for any value of $\varepsilon$ .

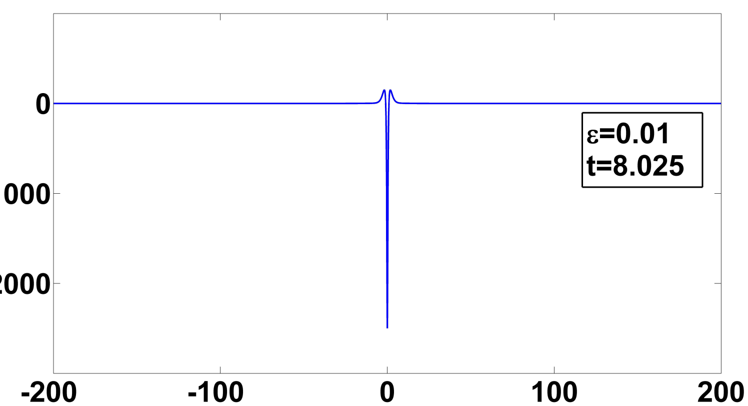
Table 1: Computed initial energy $E(0)$ as a function of $\varepsilon$ and $A$ and the existence time of the solutions of BPE, $x \in [-200, 200]$, $0 \le t \le 100$, $t^*$ - blow up time.

$d = 2/3 \approx 0.66666666\ldots, \quad u_1(x) = \varepsilon\tilde{u}_0(x)$

| $A = 0.95,\ I(0) \approx 0.23465000 > 0$ | | | $A = 0.99,\ I(0) \approx 0.05201164 > 0$ | | |
|---|---|---|---|---|---|
| $\varepsilon$ | $E(0)$ | existence time | $\varepsilon$ | $E(0)$ | existence time |
| 0.000 | 0.66028426 | $t = 100$ | 0.000 | 0.66635497 | $t$=100 |
| 0.010 | 0.66151554 | $t = 100$ | 0.004 | 0.66656891 | $t$=100 |
| 0.046 | 0.68633775 | $t = 100$ | 0.009 | 0.66743804 | $t$=100 |
| 0.047 | 0.68748283 | $t^* \approx 8.475$ | 0.010 | 0.66769210 | $t^* \approx 8.075$ |
| 0.050 | 0.69106579 | $t^* \approx 6.775$ | 0.020 | 0.67170345 | $t^* \approx 5.35$ |




(a) $A = 0.99$, $\varepsilon = 0.009$




(b) $A = 0.99$, $\varepsilon = 0.01$

## **Numerical test 2.** Aim: To study the importance of the initial velocity $u_1(x) = \varepsilon\tilde{u}_0(x)$ to the behavior of the solution.

Table 2: Computed initial energy $E(0)$ as a function of $\varepsilon$ and the existence time of the solutions of BPE; $A = 1.01$, $I(u_0) \approx -0.05467736 < 0$, $x \in [-220, 220]$, $0 \leq t \leq 150$, $t^*$ - blow up time.

| $\varepsilon$ | $E(0)$ | existence time |
|---|---|---|
| 0.010 | 0.6742553634 | $t^* \approx 4.675$ |
| 0.000 | 0.6663476942 | $t^* \approx 5.45$ |
| -0.001 | 0.6664267711 | $t^* \approx 5.55$ |
| -0.009 | 0.6727529162 | $t^* \approx 10.875$ |
| -0.010 | 0.6742553687 | $t = 150$ |

For $\varepsilon = 0.01$ and $\varepsilon = -0.01$ the solutions have different behavior although their initial energies coincide up to the eighth digit. In the first case (when $\varepsilon = 0.01$) the solution blows up for $t^* \approx 4.675$ and in the second case (when $\varepsilon = -0.01$) the solution is bounded up to $t = 150$.

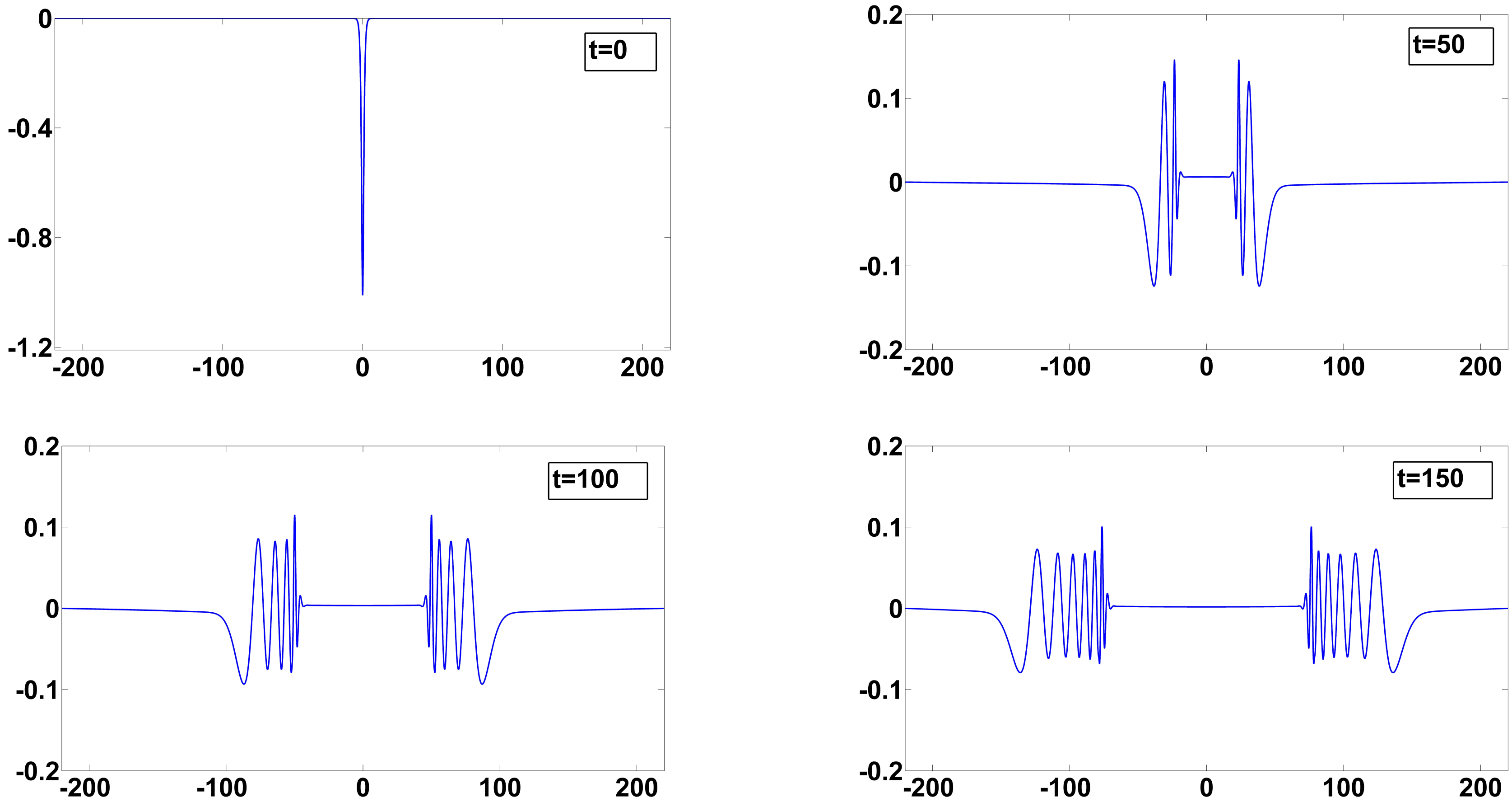



Figure 1: $A = 1.01$, $\varepsilon = -0.01$, $t \in [0, 150]$.

The different sign of the scalar product of the initial data $u_0$, $u_1$, i.e. the sign of $\beta_2 \left( (-\Delta)^{-1/2} u_1, (-\Delta)^{-1/2} u_0 \right) + \beta_1 (u_1, u_0)$ is a possible explanation of the different behaviour of the solutions for supercritical initial energy $E(0) > d$. This means that some additional conditions on the initial data for $E(0) > d$ are needed for proving global existence or finite time blow up of the solution.

# Generalized Nehari functional

Our numerical experiments do not confirm the conjecture that only the sign of $I(u_0)$ is essential for the behavior of the solutions for large initial energy $E(0) > d$. This motivates us to consider a more general functional $I_\gamma(u)$ depending of the parameter $\gamma$:

$$I_\gamma(u) = (1-\gamma)\|u\|^2_{\mathrm{H}^1} + \alpha \int_{\mathbb{R}^n} |u|^p u \, dx = I(u) - \gamma \|u\|^2_{\mathrm{H}^1}, \quad \text{for} \quad \gamma > -\frac{p-1}{2}.$$

We define the constant $D_\gamma$ and the set $\mathrm{N}_\gamma$

$$D_\gamma = \inf_{u \in \mathrm{N}_\gamma} J(u), \qquad \mathrm{N}_\gamma = \{u \in \mathrm{H}^1;\ I_\gamma(u) = 0,\ \|u\|_{\mathrm{H}^1} \neq 0\}.$$

It is clear that if $\gamma = 0$, then $I_0(u) = I(u)$ and $D_0 = d$.

**Theorem 2:** For arbitrary $n$ and $\gamma > -\frac{p-1}{2}$ the equality

$$D_\gamma = \frac{p-1+2\gamma}{2(p+1)} \left( \frac{|1-\gamma|}{\alpha C_\star^{p+1}} \right)^{2/(p-1)} = \frac{p-1+2\gamma}{p-1} \, d \, |1-\gamma|^{2/(p-1)} \qquad \text{holds.}$$

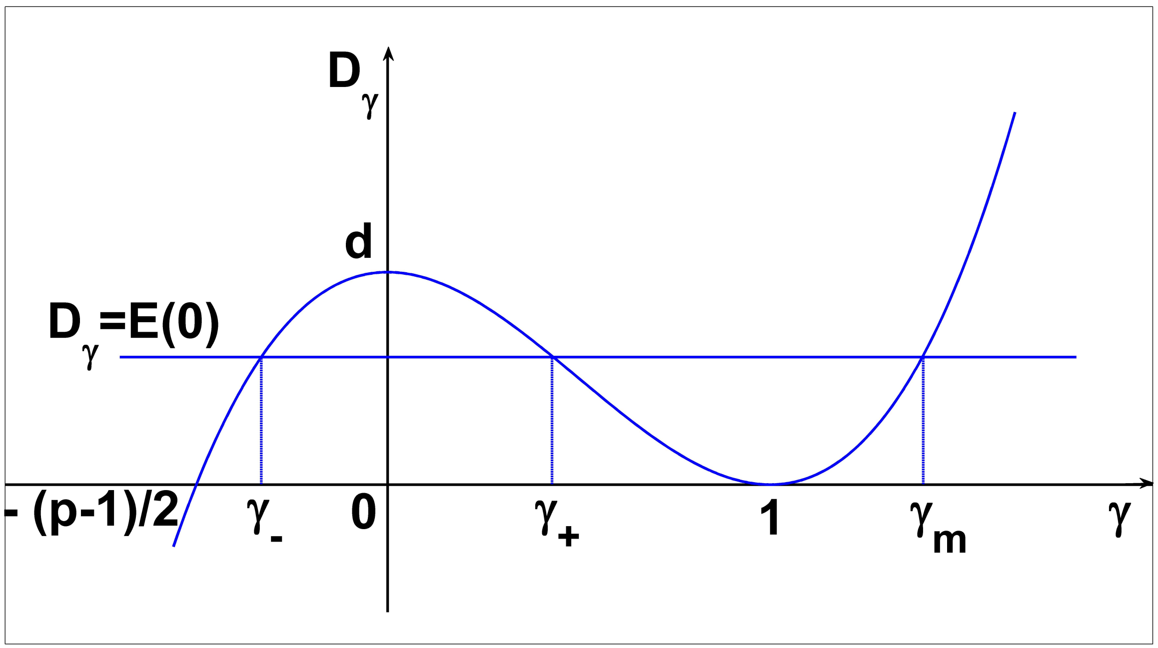



Figure 2: The graphics of $D_\gamma$ as a function of $\gamma$ for $n = 1$, $p = 2$.

# Sign preserving properties of $I_\gamma(u)$

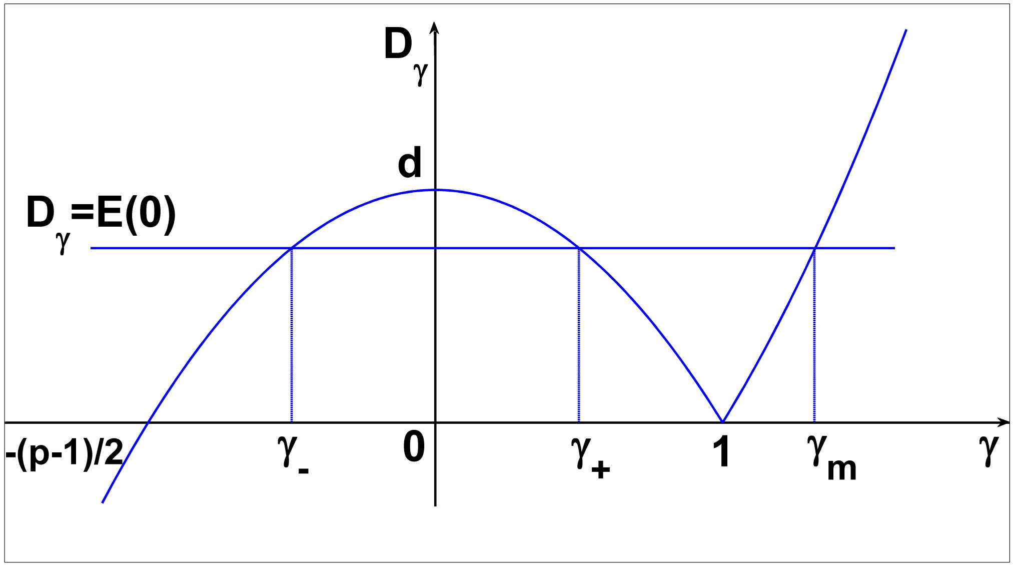



Figure 3: The graphics of $D_\gamma$ as a function of $\gamma$ for $n = 1$, $p = 3$.

**Theorem 3:** Let $u(x,t)$ be a weak solution of BPE with maximal existence time interval $[0, T)$, $T \leq \infty$ and let $0 < E(0) < d$. Then for every $\gamma \in (\gamma_-, \gamma_+]$ the following statements are true:

(i) if $I_\gamma(u_0) < 0$, then $I_\gamma(u(t)) < 0$ for every $t \in [0, T)$;

(ii) if $I_\gamma(u_0) \geq 0$, then $I_\gamma(u(t)) \geq 0$ for every $t \in [0, T)$.

**Remark 1:** The same result as in Theorem 3 holds for $E(0) = d$ in the one-dimensional case $n = 1$.

**Theorem 4:** Let $u(x, t)$ be a weak solution of BPE with maximal existence time interval $[0, T)$, $T \leq \infty$. If $E(0) > 0$, then $I_\gamma(u(t)) \leq 0$ for every $\gamma \geq \gamma_m$ and every $t \in [0, T)$.

**Remark 2:** Our idea was to prove a similar to Theorem 1 global existence result for supercritical initial energy $E(0) > d$ using the sign of $I_\gamma(u_0)$ instead of $I_0(u_0)$. For this purpose the sign preserving properties of $I_\gamma(u)$ under the flow of BPE) should be proved. We can prove the sign invariance of $I_\gamma(u)$. But unfortunately Theorem 4 states that $I_{\gamma_m}(u_0)$ is always non positive for initial energy $E(0) = D_{\gamma_m}$, $\gamma_m > 1$. Moreover $I_{\gamma_m}(u(t))$ is non positive under the flow of BPE. Therefore, initial data with $I_\gamma(u_0) > 0$ do not exist and global existence of the weak solution could not be proved following this idea.

# Global existence

In order to study the global existence of the solution we introduce a new more general Nehari functional $\overline{K}(v,w)$ as

$$\overline{K}(v,w) = I_0(v) - \beta_2\left((-\Delta)^{-1/2}w, (-\Delta)^{-1/2}w\right) - \beta_1(w,w)$$

for every $v \in \mathrm{H}^1$ and for every $w$, $w \in \mathrm{L}^2$ and $(-\Delta)^{-1/2}w \in \mathrm{L}^2$. For simplicity we denote

$$K(u,t) = \overline{K}(u(\cdot,t), u_t(\cdot,t)).$$

In the following Theorem we adapt the nonexistence technique of H. Levine (see also its generalization of V. K. Kalantarov and O. A. Ladyzhenskaya) for global existence. This is untypical application of the so called convex method of Levine.


- H. A. Levine, *TAMS* 192 (1974) 1–21.

- V. Kalantarov and O. Ladyzhenskaya, *Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya matematicheskogo Institute im. V. A. Steklova AN SSSR* 71 (1977) 77–102. (in Russian)

**Theorem 5:** Let $u(x,t)$ be a weak solution of BPE with maximal existence time interval $[0,T)$, $T \le \infty$. Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$, $(-\Delta)^{-1/2}u_1 \in \mathrm{L}^2$ and $E(0) > 0$. Suppose that the following condition is true

$$\beta_2 \left((-\Delta)^{-1/2}u_1, (-\Delta)^{-1/2}u_0\right) + \beta_1(u_1,u_0) + \frac{\beta_2}{2}\|(-\Delta)^{-1/2}u_0\|^2$$
$$+\frac{\beta_1}{2}\|u_0\|^2 + \frac{(p+1)\gamma}{p-1+(p+3)\gamma}E(0) \le 0$$

for some $\gamma \ge \gamma_m$. If $K(u,0) > 0$, then $K(u,t) > 0$ for every $t \in [0,T)$.

As a consequence of the sign preserving properties of the functional $K(u,t)$ we get immediately global existence of the weak solutions of BPE with initial energy $E(0) > d$.

**Theorem 6:** Let us consider the one dimensional BPE problem with initial data $u_0 \in \mathrm{H}^1(\mathbb{R})$, $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2(\mathbb{R})$, $u_1 \in \mathrm{L}^2(\mathbb{R})$, $(-\Delta)^{-1/2}u_1 \in \mathrm{L}^2(\mathbb{R})$. Under the conditions of Theorem 6 if $K(u,0) > 0$ then the weak solution of BPE is globally defined for every $t \in [0,\infty)$.

**Corollary:** Under the conditions of Theorem 6 the following norm

$$\beta_2 \|(-\Delta)^{-1/2} u\|^2 + \beta_1 \|u\|^2$$

strictly monotone decreases to a nonnegative constant when $t \to \infty$.

# More general Nehari functionals

Let us define a family of general Nehari functionals

$$K_\delta(u,t) = K(u,t) - \delta(\beta_2 \|(-\Delta)^{-1/2} u\|^2 + \beta_1 \|u\|^2)$$

for every $\delta \geq 0$. Let $\gamma_m > 1$ be the maximal positive root of $D_\gamma = E(0)$.

**Theorem 7:** Let $u(x,t)$ be a weak solution of BPE with maximal existence time interval $(0,T]$, $T \le \infty$. Suppose $u_0 \in \mathrm{H}^1(\mathbb{R})$, $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2(\mathbb{R})$, $u_1 \in \mathrm{L}^2(\mathbb{R})$, $(-\Delta)^{-1/2}u_1 \in \mathrm{L}^2(\mathbb{R})$, $E(0) > d$ and the following condition

$$([(p+3)\gamma + p - 1](2\gamma + p - 1)\delta)^{1/2}\,[\beta_2((-\Delta)^{-1/2}u_1, (-\Delta)^{-1/2}u_0) + \beta_1(u_1, u_0)] + (p+1)\gamma E(0) \le 0$$

holds for some $\gamma \ge \gamma_m$. If $K_\delta(u,0) > 0$ then $K_\delta(u,t) > 0$ for every $t \in [0,T]$.

**Theorem 8:** Under the conditions of Theorem 7 the weak solution of the one dimensional BPE is globally defined for every $t \in [0,T]$.

## Conclusions

- The special property of the critical energy constant $d$ as a threshold between the global existence or nonexistence of the solution to BPE is clarified. In fact, $d$ is a local maximum of the family of generalized Nehari functionals $I_\gamma(u)$, $\gamma \in \mathbb{R}$.

- For subcritical energy $0 < E(0) \leq d$ the global existence or finite time blow up of the solutions to BPE depends on the initial profile and are independent of the initial velocity.

- For supercritical energy $E(0) > d$ the initial velocity is essential for global solvability or finite time blow up of the solutions to BPE by means of the sign of the scalar product $\beta_2((-\Delta)^{-1/2}u_1, (-\Delta)^{-1/2}u_0) + \beta_1(u_1, u_0)$.

- Generalized Nehari functionals $K_\delta$ (depending on the solution $u(x,t)$ and the time derivative $u_t(x,t)$) and their sign preserving properties are crucial for the global existence of BPE.