Global existence to generalized Boussinesq equation with combined power-type nonlinearities

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We study the Cauchy problem to the generalized Boussinesq equation (Boussinesq Paradigm Equation)

$$\begin{aligned} &\beta_2 u_{tt} - u_{xx} - \beta_1 u_{ttxx} + u_{xxxx} = (f(u))_{xx} && \text{for } x \in \mathbb{R},\ t \in [0, T),\ T \le \infty, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) && \text{for } x \in \mathbb{R},\ \beta_1 \ge 0,\ \beta_2 > 0, \end{aligned}$$

$u_0 \in \mathrm{H}^1(\mathbb{R})$, $u_1 \in \mathrm{L}^2(\mathbb{R})$, $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2(\mathbb{R})$;

$(-\Delta)^{-s} u = \mathcal{F}^{-1}\left(|\xi|^{-2s}\mathcal{F}(u)\right)$, $s > 0$

$$f(u) = \sum_{k=1}^{m} a_k |u|^{p_k - 1} u, \quad a_k = const,\ a_m \neq 0, \quad 1 < p_1 < \ldots < p_m$$

This problem arises in a number of mathematical models of physical processes, for example in the modeling of surface waves in shallow waters.

- Christov, C.I., Wave motion, 34 (2001) 161 – 174

$$f(u) = a|u|^p, \qquad f(u) = a|u|^{p-1}u, \quad a = const, \quad p > 1$$

$$\beta_1 = 0,\ \beta_2 = 1 \quad \text{or} \quad \beta_1 = 1,\ \beta_2 = 1$$

- F. Linares, J. Differential Equations 106 (1993) 257–293
- S. Wang, G. Chen, Nonlinear Anal. 64 (2006) 159–173
- Y. Liu, R. Xu , Physica D. 237 (2008) 721–731
- Y. Liu, R. Xu , J. Math. Anal. Appl. 338 (2008) 1169–1187
- R. Xu, Y. Liu, J. Math. Anal. Appl. 359 (2009) 739–751

$$f(u) = \sum_{k=1}^{m} a_k |u|^{p_k - 1} u, \qquad 1 < p_1 < \ldots < p_m, \qquad \beta_1 = 0,\ \beta_2 = 1$$

$$\exists\ \overline{s}:\ \overline{s} \in [1, m-1]: \begin{cases} a_i \geq 0 & \text{for} \quad i \in [1, \overline{s}]; \\ a_i \leq 0 & \text{for} \quad i \in [\overline{s}+1, m-1]; \qquad a_m < 0 \end{cases}$$

- R. Xu, Math. Methods in Appl. Sci. 34 (2011) 2318–2328

- Original Boussinesq equation – $f(u) = au^2$
- In some models in the theory of atomic chains and shape-memory alloys – cubic-quintic, i.e. $f(u) = au^3 + bu^5$
- Generalized Lienard (or generalized Bernoulli) nonlinearities:

$$f(u) = a|u|^p u + b|u|^{2p} u, \quad p > 0, \; a, b = const \neq 0.$$

Stationary equation corresponding to **BPE**:

$$\psi''(x) = \psi(x) + a|\psi(x)|^p \psi(x) + b|\psi(x)|^{2p} \psi(x) \tag{1}$$

- Equation (1) is known in the literature as the generalized Lienard (or generalized Bernoulli) equation and it is explicitly solvable
- Applications to: exact solutions of the nonlinear Schrödinger equation, the nonlinear Helmholtz equation, the generalized KP equation, the compound KdV the generalized Pochhammer - Chree equation etc.

$$\beta_2 u_{tt} - u_{xx} - \beta_1 u_{ttxx} + u_{xxxx} = (f(u))_{xx} \quad \text{for } x \in \mathbb{R}, \quad t \in [0, T), \quad T \leq \infty,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad \text{for } x \in \mathbb{R}, \ \beta_1 \geq 0, \ \beta_2 > 0,$$

$$f(u) = \sum_{k=1}^{m} a_k |u|^{p_k - 1} u, \ a_k = const, \ a_m \neq 0, \quad 1 < p_1 < \ldots < p_m$$

Main assumption on the nonlinear terms

There exists $\zeta = \inf\{z > 0 : \ \frac{1}{2} z^2 + F(z) = 0\} > 0$
such that $\zeta + f(\zeta) < 0$, where $F(u) = \int_0^u f(s)\, ds.$

This is a necessary and sufficient condition for existence of a nontrivial H^1 solution of second order equation

$$\psi''(x) - \psi(x) = f(\psi) \quad \text{for} \quad x \in \mathbb{R}.$$

H. Berestycki, P.-L. Lions, Arch. Rational Mech. Anal. 82(4) (1983) 313–345.

Important functionals

Conservation low: $E(t) = E(0)$ for every $t \in [0, T)$,

$$E(t) = E(u(\cdot,t), u_t(\cdot,t)) =$$
$$\frac{1}{2}\left(\beta_2 \left\|(-\Delta)^{-1/2} u_t(\cdot,t)\right\|^2 + \beta_1 \|u_t(\cdot,t)\|^2 + \|u(\cdot,t)\|^2_{\mathrm{H}^1}\right) + \int_{\mathbb{R}} F(u(x,t))\, dx$$

Potential energy functional $J(u)$:

$$J(u) = \frac{1}{2}\|u\|^2_{\mathrm{H}^1} + \int_{\mathbb{R}} F(u)\, dx$$

Nehari functional $I(u)$:

$$I(u) = J'(u)u = \|u\|^2_{\mathrm{H}^1} + \int_{\mathbb{R}} uf(u)\, dx$$

Potential well method

- Nehari manifold $\mathcal{N}$:

$$\mathcal{N} = \{u \in \mathrm{H}^1 : \ I(u) = 0, \ \|u\|_{\mathrm{H}^1} \neq 0\}$$

- Critical energy constant d:

$$d = \inf_{u \in \mathcal{N}} J(u)$$

- Depth D of the potential well:

$$D = \inf_{u \in \mathrm{H}^1 \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u) > 0$$

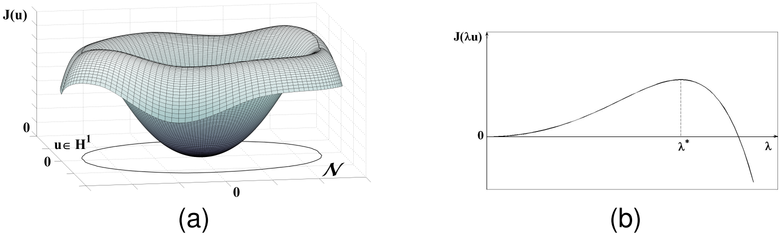


Figure: (a) Schematic illustration of $J(u)$ as a function of $u \in \mathrm{H}^1$; (b) Cross section of $J(\lambda u)$ as a function of λ for a fixed $u \in \mathrm{H}^1$.

Generalized Lienard (or generalized Bernoulli) nonlinearities:

$$f(u) = a|u|^p u + b|u|^{2p} u, \quad p > 0,\ a, b = const \neq 0.$$

Case 1: $b < 0$

Case 2: $b > 0, \quad a < 0, \quad a^2 - \frac{(p+2)^2}{p+1} b > 0$

Case 3: $b > 0, \quad a < 0, \quad a^2 - \frac{(p+2)^2}{p+1} b = 0$

Case 4: $b > 0, \quad a < 0, \quad a^2 - \frac{(p+2)^2}{p+1} b < 0$

Case 5: $b > 0, \quad a > 0$

Remark

Only in **Case 1** and **Case 2** our main assumption on the nonlinearities $f(u)$ is satisfied.

Case 1: $b < 0$, Potential well method

$$I(u) = \|u\|^2_{\mathrm{H}^1} + a \int_{\mathbb{R}} |u|^{p+2}\, dx + b \int_{\mathbb{R}} |u|^{2p+2}\, dx$$

Theorem (Case 1, Global existence)

Let $f(u) = a|u|^p u + b|u|^{2p} u$ and $b < 0$. Suppose $u_0 \in \mathrm{H}^1$, $u_1 \in \mathrm{L}^2$, and $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$. If $E(0) < d$ and $I(u_0) > 0$ or $\|u_0\|_{\mathrm{H}^1} = 0$, then problem **BPE** has a unique global solution defined for every $t \in [0, \infty)$.

Theorem (Case 1, Finite time blow up)

Let $f(u) = a|u|^p u + b|u|^{2p} u$, $b < 0$ and $u_0 \in \mathrm{H}^1$, $u_1 \in \mathrm{L}^2$, $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$. If $E(0) < d$ and $I(u_0) < 0$, then the weak solution of **BPE** blows up in a finite time.

All nontrivial critical points of variational problem

$$d = D = \inf_{u \in \mathrm{H}^1 \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u) = J(\psi)$$

are nontrivial solutions of the following Euler–Lagrange equation:

$$\begin{aligned} &\psi''(x) = \psi(x) + a|\psi(x)|^p \psi(x) + b|\psi(x)|^{2p} \psi(x) \\ &|\psi(x)| \to 0 \text{ as } |x| \to \infty \end{aligned} \qquad (2)$$

Lemma (Ground state solution)

Suppose $f(u) = a|u|^p u + b|u|^{2p} u$ and $b < 0$. Then problem (2) has a unique (up to the sign and translation of the coordinate system) ground state solution $\psi(x)$ defined by

$$\psi(x) = (p+2)^{1/p} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b} \cosh(px) - a \right)^{-1/p}.$$

Explicitly evaluation of the critical energy constant d

Theorem

Let $f(u) = a|u|^p u + b|u|^{2p} u$ and $b < 0$. Then the critical energy constant d is given by:

$$d = -\frac{a}{2}(p+2)^{2/p} \int_{\mathbb{R}} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b \cosh(y)} - a \right)^{-(p+2)/p} dy$$

$$- b \frac{(p+2)^{2(p+1)/p}}{2(p+1)} \int_{\mathbb{R}} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b \cosh(y)} - a \right)^{-2(p+1)/p} dy.$$

Corollary

Let $f(u) = au^3 + bu^5$ and $b < 0$. Then the critical energy constant d is equal to:

$$d\Big|_{p=2} = -\frac{12a}{(3a^2 - 16b)} - \frac{384ab}{(3a^2 - 16b)^2} - \frac{3a}{8b(3a^2 - 16b)^2}(9a^4 - 256b^2 - 192a^2 b)$$

$$+ \frac{\sqrt{3}(3a^2 - 16b)}{32(-b)^{3/2}} \left(\frac{\pi}{2} + \arctan \frac{a}{4}\sqrt{\frac{3}{-b}} \right).$$

Table: Exact values of the critical energy constant d and the lower bound d_0 computed for $b = -3$ and for various values of a.

a	d	d_0	a	d	d_0
0	**0.78539816**	**0.78539816**			
0.0001	0.78542316	0.78539816	-0.0001	0.78537316	0.58903561
1	1.08963046	0.78539816	-1	0.57934063	0.47313158
3	2.10491987	0.78539816	-3	0.34944939	0.31629775
20	40.77456221	0.78539816	-20	0.06614228	0.06583393

$$d_0 = \begin{cases} \frac{p r_0^2}{2(p+2)} & \text{for} \quad a < 0 \\ \frac{p r_0^2}{2(p+1)} & \text{for} \quad a > 0 \end{cases}$$

(d_0 - introduced by R. Xu),

where r_0 is the unique positive root of the equation $\varphi(r) = 1$,

$$\varphi(r) = -a^- C_1^{p+2} r^p - b^- C_2^{2p+2} r^{2p}, \qquad a^- = \min(a,0),\ b^- = \min(b,0),$$

$$C_1 = \sup_{u \in \mathrm{H}^1 \setminus \{0\}} \frac{\|u\|_{\mathrm{L}^{p+1}}}{\|u\|_{\mathrm{H}^1}}, \quad C_2 = \sup_{u \in \mathrm{H}^1 \setminus \{0\}} \frac{\|u\|_{\mathrm{L}^{2p+1}}}{\|u\|_{\mathrm{H}^1}}$$

Numerical experiments, $f(u) = au^3 + bu^5$

Initial data:

$$u_0(x) = -\delta\psi(x), \qquad u_1(x) = -\varepsilon\delta\widetilde{\psi}(x),$$
$$\widetilde{\psi}(x) = \psi(x) - \frac{\nu}{2}\psi(\nu(x-M)) - \frac{\nu}{2}\psi(\nu(x+M)),$$
$$\psi(x) = 2\left(\sqrt{a^2 - \tfrac{16}{3}b}\,\cosh(2x) - a\right)^{-1/2},$$

$\psi(x)$ – ground state solution; δ, ε, ν and M – constants, $\delta > 0$

$$\int_{-\infty}^{\infty} \widetilde{\psi}(x)\,dx = 0 \Rightarrow u_1 \in \mathrm{L}^2 \text{ and } (-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$$

Example: $\beta_1 = 1.5$, $\beta_2 = 0.5$, $a = 1$, $b = -3$;
$\varepsilon = 0.1$, $\nu = 0.1$, $M = 10$

A regular mesh defined in [-250,250] with space step $h = 0.01$ and time step $\tau = 0.01$ is used. In addition mesh refinement analysis is performed.

Numerical experiments, $f(u) = au^3 + bu^5$, $a = 1$, $b = -3$, $d \approx 1.08963046$

- $\delta = 0.8$, $\widetilde{E}(0) \approx 0.92808881 < d$, $\widetilde{I}(u_0) \approx 1.31204620 > 0$

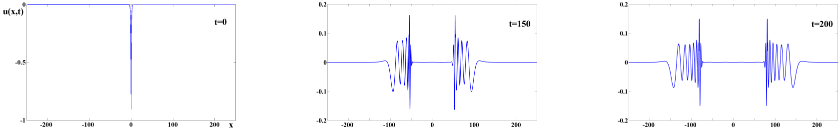


Figure: Profiles of the numerical solution $u(x, t)$ of **BPE** computed for $a = 1$, $b = -3$, $\delta = 0.8$ at different evolution times: (a) t=0; (b) t=150; (c) t=200.

Numerical experiments, $f(u) = au^3 + bu^5$, $a = 1$, $b = -3$, $d \approx 1.08963046$

- $\delta = 1.2$, $\widetilde{E}(0) \approx 0.76106892 < d$, $\widetilde{I}(u_0) \approx -5.82783925 < 0$

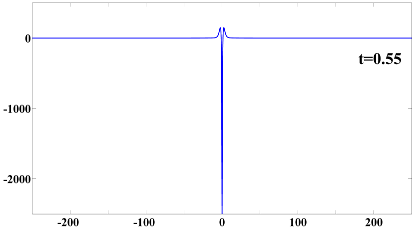


Figure: Profiles of the numerical solution $u(x,t)$ of **BPE** computed for $a = 1$, $b = -3$, $\delta = 1.2$ at evolution time t=0.55; $t^* \approx 0.56$ – blow up time

Case 2, Nonstandard potential well method

Case 1: $b < 0$

Case 2: $b > 0, \quad a < 0, \quad a^2 - \frac{(p+2)^2}{p+1} b > 0$

Main differences between **Case 1** and **Case 2**:

	Case 1 **Potential well method**	**Case 2** **Nonstandard potential well method**
d	$d > 0$	$d = -\infty$
D	$D = d$	$D = +\infty$
$\mathcal{N}$	simply connected set	unbounded set with complicated structure

$$I(\lambda u) = 0, \tag{3}$$

$$I(\lambda u) = \lambda^2 \left(\|u\|^2_{\mathrm{H}^1} + a\lambda^p \int_{\mathbb{R}} |u|^{p+2}\, dx + b\lambda^{2p} \int_{\mathbb{R}} |u|^{2p+2}\, dx \right)$$

$$G(u) = \|u\|^{2(p+2)}_{\mathrm{L}^{p+2}} - \frac{4b}{a^2} \|u\|^2_{\mathrm{H}^1} \|u\|^{2p+2}_{\mathrm{L}^{2p+2}} \text{ – the discriminant of (3)}$$

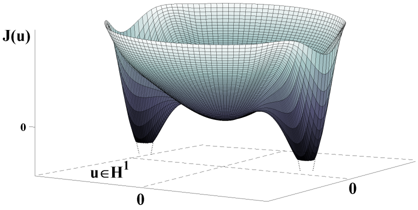
$G(u) < 0$	no real nonzero roots
$G(u) = 0$	$\lambda_0(u) = (-a)^{1/p} \left(2b\|u\|^{2p+2}_{\mathrm{L}^{2p+2}} \right)^{-1/p}$
$G(u) > 0$	$\lambda_\pm(u) = \left[a \left(2b\|u\|^{2p+2}_{\mathrm{L}^{2p+2}} \right)^{-1} \left(-\|u\|^{p+2}_{\mathrm{L}^{p+2}} \pm G^{1/2}(u) \right) \right]^{1/p}$ $\lambda_+ < \lambda_-$

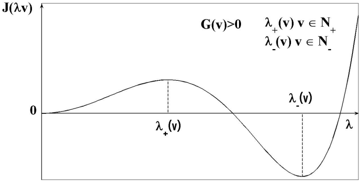
Structure of the Nehary manifold $\mathcal{N}$

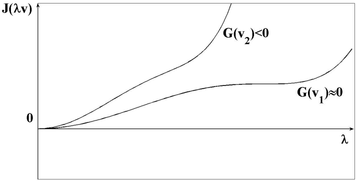
Nehari manifold $\mathcal{N}$: $\quad \mathcal{N} = \mathcal{N}_+ \cup \mathcal{N}_- \cup \mathcal{N}_0,$

$$\mathcal{N}_\pm = \{\lambda_\pm(v)v : \ v \in \mathrm{H}^1, \|v\|_{\mathrm{H}^1} = 1,\ G(v) > 0,\ I(\lambda_\pm(v)v) = 0\}$$

$$\mathcal{N}_0 = \{\lambda_0(v)v : \ v \in \mathrm{H}^1, \|v\|_{\mathrm{H}^1} = 1,\ G(v) = 0,\ I(\lambda_0(v)v) = 0\}$$







New energy constant d_+ (analog of d)

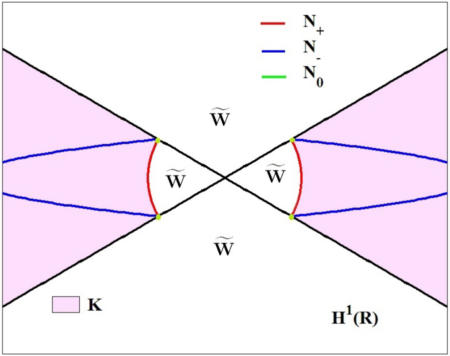
$$d_+ = \inf_{u \in \mathcal{N}_+ \cup \mathcal{N}_0} J(u)$$

$$d_+ = J(\psi) = -\frac{a}{2}(p+2)^{2/p} \int_{\mathbb{R}} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b} \cosh(y) - a \right)^{-(p+2)/p} dy$$
$$- b \frac{(p+2)^{2(p+1)/p}}{2(p+1)} \int_{\mathbb{R}} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b} \cosh(y) - a \right)^{-2(p+2)/p} dy > 0$$

$$d_+ \Big|_{p=2} = -\frac{12a}{(3a^2 - 16b)} - \frac{384ab}{(3a^2 - 16b)^2} - \frac{3a}{8b(3a^2 - 16b)^2}(9a^4 - 256b^2 - 192a^2 b)$$
$$+ \frac{\sqrt{3}(3a^2 - 16b)}{64 b^{3/2}} \ln \frac{\sqrt{3}a + 4\sqrt{b}}{\sqrt{3}a - 4\sqrt{b}}$$

$$\widetilde{W} = \mathrm{H}^1 \setminus \overline{K},$$

$$K = \{\lambda v : \ v \in \mathrm{H}^1,\ \|v\|_{\mathrm{H}^1} = 1,\ G(v) > 0 \text{ and } \lambda > \lambda_+(v)\}$$



Theorem (Case 2, Sign preserving property of the Nehari functional)

Let $f(u) = a|u|^p u + b|u|^{2p} u$, constants a and b satisfy the conditions listed in **Case 2** (i.e. $b > 0$, $a < 0$, $a^2 - \frac{(p+2)^2}{p+1} b > 0$). Suppose $u_0 \in \mathrm{H}^1$, $u_1 \in \mathrm{L}^2$, and $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$. If $E(0) < d_+$ and $u_0 \in \widetilde{W}$, then $u(x,t) \in \widetilde{W}$ for every weak solution $u(x,t)$ of **BPE** on its maximal existence time interval.

Theorem (Case2, Global existence)

Let $f(u) = a|u|^p u + b|u|^{2p} u$, constants a and b satisfy the conditions listed in **Case 2** (i.e. $b > 0$, $a < 0$, $a^2 - \frac{(p+2)^2}{p+1} b > 0$). Suppose $u_0 \in \mathrm{H}^1$, $u_1 \in \mathrm{L}^2$, and $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$. If $E(0) < d_+$ and $u_0 \in \widetilde{W}$ then **BPE** has a unique global solution $u(x,t)$ defined for every $t \in [0, \infty)$.

Numerical experiments, $f(u) = au^3 + bu^5$

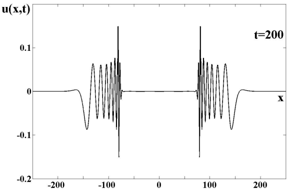
Initial data:

$$u_0(x) = -\delta\overline{\psi}(x), \qquad u_1(x) = 0,$$
$$\overline{\psi}(x) = 2\left(\sqrt{a^2 - \frac{16}{3}b}\ \cosh(2x) - a(1+\varepsilon)\right)^{-1/2}$$

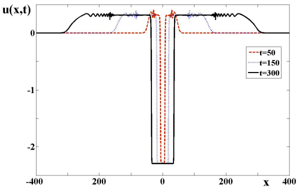
δ, ε, – constants, $\delta > 0$; when $\varepsilon = 0$ $\overline{\psi}(x)$ – ground state solution ;
$E(0) \equiv J(u_0)$

Numerical experiments, $f(u) = au^3 + bu^5$, $a = -1$, $b = 0.15$, $d_+ \approx 1.69298324$

- $\varepsilon = 5$, $\delta = 1.4$, $\widetilde{E}(0) \approx 1.59242338 < d_+$, $u_0 \in \widetilde{W}$ – Figure (a)
- $\varepsilon = 5$, $\delta = 3$, $\widetilde{E}(0) \approx 0.44503829 < d_+$, $u_0 \notin \widetilde{W}$ – Figure (b)
- $\varepsilon = -0.8$, $\delta = 1$, $\widetilde{E}(0) \approx 2.0496858 > d_+$, $u_0 \in \widetilde{W}$ – Figure (b)



(a)



(b)

Figure: Profiles of the numerical solution $u(x,t)$: (a) when the conditions of the Theorem are satisfied; (b) when the conditions of the Theorem fail.

Conjecture

Let $f(u) = a|u|^p u + b|u|^{2p} u$, constants a and b satisfy the conditions listed in **Case 2** (i.e. $b > 0$, $a < 0$, $a^2 - \frac{(p+2)^2}{p+1} b > 0$). If the initial data satisfy either $E(0) > d_+$ or $E(0) < d_+$ and $u_0 \in \overline{K}$ then solutions of **BPE** are globally defined for $t \in [0, \infty)$ and possibly blow up at $t = \infty$.

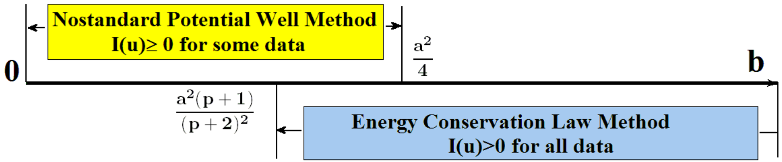
Numerical experiments in the two-dimensional case for $f(u) = au^3 + bu^5$, $a < 0$ and $b > 0$ confirm this conjecture.

- Christov, C.I., Todorov, M.T., Christou, M.A., AIP Conference Proceedings 1404, 49–56 (2011)
- M. Dimova, D. Vasileva, Proc. 5th International Conference on Numerical Methods and Applications, Lozenetz, Bulgaria, 2012, LNCS (accepted)

Case 2: $b > 0, \quad a < 0, \quad a^2 - \frac{(p+2)^2}{p+1} b > 0$

Case 3: $b > 0, \quad a < 0, \quad a^2 - \frac{(p+2)^2}{p+1} b = 0$

Case 4: $b > 0, \quad a < 0, \quad a^2 - \frac{(p+2)^2}{p+1} b < 0$



Thank you
for your attention!