Global behaviour of the solutions to Boussinesq type equation with linear restoring force

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AMiTaNS'14

June 26 - July 1, 2014, Albena, Bulgaria

Supported by National Science Fund Grant DDVU 02/71

Presentation outline

Boussinesq type equation with linear restoring force **(BE_{rf})**

$$\beta_2 u_{tt} - u_{xx} - \beta_1 u_{ttxx} + u_{xxxx} + mu + (f(u))_{xx} = 0,$$

$$x \in \mathbb{R},\ t \in [0, T_m),\ T_m \leq \infty,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \mathbb{R}$$

$\beta_1 \geq 0, \quad \beta_2 > 0,$ – dispersion coefficients, $\quad m > 0$

$u_0 \in \mathrm{H}^1(\mathbb{R}),\ (-\Delta)^{-1/2} u_0 \in \mathrm{L}^2(\mathbb{R}) \quad u_1 \in \mathrm{H}^1(\mathbb{R}),\ (-\Delta)^{-1/2} u_1 \in \mathrm{L}^2(\mathbb{R})$

Nonlinearities

$f(u) = a|u|^{p-1}u, \quad p \geq 2, \quad a = const > 0$

- transverse deflections of an elastic rod on elastic foundation

Modeling

- A.M. Samsonov, Nonlinear waves in elastic waveguides, Springer, 1994
- A.D. Mishkis, P.M. Belotserkovskiy, *ZAMM Z. Angew. Math. Mech.* 79 (1999) 645–647
- A.V. Porubov, Amplification of nonlinear strain waives in solids, World Scientific, 2003

Numerical Study

- C.I. Christov, T.T. Marinov, R.S. Marinova, *Math. Comp. Simulation* 80 (2009) 56–65
- M.A. Cristou, *AIP Conf. Proc.* 1404 (2011) 41–48

Theoretical Investigations

- F. Gazzola, M.Squassina, *Ann. Inst. Henri Poincaré, Analyse non Linéaire* 23 (2006) 185–207 (nonlinear wave equation)
- R. Xu, Y.Ding, *Acta Mathematica Scienta* 33B(3) (2013) 643–652 (Klein-Gordon equation)

Important functionals

Conservation low: $E(t) = E(0)$ for every $t \in [0, T_m)$,

$$E(t) = E(u(\cdot,t), u_t(\cdot,t)) = \frac{1}{2}\left(\beta_2 \left\|(-\Delta)^{-1/2} u_t(\cdot,t)\right\|^2_{L^2} + \beta_1 \|u_t(\cdot,t)\|^2_{L^2} + \|u(\cdot,t)\|^2_{H^1} + m\left\|(-\Delta)^{-1/2} u(\cdot,t)\right\|^2_{L^2}\right) - \frac{a}{p+1}\int_{\mathbb{R}} |u(x,t)|^{p+1}\, dx$$

Potential energy functional $J(u)$:

$$J(u) = \frac{1}{2}\|u\|^2_{H^1} + \frac{m}{2}\left\|(-\Delta)^{-1/2} u(\cdot,t)\right\|^2_{L^2} - \frac{a}{p+1}\int_{\mathbb{R}} |u(x,t)|^{p+1}\, dx$$

Nehari functional $I(u)$:

$$I(u) = J'(u)u = \|u\|^2_{H^1} + m\left\|(-\Delta)^{-1/2} u(\cdot,t)\right\|^2_{L^2} - a\int_{\mathbb{R}} |u(x,t)|^{p+1}\, dx$$

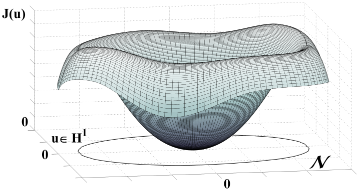
Potential well method

Nehari manifold $\mathcal{N}$: $\mathcal{N} = \{u \in \mathrm{H}^1 : \ I(u) = 0, \ \|u\|_{\mathrm{H}^1} \neq 0\}$

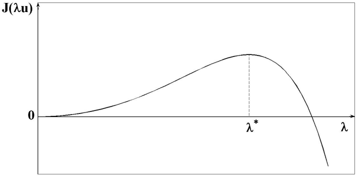
Critical energy constant d: $d = \inf_{u \in \mathcal{N}} J(u)$

Depth D of the potential well: $D = \inf_{u \in \mathrm{H}^1 \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u) > 0$

$d = D$



(a)



(b)

Figure: (a) Schematic illustration of $J(u)$ as a function of $u \in \mathrm{H}^1$; (b) Cross section of $J(\lambda u)$ as a function of λ for a fixed $u \in \mathrm{H}^1$.

Potential well method

Theorem (Global existence)

Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$, and $(-\Delta)^{-1/2}u_1 \in \mathrm{L}^2$. If $0 < E(0) < d$ and $I(u_0) > 0$ or $\|u_0\|_{\mathrm{H}^1} = 0$ then problem $\mathbf{BE_{rf}}$ has a unique global solution defined for every $t \in [0, \infty)$.

Theorem (Finite time blow up)

Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$, and $(-\Delta)^{-1/2}u_1 \in \mathrm{L}^2$. If $0 < E(0) < d$ and $I(u_0) < 0$ then the weak solution of $\mathbf{BE_{rf}}$ blows up in a finite time.

$$d = \inf_{u \in \mathcal{N}} J(u) \geq d_0 = \frac{(p-1)}{2(p+1)} \frac{1}{\left(aC_p^{(p+1)}\right)^{\frac{2}{p-1}}} > 0$$

$$C_p = \sup_{v \in \mathrm{H}^1, \|v\|_1 \neq 0} \frac{\|v\|_{\mathrm{L}^{p+1}}}{\|v\|_1} = \frac{1}{\sqrt{2(p+1)}} \left((p-1)(p+3) \frac{\Gamma\left(\frac{4}{p-1}\right)}{\Gamma^2\left(\frac{2}{p-1}\right)} \right)^{\frac{p-1}{2(p+1)}}$$

Potential well method

$$W = \{u \in \mathrm{H}^1 : \ I(u) > 0\} \cup \{0\}, \quad V = \{u \in \mathrm{H}^1 : \ I(u) < 0\}$$

Theorem (Sign preserving properties of $I(u(t))$)

Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2} u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$, $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$ and $0 < E(0) < d$.

- If $u_0(x) \in W$ then $u(x,t) \in W$ for every $t \in [0, T_m)$.
- If $u_0(x) \in V$ then $u(x,t) \in V$ for every $t \in [0, T_m)$.

Here T_m is the maximal existence time interval of the weak solution $u(x,t)$.

High energy blow up method

$$\beta_2 u_{tt} - u_{xx} - \beta_1 u_{ttxx} + u_{xxxx} + mu + (a|u|^{p-1}u)_{xx} = 0,$$
$$x \in \mathbb{R},\ t \in [0, T_m), T_m \le \infty,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \mathbb{R}$$

Definitions

$$(u,v) = \int_{\mathbb{R}} uv\,dx, \qquad u, v \in \mathrm{H}^1,$$
$$\langle u,v\rangle = \beta_2 \left((-\Delta)^{-1/2}u, (-\Delta)^{-1/2}v\right) + \beta_1 (u,v),$$
$$u, v \in \mathrm{H}^1, \quad (-\Delta)^{-1/2}u \in \mathrm{L}^2, (-\Delta)^{-1/2}v \in \mathrm{L}^2$$

$$E(0) = \frac{1}{2}\left(\beta_2 \left\|(-\Delta)^{-1/2}u_1\right\|^2_{\mathrm{L}^2} + \beta_1 \|u_1\|^2_{\mathrm{L}^2} + \|u_0\|^2_{\mathrm{H}^1} + m\left\|(-\Delta)^{-1/2}u_0\right\|^2_{\mathrm{L}^2}\right) - \frac{a}{p+1}\int_{\mathbb{R}} |u_0(x)|^{p+1}\,dx$$

High energy blow up method

Theorem (Sign preserving properties of $I(u(t))$)

Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$, $(-\Delta)^{-1/2}u_1 \in \mathrm{L}^2$, and $\|u_0\|_{\mathrm{H}^1} \neq 0$. If either

(i) $\langle u_0, u_1\rangle \geq 0, \quad m_0\sqrt{\frac{p-1}{p+1}}\langle u_0, u_1\rangle \geq E(0) > 0$

or $\qquad m_0 = \min\left(\frac{m}{\beta_2}, \frac{1}{\beta_1}\right)^{1/2}$

(ii) $0 \leq \langle u_0, u_1\rangle \leq m_0\sqrt{\frac{p-1}{p+1}}\langle u_0, u_0\rangle,$

$$\frac{m_0^2(p-1)}{2(p+1}\langle u_0, u_0\rangle + \frac{\langle u_0, u_1\rangle^2}{2\langle u_0, u_0\rangle} \geq E(0) > 0$$

is satisfied then $I(u(t)) < 0$ for every $t \in [0, T_m)$.

Moreover if $\widetilde{t} < T_m$, where $\widetilde{t} = \begin{cases} \frac{1}{m_0}\sqrt{\frac{p-1}{p+1}} & \text{in case (i);} \\ \frac{p+1}{2(p-1)m_0^2}\frac{\langle u_0,u_1\rangle}{\langle u_0,u_0\rangle} & \text{in case (ii),} \end{cases}$

then $I(u(t)) \leq -\frac{p+1}{2}\langle u_t(t), u_t(t)\rangle$ for $t \in [\widetilde{t}, T_m)$.

High energy blow up method

Theorem (Finite time blow up)

Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$, $(-\Delta)^{-1/2}u_1 \in \mathrm{L}^2$, and $\|u_0\|_{\mathrm{H}^1} \neq 0$. If either

(i) $\langle u_0, u_1\rangle \geq 0, \quad m_0\sqrt{\frac{p-1}{p+1}}\langle u_0, u_1\rangle \geq E(0) > 0$

or

$$m_0 = \min\left(\frac{m}{\beta_2}, \frac{1}{\beta_1}\right)^{1/2}$$

(ii) $0 \leq \langle u_0, u_1\rangle \leq m_0\sqrt{\frac{p-1}{p+1}}\langle u_0, u_0\rangle,$

$$\frac{m_0^2(p-1)}{2(p+1}\langle u_0, u_0\rangle + \frac{\langle u_0, u_1\rangle^2}{2\langle u_0, u_0\rangle} \geq E(0) > 0$$

is satisfied then every weak solution of $\mathbf{BE_{rf}}$ blows up for a finite time $t_* < \infty$, where either $t_* < \tilde{t}$ or

$$t_* \leq \frac{2\langle u(\tilde{t}), u(\tilde{t})\rangle}{(p-1)\langle u(\tilde{t}), u_t(\tilde{t})\rangle}, \quad \tilde{t} = \begin{cases} \frac{1}{m_0}\sqrt{\frac{p-1}{p+1}} & \text{in case (i);} \\ \frac{p+1}{2(p-1)m_0^2}\frac{\langle u_0,u_1\rangle}{\langle u_0,u_0\rangle} & \text{in case (ii).} \end{cases}$$

Comparison between Potential well method (**PWM**) and High energy blow up method (**HEBM**)

method	$u_0,\ u_1$	$I(u_0)$	$E(0)$
PWM	no conditions	$I(u_0) < 0$	$0 < E(0) < d$
HEBM (i)	$\langle u_0, u_1 \rangle \geq 0$ $m_0\sqrt{\frac{p-1}{p+1}}\langle u_0, u_1 \rangle \geq E(0) > 0$	no conditions	arbitrary high positive
HEBM (ii)	$0 \leq \langle u_0, u_1 \rangle \leq m_0\sqrt{\frac{p-1}{p+1}}\langle u_0, u_0 \rangle$ $\frac{m_0^2(p-1)}{2(p+1}\langle u_0, u_0 \rangle + \frac{\langle u_0, u_1 \rangle^2}{2\langle u_0, u_0 \rangle} \geq E(0) > 0$	no conditions	arbitrary high positive

Conclusions:

- Subcritical initial energy: $0 < E(0) < d$ – **PWM**
- Supercritical initial energy: $E(0) > d$ – **HEBM**

Choice of initial data

$$u_0(x) = r(w(\theta x))'_x, \quad u_1(x) = rq(w(\theta x))'_x,$$

$w: \quad w \in \mathrm{H}^2, \ \|w\|_{\mathrm{H}^2} \neq 0;$

θ, r, q – constants

parameters:

$\beta_1 = \beta_2 = 1$, $m = 1$, $m_0 = 1$, $a = 2$, $p = 2$ $(f(u) = 2|u|u)$

$$d_0 = \frac{6}{5a^2} \leq d = \inf_{u \in \mathcal{N}} J(u) \leq J(v_0), \forall v_0 \in \mathcal{N}$$

$$0.3 \leq d \leq 9.59601581$$

Example 1 : $w(x) = -\dfrac{2}{e^{2x}+1} + \dfrac{1}{e^{(0.4x-2)}+1} + \dfrac{1}{e^{(0.4x+2)}+1}$

Example 2 : $w(x) = \dfrac{1}{\cosh(x)}$

Family of finite difference schemes

$$B\left(\frac{v_i^{n+1}-2v_i^n+v_i^{n-1}}{\tau^2}\right)-\Lambda^{xx}v_i^n+\beta_2\Lambda^{xxxx}v_i^n-mv_i^n = \frac{a}{3}\Lambda^{xx}\left(\frac{|v_i^{n+1}|^3-|v_i^{n-1}|^3}{v_i^{n+1}-v_i^{n-1}}\right),$$

$$B=(1+\frac{m}{2}\tau^2)I-(\beta_1+\sigma\tau^2)\Lambda^{xx}+\sigma\tau^2\beta_2\Lambda^{xxxx}$$

- v_i^n – a discrete approximation to u at (x_i,t_n), τ is a time-step
- $\Lambda^{xx}v_i=\frac{v_{i+1}-2vu_i+v_{i-1}}{h^2}$, $\Lambda^{xxxx}v_i=\frac{v_{i+2}-4v_{i+1}+6v_i-4v_{i-1}+v_{i-2}}{h^4}$, σ - parameter

Properties:

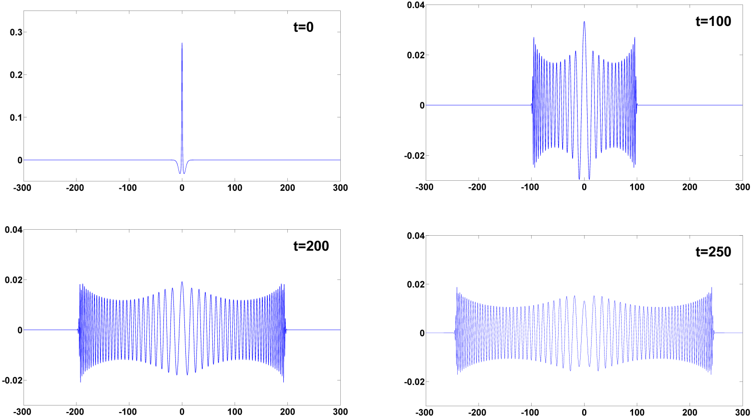
- **Convergence:** The schemes have second order of convergence in space and time $O(|h|^2+\tau^2)$.
- **Stability:** The schemes are unconditionally stable for $\sigma>1/4$.
- **Conservativeness:** The discrete energy is conserved in time, i.e. $E_h(v^{(n)})=E_h(v^{(0)}),\quad n=1,2,\dots$

Potential well method, numerical experiments

Example 1: $\beta_1 = \beta_2 = 1$, $m = 1$, $m_0 = 1$, $a = 2$, $p = 2$, $d \geq d_0 = 0.3$

$$\theta = 1.2, \quad r = 0.3, \quad q = 0.0057$$

$$\widetilde{E}(0) \approx 0.22850225 < d_0 \leq d, \quad \widetilde{I}(u_0) \approx 0.44530787 > 0$$



Example 1: $\beta_1 = \beta_2 = 1$, $m = 1$, $m_0 = 1$, $a = 2$, $p = 2$, $d \geq d_0 = 0.3$

$$\theta = 1.2, \quad r = 6.1515, \quad q = 0.0057$$

$$\widetilde{E}(0) \approx 0.24786145 < d_0 \leq d, \quad \widetilde{I}(u_0) \approx -100.24924380 < 0$$

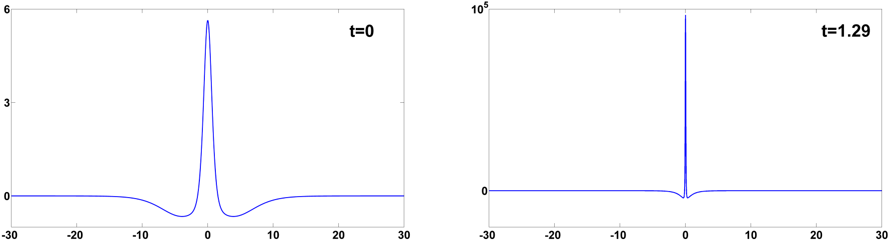


Figure: Profiles of the numerical solution $u(x,t)$ at evolution time $t = 0$ and t=1.29; $t_* \approx 1.30$ – blow up time.

High energy blow up method, numerical experiments

Example 1: $\beta_1 = \beta_2 = 1$, $m = 1$, $m_0 = 1$, $a = 2$, $p = 2$,
$\widetilde{t} = \sqrt{3}/3 \approx 0.57735027$, $d < 9.56015811$

$S = 10 > d$ – fixed, $\theta = 1.2$, $r = 5.7789$, $q = 0.27$:

$\widetilde{E}(0) \approx 10.50341845 > S > d$, $\widetilde{t} \approx 0.58$, $t_* \leq \dfrac{2\langle u(\widetilde{t}), u(\widetilde{t})\rangle}{\langle u(\widetilde{t}), u_t(\widetilde{t})\rangle} \approx 1.91$

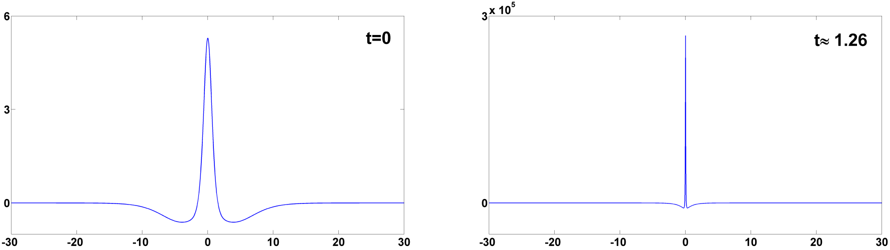


Figure: Profiles of the numerical solution $u(x, t)$ at evolution time $t = 0$ and t=1.26; $t_* \approx 1.27 < 1.91$ – blow up time.

High energy blow up method, numerical experiments

Example 2: $\beta_1 = \beta_2 = 1$, $m = 1$, $m_0 = 1$, $a = 2$, $p = 2$,
$\widetilde{t} = \sqrt{3}/3 \approx 0.57735027$, $d < 9.56015811$

$S = 10 > d$ – fixed, $\theta = 1.2$, $r = 9.0881$, $q = 0.25$:

$\widetilde{E}(0) \approx 10.09279508 > S > d$, $\widetilde{t} \approx 0.58$, $t_* \leq \dfrac{2\langle u(\widetilde{t}), u(\widetilde{t})\rangle}{\langle u(\widetilde{t}), u_t(\widetilde{t})\rangle} \approx 1.60$

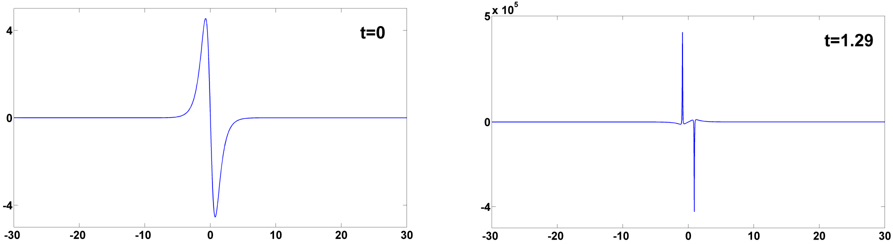


Figure: Profiles of the numerical solution $u(x, t)$ at evolution time $t = 0$ and t=1.29; $t^* \approx 1.30 < 1.60$ – blow up time.

Thank you
for your attention!