Theoretical and numerical aspects to the generalized sixth order Boussinesq equation

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BGSIAM'13

December 19th, 2013, Sofia, Bulgaria

Supported by National Science Fund Grant DDVU 02/71

Generalized sixth order Boussinesq equation (**6GBE**)

$$u_{tt} - u_{xx} - \beta_1 u_{ttxx} + \beta_2 u_{xxxx} + \beta_3 u_{ttxxxx} = f(u)_{xx},$$
$$x \in \mathbb{R},\ t \in [0, T), T \leq \infty,$$
$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \qquad x \in \mathbb{R}$$

$\beta_1 \geq 0, \quad \beta_2 > 0, \quad \beta_3 \geq 0$ – dispersion coefficients

$u_0 \in \mathrm{H}^1(\mathbb{R}), \quad u_1 \in \mathrm{H}^1(\mathbb{R}), \quad (-\Delta)^{-1/2} u_1 \in \mathrm{L}^2(\mathbb{R})$

Generalized Bernoulli nonlinearities

$$f(u) = a|u|^p u + b|u|^{2p} u, \quad p > 0, \quad a, b = const \neq 0$$

- $\beta_2 = 1, \beta_1 = \beta_3 = 0$ – "good" Boussinesq equation
- $\beta_3 = 0, \beta_i \neq 0, i = 1, 2$ – Boussinesq Paradigm equation
- $\beta_1 = 0, \beta_i \neq 0, i = 2, 3$ – Rosenau equation
- $\beta_2 = 0, \beta_3 = 0, \beta_1 \neq 0$ – Pochhammer – Chree equation
- $\beta_i \neq 0, i = 1, 2, 3$ – Generalized sixth order Boussinesq equation
  - G. Schneidera, C. E. Wayne, *Phys D*, v. 152-153 (2001) 384–394

History

- Y. Wang, C. Mu, *Applied Mathematics and Computation* 188 (2007) 1131–1141
- Y. Wang, C. Mu, Y. Wu, *J. Differential Equations* 247 (2009) 2380–2394
- S. Xia, J. Yuan, *Nonlinear Analysis* 73 (2010) 1015–1027

$$f(u) = a|u|^p, \qquad f(u) = a|u|^{p-1}u, \quad a = const, \quad p > 1$$

- S. Wang, H. Xue, *Applied Mathematics and Computation* 204 (2008) 130–136
- X. Jiang, Y. Ding, Y. Liu and R. Xu, *AIP Conf. Proc.* 1389 (2011) 1640–1643
- H. Taskesen, N. Polat, *Contemporary Analysis and Applied Mathematics* 1(1) (2013) 60–69

$$f(u) = \sum_{k=1}^{m} a_k |u|^{p_k - 1} u, \qquad 1 < p_1 < \ldots < p_m$$

- R. Xu, M*ath. Methods in Appl. Sci.* 34 (2011) 2318–2328
- N. Kutev, N. Kolkovska, M. Dimova, *J. Math. Anal. Appl.* 410 (2014) 427–444

Generalized Bernoulli nonlinearities

$$f(u) = a|u|^p u + b|u|^{2p} u, \qquad a, b = cost \neq 0, \quad p > 0$$

$b < 0$ **Potential well method**

- S. Wang, H. Xue, *Applied Mathematics and Computation* 204 (2008) 130–136
- X. Jiang, Y. Ding, Y. Liu and R. Xu, *AIP Conf. Proc.* 1389 (2011) 1640–1643

$b > 0,\ a < 0,\ a^2 - \frac{(p+2)^2}{p+1} b > 0$ **Nonstandard potential well method**

- N. Kutev, N. Kolkovska, M. Dimova, *J. Math. Anal. Appl.* 410 (2014) 427–444

$a^2 - \frac{(p+2)^2}{p+1} b \leq 0$ **Conservation law's method**

Important functionals

Conservation low: $E(t) = E(0) \quad \text{for every} \quad t \in [0, T)$,

$$E(t) = E(u(\cdot,t), u_t(\cdot,t)) =$$
$$\frac{1}{2}\left(\beta_2 \left\|(-\Delta)^{-1/2} u_t(\cdot,t)\right\|^2 + \beta_1 \|u_t(\cdot,t)\|^2 + \beta_3 \|u_{tx}(\cdot,t)\|^2 + \|u(\cdot,t)\|^2_{\mathrm{H}^1}\right)$$
$$+ \int_{\mathbb{R}} F(u(x,t))\, dx, \qquad \text{where } F(u) = \int_0^u f(s)\, ds$$

Potential energy functional $J(u)$:

$$J(u) = \frac{1}{2}\|u\|^2_{\mathrm{H}^1} + \int_{\mathbb{R}} F(u)\, dx$$

Nehari functional $I(u)$:

$$I(u) = J'(u)u = \|u\|_1^2 + \int_{\mathbb{R}} u f(u)\, dx$$

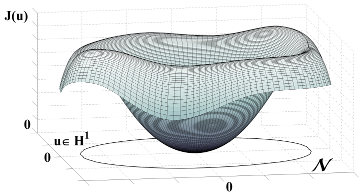
Potential well method

Nehari manifold $\mathcal{N}$: $\mathcal{N} = \{u \in \mathrm{H}^1 : \; I(u) = 0, \; \|u\|_{\mathrm{H}^1} \neq 0\}$

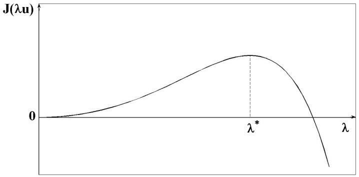
Critical energy constant d: $d = \inf_{u \in \mathcal{N}} J(u)$

Depth D of the potential well: $D = \inf_{u \in \mathrm{H}^1 \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u) > 0$

$d = D$



(a)



(b)

Figure: (a) Schematic illustration of $J(u)$ as a function of $u \in \mathrm{H}^1$; (b) Cross section of $J(\lambda u)$ as a function of λ for a fixed $u \in \mathrm{H}^1$.

Potential well method, $b < 0$

Theorem (Global existence)

Let $f(u) = a|u|^p u + b|u|^{2p} u$, $b < 0$, and $u_0 \in \mathrm{H}^1$, $u_1 \in \mathrm{H}^1$, and $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$. If $E(0) < d$ and $I(u_0) > 0$ or $\|u_0\|_{\mathrm{H}^1} = 0$, then problem **6GBE** has a unique global solution defined for every $t \in [0, \infty)$

Theorem (Finite time blow up)

Let $f(u) = a|u|^p u + b|u|^{2p} u$, $b < 0$, and $u_0 \in \mathrm{H}^1$, $u_1 \in \mathrm{H}^1$, and $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$. If $E(0) < d$ and $I(u_0) < 0$, then the weak solution of **6GBE** blows up in a finite time.

$$d = D = \inf_{u \in \mathrm{H}^1 \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u) = J(\psi)$$

Ground state solution:

$$\psi(x) = (p+2)^{1/p} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b} \cosh(px) - a \right)^{-1/p}$$

Explicitly evaluation of the critical energy constant d

Theorem

Let $f(u) = a|u|^p u + b|u|^{2p} u$ and $b < 0$. Then the critical energy constant d is given by:

$$d = -\frac{a}{2}(p+2)^{2/p} \int_{\mathbb{R}} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b \cosh(y)} - a \right)^{-(p+2)/p} dy$$
$$- b\frac{(p+2)^{2(p+1)/p}}{2(p+1)} \int_{\mathbb{R}} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b \cosh(y)} - a \right)^{-2(p+1)/p} dy.$$

Corollary

Let $f(u) = a|u|u + b|u|^2 u$ and $b < 0$. Then the critical energy constant d is equal to:

$$d\Big|_{p=1} = \frac{3k^2(2+k^2)}{a^2(1-k^2)^2} + \frac{9k^3}{a^2(1-k^2)^{3/2}} \left(\frac{\pi}{2} + \arctan \frac{k}{\sqrt{1-k^2}} \right), \quad k = \frac{a}{\sqrt{a^2 - \frac{9}{2} b}}$$

Potential well method, numerical experiments

Initial data:

$$u_0(x) = -\delta\psi(x), \qquad \psi(x) = 3\left(\sqrt{a^2 - \frac{9}{2}b}\,\cosh(x) - a\right)^{-1},$$
$$u_1(x) = 0$$

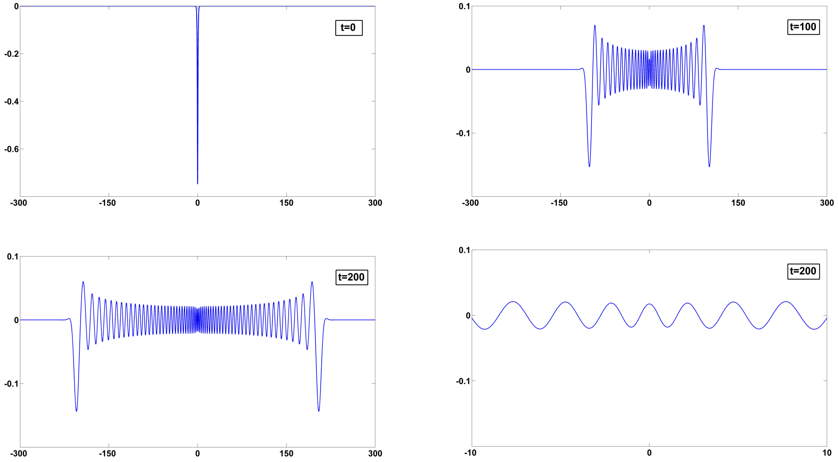
$\psi(x)$ – ground state solution; $\delta > 0$ – constants

To solve numerically the problem we use a conservative finite difference scheme.

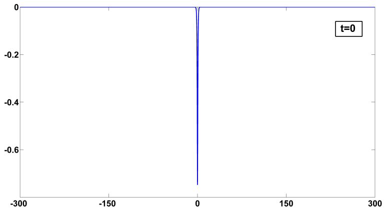
- **Conservativeness:** The discrete energy is conserved in time, i.e. $E_h(v^{(n)}) = E_h(v^{(0)}), \quad n = 1, 2, \ldots$
- **Convergence:** The schemes have second order of convergence in space and time $O(|h|^2 + \tau^2)$.

A regular mesh defined in [-300,300] with space step $h = 0.01$ and time step $\tau = 0.01$ is used. In addition mesh refinement analysis is performed.

- $\delta = 0.7, \quad \widetilde{E}(0) \approx 0.61794693 < d, \quad \widetilde{I}(u_0) \approx 0.88402820 > 0$



- $\delta = 1.2, \quad \widetilde{E}(0) \approx 0.67082827 < d, \quad \widetilde{I}(u_0) \approx -2.38266613 < 0$



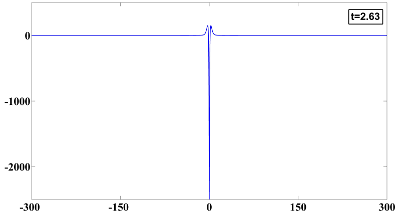
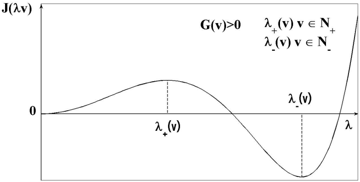


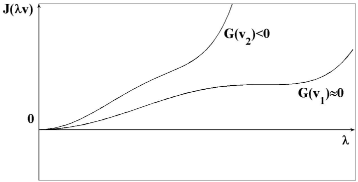
Figure: Profiles of the numerical solution $u(x,t)$ computed for $a = 1$, $b = -3$, $\delta = 1.2$ at evolution time t=2.63; $t^* \approx 2.65$ – blow up time

Nonstandard potential well method

	Potential well method $b < 0$	**Nonstandard potential well method** $b > 0, a < 0, a^2 - \frac{(p+2)^2}{p+1} b > 0$
d	$d > 0$	$d = -\infty$
D	$D = d$	$D = +\infty$
$\mathcal{N}$	simply connected set	unbounded set with complicated structure

Nonstandard potential well method





$$I(\lambda u) = \lambda^2 \left(\|u\|^2_{\mathrm{H}^1} + a\lambda^p \int_{\mathbb{R}} |u|^{p+2}\, dx + b\lambda^{2p} \int_{\mathbb{R}} |u|^{2p+2}\, dx \right) = 0$$

$$G(u) = \|u\|^{2(p+2)}_{\mathrm{L}^{p+2}} - \frac{4b}{a^2} \|u\|^2_{\mathrm{H}^1} \|u\|^{2p+2}_{\mathrm{L}^{2p+2}} - \text{discriminant}$$

Nehari manifold $\mathcal{N}$: $\mathcal{N} = \mathcal{N}_+ \cup \mathcal{N}_- \cup \mathcal{N}_0,$

$$\mathcal{N}_\pm = \{\lambda_\pm(v)v : \ v \in \mathrm{H}^1, \|v\|_{\mathrm{H}^1} = 1, \ G(v) > 0, \ I(\lambda_\pm(v)v) = 0\}$$

$$\mathcal{N}_0 = \{\lambda_0(v)v : \ v \in \mathrm{H}^1, \|v\|_{\mathrm{H}^1} = 1, \ G(v) = 0, \ I(\lambda_0(v)v) = 0\}$$

Nonstandard potential well method, $b > 0, a < 0, a^2 - \frac{(p+2)^2}{p+1} b > 0$

New energy constant d_+ (analog of d)

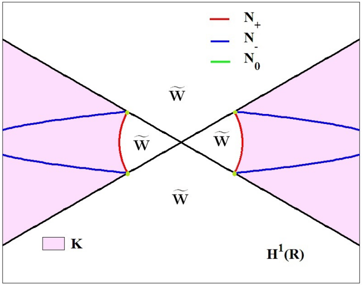
$$d_+ = \inf_{u \in \mathcal{N}_+ \cup \mathcal{N}_0} J(u)$$

$$d_+ = J(\psi) = -\frac{a}{2}(p+2)^{2/p} \int_{\mathbb{R}} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b} \cosh(y) - a \right)^{-(p+2)/p} dy$$
$$- b \frac{(p+2)^{2(p+1)/p}}{2(p+1)} \int_{\mathbb{R}} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b} \cosh(y) - a \right)^{-2(p+2)/p} dy > 0$$

$$d_+\Big|_{p=1} = \frac{3k^2(2+k^2)}{a^2(1-k^2)^2} + \frac{9k^3}{a^2(k^2-1)^{3/2}} \ln \frac{k - \sqrt{k^2-1}}{k + \sqrt{k^2-1}}, \quad k = \frac{a}{\sqrt{a^2 - \frac{9}{2} b}}$$

$$\widetilde{W} = \mathrm{H}^1 \setminus \overline{K},$$

$$K = \{\lambda v : \ v \in \mathrm{H}^1, \ \|v\|_{\mathrm{H}^1} = 1, \ G(v) > 0 \text{ and } \lambda > \lambda_+(v)\}$$



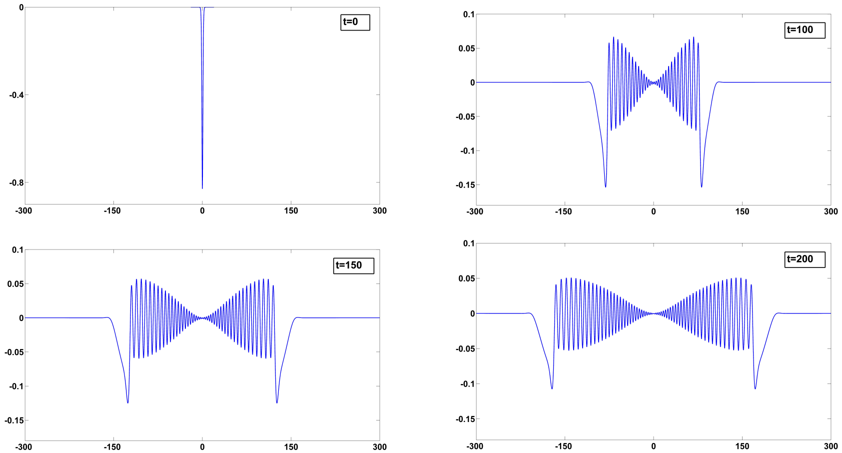
Theorem (Global existence)

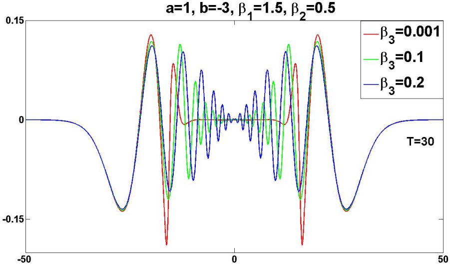
Let $f(u) = a|u|^p u + b|u|^{2p} u$, $b > 0$, $a < 0$, $a^2 - \frac{(p+2)^2}{p+1} b > 0$. Suppose $u_0 \in \mathrm{H}^1$, $u_1 \in \mathrm{H}^1$, and $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$. If $E(0) < d_+$ and $u_0 \in \widetilde{W}$ then **6GBE** has a unique global solution $u(x, t)$ defined for every $t \in [0, \infty)$.

$f(u) = a|u|u + bu^3$, $a = -1.5$, $b = 0.4$, $d_+ \approx 0.89912476$

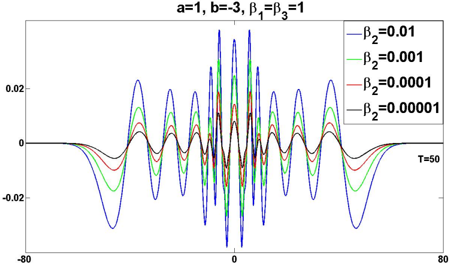
$$u_0(x) = -3\delta \left(\sqrt{a^2 - \frac{9}{2} b}\ \cosh(x) - a \right)^{-1}, \qquad u_1(x) = 0$$

$\delta = 0.6$, $\quad \widetilde{E}(0) \approx 0.62999953 < d_+$, $\quad u_0 \in \widetilde{W}$





Dependence on the surface tension β_3



Limited case, $\beta_2 \to 0$: Pochhammer – Chree equation with surface tension

Thank you
for your attention!