Finite time blow up of the solutions to Klein–Gordon and Boussinesq type equations with arbitrary positive initial energy

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Presentation outline

Nonlinear dispersive equation

$$\alpha_1 U_{tt} - \alpha_2 \Delta U - \alpha_3 \Delta U_{tt} + \alpha_4 \Delta^2 U + \alpha_5 \Delta^2 U_{tt} + Mu + \Delta f(u) = 0,$$

$$X \in \mathbb{R}^n,\ t \in [0, T_m),\ T_m \leq \infty,$$

$$U(X,0) = U_0(X), \quad U_t(X,0) = U_1(X), \quad X \in \mathbb{R}^n$$

Nonlinearities

$$f(U) = \alpha |U|^{p-1} U, \quad \alpha = const \neq 0, \quad 1 < p < \begin{cases} \infty & \text{for } n = 1, 2 \\ \frac{n+2}{n-2} & \text{for } n \geq 3 \end{cases}$$

- Nonlinear Klein–Gordon equation: $\alpha_1 = 0, \alpha_2 = -m < 0, \alpha_3 = -1, \alpha_4 = 1, \alpha_5 = 0, M = 0$
- Rosenau equation: $\alpha_1 = 1, \alpha_2 > 0, \alpha_3 = 0, \alpha_4 > 0, \alpha_5 > 0, M = 0, \alpha > 0$
- Pochhammer–Chree equation: $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1, \alpha_4 = \alpha_5 = 0, M = 0$
- Boussinesq paradigm equation: $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 \geq 0, \alpha_4 > 0, \alpha_5 = 0, M = 0$
- Generilized Bretherton equation (nonlinear beam equation): $\alpha_1 = 1, \alpha_3 = 0, \alpha_4 > 0, \alpha_5 = 0$
- Boussinesq type equation with linear restoring force: $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 \geq 0, \alpha_4 > 0, \alpha_5 = 0, M > 0, \alpha > 0$

Nonlinear Klein-Gordon equation (**KG**)

$$u_{tt} - \Delta u + u - a|u|^{p-1}u = 0, \qquad x \in \mathbb{R}^n,\ t \in [0, T_m), T_m \le \infty$$
$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad u_0 \in \mathrm{H}^1(\mathbb{R}^n), u_1 \in \mathrm{L}^2(\mathbb{R}^n),\ a > 0$$

- H. Pesher 1976 (local existence)
- W.A. Strauss 1989 (global existence with small data)
- H.A. Levine 1974; J.M. Ball 1978 (finite time blow up with negative initial energy)
- V. Georgiev, G. Todorova, *J. Differential Equations* 109(2) (1994) 295–308 (damped wave equation with source)

Finite time blow up for subcritical initial positive initial energy (Potential well method)

- L.E. Payne, D.H. Sattinger 1975
- J. Zhang 2002; Liu Yacheng 2006; R. Xu 2010; R.Xu, Y.Ding 2013

Finite time blow up for arbitrary high positive initial energy

- F. Gazzola, M.Squassina, *Ann. Inst. Henri Poincaré, Analyse non Linéaire* 23 (2006) 185–207 (nonlinear wave equation)
- Y. Wang, *Proc. Amer. Math. Soc.* 136(2008) 3477–3482 (Klein-Gordon equation)
- R. Xu, Y.Ding, *Acta Mathematica Scienta* 33B(3) (2013) 643–652 (damped Klein-Gordon equation)

Boussinesq type equation with linear restoring force ($\mathbf{BE_{lrf}}$)

$$\beta_2 u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \Delta^2 u + u + a\Delta(|u|^{p-1}u) = 0,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \quad t \in [0, T_m), T_m \leq \infty$$

$$\beta_1 \geq 0, \quad \beta_2 > 0, \quad a > 0$$

$$u_0 \in \mathrm{H}^1(\mathbb{R}^n), (-\Delta)^{-1/2} u_0 \in \mathrm{L}^2(\mathbb{R}^n), \quad u_1 \in \mathrm{L}^2(\mathbb{R}^n), (-\Delta)^{-1/2} u_1 \in \mathrm{L}^2(\mathbb{R}^n)$$

- transverse deflections of an elastic rod on elastic foundation

Modeling

- A.M. Samsonov, Nonlinear waves in elastic waveguides, Springer, 1994
- A.D. Mishkis, P.M. Belotserkovskiy, *ZAMM Z. Angew. Math. Mech.* 79 (1999) 645–647
- A.V. Porubov, Amplification of nonlinear strain waives in solids, World Sci., 2003

Numerical Study

- C.I. Christov, T.T. Marinov, R.S. Marinova, *Math. Comp. Simulation* 80 (2009) 56–65
- M.A. Cristou, *AIP Conf. Proc.* 1404 (2011) 41–48

Theoretical Investigations

- N. Kutev, N. Kolkovska, M. Dimova, 2014

Short notations: $\|u\|_{\mathrm{L}^2(\mathbb{R}^n)} = \|u\|_{\mathrm{L}^2}, \quad \|u\|_{\mathrm{H}^1(\mathbb{R}^n)} = \|u\|_{\mathrm{H}^1}$

$$\langle u, v\rangle = \begin{cases} \displaystyle\int_{\mathbb{R}^n} u(x)\, v(x)\, dx & \textbf{KG} \\ \displaystyle\beta_1 \int_{\mathbb{R}^n} u(x)\, v(x)\, dx + \beta_2 \int_{\mathbb{R}^n} (-\Delta)^{-1/2} u(x).(-\Delta)^{-1/2} v(x)\, dx & \textbf{BE}_{\textbf{lrf}} \end{cases}$$

Conservation low: $E(t) = E(0) \ \ \forall\, t \in [0, T_m),$
$E(t) := E(u(\cdot, t), u_t(\cdot, t))$

$$E(t) = \begin{cases} \dfrac{1}{2}\langle u_t(\cdot,t), u_t(\cdot,t)\rangle^2 + \dfrac{1}{2}\|u(\cdot,t)\|^2_{\mathrm{H}^1} - \dfrac{a}{p+1}\displaystyle\int_{\mathbb{R}^n} |u(x,t)|^{p+1}\, dx & \textbf{KG} \\ \dfrac{1}{2}\langle u_t(\cdot,t), u_t(\cdot,t)\rangle^2 + \dfrac{1}{2}\|u(\cdot,t)\|^2_{\mathrm{H}^1} - \dfrac{a}{p+1}\displaystyle\int_{\mathbb{R}^n} |u(x,t)|^{p+1}\, dx & \\ \qquad + \dfrac{1}{2}\left\|(-\Delta)^{-1/2} u(\cdot,t)\right\|^2_{\mathrm{L}^2} & \textbf{BE}_{\textbf{lrf}} \end{cases}$$

Important functionals

Potential energy functional $J(u)$:

$$J(u) = \begin{cases} \frac{1}{2}\|u\|^2_{\mathrm{H}^1} - \frac{a}{p+1}\int_{\mathbb{R}^n} |u(x,t)|^{p+1}\,dx & \textbf{KG} \\ \frac{1}{2}\|u\|^2_{\mathrm{H}^1} - \frac{a}{p+1}\int_{\mathbb{R}^n} |u(x,t)|^{p+1}\,dx + \frac{1}{2}\left\|(-\Delta)^{-1/2}u\right\|^2_{\mathrm{L}^2} & \textbf{BE}_{\textbf{lrf}} \end{cases}$$

Nehari functional $I(u)$:

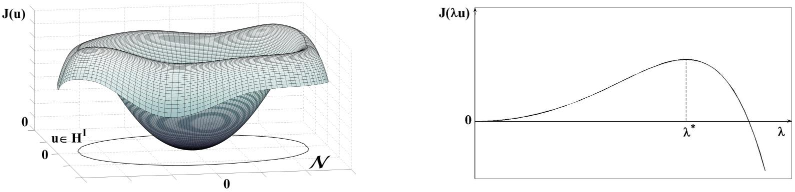
$$I(u) = J'(u)u = \begin{cases} \|u\|^2_{\mathrm{H}^1} - a\int_{\mathbb{R}^n} |u(x,t)|^{p+1}\,dx & \textbf{KG} \\ \|u\|^2_{\mathrm{H}^1} - a\int_{\mathbb{R}^n} |u(x,t)|^{p+1}\,dx + \left\|(-\Delta)^{-1/2}u\right\|^2_{\mathrm{L}^2} & \textbf{BE}_{\textbf{lrf}} \end{cases}$$

Nehari manifold $\mathcal{N}$: $\mathcal{N} = \{u \in \mathrm{H}^1 : \; I(u) = 0, \; \|u\|_{\mathrm{H}^1} \neq 0\}$

Critical energy constant d: $d = \inf_{u \in \mathcal{N}} J(u)$

Depth D of the potential well: $D = \inf_{u \in \mathrm{H}^1 \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u) > 0$

$d = D$



(a) (b)

Figure: (a) Schematic illustration of $J(u)$ as a function of $u \in \mathrm{H}^1$; (b) Cross section of $J(\lambda u)$ as a function of λ for a fixed $u \in \mathrm{H}^1$.

Potential well method – KG

Theorem (Global existence)

Suppose $u_0 \in \mathrm{H}^1$ and $u_1 \in \mathrm{L}^2$. If $0 < E(0) < d$ and $I(u_0) > 0$ or $\|u_0\|_{\mathrm{H}^1} = 0$ then problem **KG** has a unique global solution defined for every $t \in [0, \infty)$.

Theorem (Finite time blow up)

Suppose $u_0 \in \mathrm{H}^1$ and $u_1 \in \mathrm{L}^2$. If $0 < E(0) < d$ and $I(u_0) < 0$ then the weak solution of **KG** blows up in a finite time.

$$n = 1, \qquad d = \inf_{u \in \mathcal{N}} J(u) = \frac{(p-1)}{2(p+1)} \frac{1}{(aC_p^{(p+1)})^{\frac{2}{p-1}}} > 0$$

$$C_p = \sup_{v \in \mathrm{H}^1, \|v\|_1 \neq 0} \frac{\|v\|_{\mathrm{L}^{p+1}}}{\|v\|_1} = \frac{1}{\sqrt{2(p+1)}} \left((p-1)(p+3) \frac{\Gamma(\frac{4}{p-1})}{\Gamma^2(\frac{2}{p-1})} \right)^{\frac{p-1}{2(p+1)}}$$

Potential well method – $\mathbf{BE_{lrf}}$

Theorem (Global existence) [Kutev,Kolkovska,Dimova]

Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$, and $(-\Delta)^{-1/2}u_1 \in \mathrm{L}^2$. If $0 < E(0) < d$ and $I(u_0) > 0$ or $\|u_0\|_{\mathrm{H}^1} = 0$ then problem $\mathbf{BE_{lrf}}$ has a unique global solution defined for every $t \in [0, \infty)$.

Theorem (Finite time blow up) [Kutev,Kolkovska,Dimova]

Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$, and $(-\Delta)^{-1/2}u_1 \in \mathrm{L}^2$. If $0 < E(0) < d$ and $I(u_0) < 0$ then the weak solution of $\mathbf{BE_{lrf}}$ blows up in a finite time.

$$n = 1, \qquad d = \inf_{u \in \mathcal{N}} J(u) \geq d_0 = \frac{(p-1)}{2(p+1)} \frac{1}{\left(aC_p^{(p+1)}\right)^{\frac{2}{p-1}}} > 0$$

$$C_p = \sup_{v \in \mathrm{H}^1, \|v\|_1 \neq 0} \frac{\|v\|_{\mathrm{L}^{p+1}}}{\|v\|_1} = \frac{1}{\sqrt{2(p+1)}} \left((p-1)(p+3) \frac{\Gamma\left(\frac{4}{p-1}\right)}{\Gamma^2\left(\frac{2}{p-1}\right)} \right)^{\frac{p-1}{2(p+1)}}$$

Finite time blow up with arbitrary positive energy – **KG**

Theorem

Suppose $u_0 \in \mathrm{H}^1$ and $u_1 \in \mathrm{L}^2$. If

$$\|u_0\|_{\mathrm{L}^2} \neq 0, \quad I(u_0) < 0,$$

$$\int_{\mathbb{R}^n} u_0(x)u_1(x)\,dx \geq 0, \quad \frac{p-1}{2(p+1)}\|u_0\|_{\mathrm{L}^2} > E(0) > 0$$

then the weak solution of **KG** blows up in a finite time.

- F. Gazzola, M.Squassina, *Ann. Inst. Henri Poincaré, Analyse non Linéaire* 23 (2006) 185–207 (nonlinear wave equation)
- Y. Wang, *Proc. Amer. Math. Soc.* 136(2008) 3477–3482 (Klein-Gordon equation)
- R. Xu, Y.Ding, *Acta Mathematica Scienta* 33B(3) (2013) 643–652 (damped Klein-Gordon equation)

High energy blow up method – KG and $\mathbf{BE_{lrf}}$

Theorem (Finite time blow up)

Suppose $u_0 \in \mathrm{H}^1$, $u_1 \in \mathrm{L}^2$ (and additionally $(-\Delta)^{-1/2} u_0 \in \mathrm{L}^2$, $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$ for $\mathbf{BE_{lrf}}$). If $\|u_0\|_{\mathrm{L}^2} \neq 0$ and either

(i) $\langle u_0, u_1 \rangle \geq 0, \quad C\sqrt{\frac{p-1}{p+1}}\langle u_0, u_1 \rangle \geq E(0) > 0$

or

$$C = \begin{cases} 1 & \text{for } \mathbf{KG} \\ \left(\min\left(\frac{1}{\beta_2}, \frac{1}{\beta_1}\right)\right)^{1/2} & \text{for } \mathbf{BE_{lrf}} \end{cases}$$

(ii) $C\sqrt{\frac{p-1}{p+1}}\langle u_0, u_0 \rangle \geq \langle u_0, u_1 \rangle \geq 0,$

$$\frac{C^2(p-1)}{2(p+1)}\langle u_0, u_0 \rangle + \frac{\langle u_0, u_1 \rangle^2}{2\langle u_0, u_0 \rangle} \geq E(0) > 0$$

is satisfied then every weak solution of **KG** or $\mathbf{BE_{lrf}}$ blows up for a finite time $t_* < \infty$, where either $t_* < t_b$ or $t_* \leq \frac{2\langle u(t_b), u(t_b)\rangle}{(p-1)\langle u(t_b), u_t(t_b)\rangle}$

$$\mathbf{KG}: \ t_b = \begin{cases} \sqrt{\frac{p+3}{p-1}} & \text{(i)} \\ \frac{p+3}{2(p-1)} \frac{\langle u_0, u_1 \rangle}{\langle u_0, u_0 \rangle} & \text{(ii)}, \end{cases} \qquad \mathbf{BE_{lrf}}: \ t_b = \begin{cases} \sqrt{\frac{p-1}{p+1}} & \text{(i)} \\ \frac{p+1}{2(p-1)} \frac{\langle u_0, u_1 \rangle}{\langle u_0, u_0 \rangle} & \text{(ii)}. \end{cases}$$

Theorem (Sign preserving properties of $I(u(t))$)

Suppose $u_0 \in \mathrm{H}^1$, $u_1 \in \mathrm{L}^2$ (and additionally $(-\Delta)^{-1/2} u_0 \in \mathrm{L}^2$, $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$ for **BE$_{\mathrm{lrf}}$**). If $\|u_0\|_{\mathrm{L}^2} \neq 0$ and either

(i) $\quad \langle u_0, u_1 \rangle \geq 0, \quad C\sqrt{\frac{p-1}{p+1}}\langle u_0, u_1 \rangle \geq E(0) > 0$

or

(ii) $\quad C\sqrt{\frac{p-1}{p+1}}\langle u_0, u_0 \rangle \geq \langle u_0, u_1 \rangle \geq 0,$

$$\frac{C^2(p-1)}{2(p+1)}\langle u_0, u_0 \rangle + \frac{\langle u_0, u_1 \rangle^2}{2\langle u_0, u_0 \rangle} \geq E(0) > 0$$

is satisfied then $I(u(t))$ has a negative constant sign under the flow of **KG** or **BE$_{\mathrm{lrf}}$** with

$$C - const, \qquad C = \begin{cases} 1 & \text{for } \mathbf{KG} \\ \left(\min\left(\frac{1}{\beta_2}, \frac{1}{\beta_1}\right)\right)^{1/2} & \text{for } \mathbf{BE_{lrf}} \end{cases}.$$

Finite time blow up – comparison – **KG**

[Y.Wang 2008] and [R. Xu 2010]

$$\|u_0\|_{L^2} \neq 0, \quad I(u_0) < 0, \quad (u_0, u_1) \geq 0, \quad \frac{p-1}{2(p+1)}(u_0, u_0) > E(0) > 0$$

[K,K,D 2014]

(i) $\|u_0\|_{L^2} \neq 0, \quad (u_0, u_1) \geq 0, \quad \sqrt{\frac{p-1}{p+1}}(u_0, u_1) \geq E(0) > 0$

or

(ii) $\|u_0\|_{L^2} \neq 0, \quad \sqrt{\frac{p-1}{p+1}}(u_0, u_0) \geq (u_0, u_1) \geq 0,$

$$\frac{(p-1)}{2(p+1)}(u_0, u_0) + \frac{(u_0, u_1)^2}{2(u_0, u_0)} \geq E(0) > 0$$

Choice of initial data for $\mathbf{BE}_{\mathbf{lrf}}$, $n = 1$

$$u_0(x) = r(w(\theta x))'_x, \quad u_1(x) = rq(w(\theta x))'_x,$$

$w: \quad w \in \mathrm{H}^2, \ \|w\|_{\mathrm{H}^2} \neq 0;$

θ, r, q – constants

parameters:

$\beta_1 = \beta_2 = 1$, $a = 2$, $p = 2 \quad (f(u) = 2|u|u)$

$$d_0 = \frac{6}{5a^2} \le d = \inf_{u \in \mathcal{N}} J(u) \le J(v_0), \forall v_0 \in \mathcal{N}$$

$$0.3 \le d \le 9.596016$$

Example 1 : $w(x) = -\dfrac{2}{e^{2x}+1} + \dfrac{1}{e^{(0.4x-2)}+1} + \dfrac{1}{e^{(0.4x+2)}+1}$

Example 2 : $w(x) = \dfrac{1}{\cosh(x)}$

Family of finite difference schemes – $\mathbf{BE_{lrf}}$, $n = 1$

$$B\left(\frac{v_i^{n+1} - 2v_i^n + v_i^{n-1}}{\tau^2}\right) - \Lambda^{xx} v_i^n + \beta_2 \Lambda^{xxxx} v_i^n - v_i^n = \frac{a}{3}\Lambda^{xx}\left(\frac{|v_i^{n+1}|^3 - |v_i^{n-1}|^3}{v_i^{n+1} - v_i^{n-1}}\right),$$

$$B = (1 + \frac{1}{2}\tau^2)I - (\beta_1 + \sigma\tau^2)\Lambda^{xx} + \sigma\tau^2\beta_2\Lambda^{xxxx}$$

- v_i^n – a discrete approximation to u at (x_i, t_n), τ is a time-step
- $\Lambda^{xx} v_i = \frac{v_{i+1} - 2vu_i + v_{i-1}}{h^2}$, $\quad \Lambda^{xxxx} v_i = \frac{v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}}{h^4}$, $\quad \sigma$ - parameter

Properties:

- **Convergence:** The schemes have second order of convergence in space and time $O(|h|^2 + \tau^2)$.
- **Stability:** The schemes are unconditionally stable for $\sigma \geq 1/4$.
- **Conservativeness:** The discrete energy is conserved in time, i.e. $E_h(v^{(n)}) = E_h(v^{(0)}), \quad n = 1, 2, \ldots$

Example 1: $n = 1$, $\beta_1 = \beta_2 = 1$, $a = 2$, $p = 2$, $d < 9.596016$

$$u_0(x) = r(w(\theta x))'_x, \quad u_1(x) = rq(w(\theta x))'_x,$$

$$w(x) = -\frac{2}{e^{2x}+1} + \frac{1}{e^{(0.4x-2)}+1} + \frac{1}{e^{(0.4x+2)}+1}$$

$\theta = 1.2$, $r = 5.7789$, $q = 0.27$, $E(0) \approx 10.50230947 > d$

Theoretical result: $t_* \le T_b$, $T_b = \frac{2\langle u(t_b) u(t_b)\rangle}{\langle u(t_b), u_t(t_b)\rangle}$, $t_b = \sqrt{3}/3 \approx 0.577350$

$\widetilde{E}(0)$ – computed discrete energy; $\widetilde{t}_*$ – computed blow up time

$\widetilde{T}_b$ – computed upper bound of the blow up time,

$$x \in [-100, 100], \quad h = 0.01, \qquad \max_j |v_j^n| > M = 10^5 \Longrightarrow \widetilde{t}_* = t^n$$

τ	$\widetilde{E}(0)$	$\widetilde{T}_b$	$\widetilde{t}_*$
0.001	10.50223587	1.956122	1.299
0.0001	10.50227964	1.956762	1.2998
0.00001	10.50228009	1.956793	1.29987

Example 1: $n = 1$, $\beta_1 = \beta_2 = 1$, $a = 2$, $p = 2$, $d < 9.596016$

$$u_0(x) = r(w(\theta x))'_x, \quad u_1(x) = rq(w(\theta x))'_x,$$

$$w(x) = -\frac{2}{e^{2x}+1} + \frac{1}{e^{(0.4x-2)}+1} + \frac{1}{e^{(0.4x+2)}+1}$$

$\theta = 1.2, \quad r = 5.7789, \quad q = 0.27, \quad E(0) \approx 10.50249147 > d$

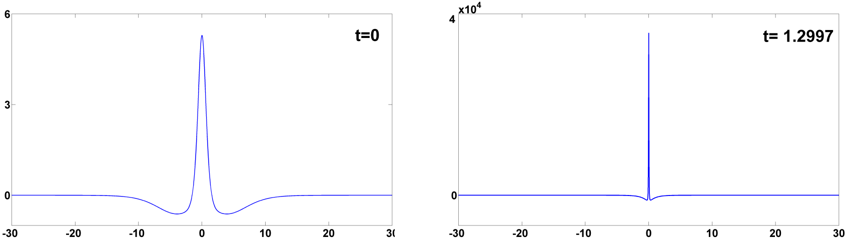


Figure: Profiles of the numerical solution $u(x,t)$ at evolution times $t = 0$ and t=1.2997; $\widetilde{t}_* = 1.2998$ – blow up time ($\tau = 0.0001$).

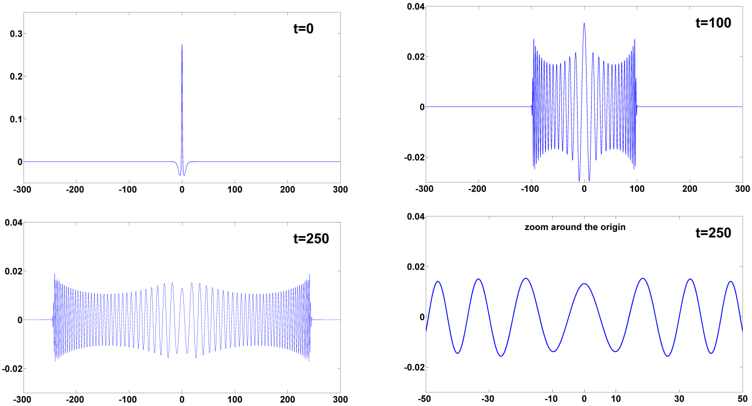
Potential well method – BE_{lrf}, numerical experiments

Example 1: $n = 1$, $\beta_1 = \beta_2 = 1$, $a = 2$, $p = 2$, $d \geq d_0 = 0.3$

$$u_0(x) = r(w(\theta x))'_x, \; u_1(x) = rq(w(\theta x))'_x, \; \theta = 1.2, \; r = 0.3, \; q = 0.0057$$

$$w(x) = -\frac{2}{e^{2x}+1} + \frac{1}{e^{(0.4x-2)}+1} + \frac{1}{e^{(0.4x+2)}+1}$$

$$\widetilde{E}(0) \approx 0.22850225 < d_0 \leq d, \quad \widetilde{I}(u_0) \approx 0.44530787 > 0$$



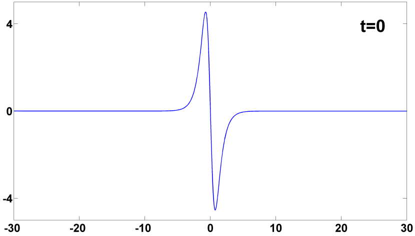
High energy blow up method – $\mathbf{BE_{lrf}}$

Example 2: $n = 1$, $\beta_1 = \beta_2 = 1$, $a = 2$, $p = 2$, $d < 9.596016$

$$u_0(x) = r(w(\theta x))'_x, \quad u_1(x) = rq(w(\theta x))'_x,$$

$$w(x) = \frac{1}{\cosh(x)}$$

$$\theta = 1.2, \quad r = 9.4, \quad q = 0.5, \quad E(0) \approx 20.09279508 > d$$



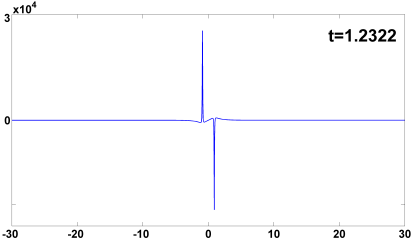


Figure: Profiles of the numerical solution $u(x,t)$ at evolution times $t = 0$ and t=1.2322; $\widetilde{t}_* = 1.2323$ – blow up time, $\widetilde{T}_b = 1.284838$ ($\tau = 0.0001$)

Thank you
for your attention!