Finite time blow up of the solutions to Klein–Gordon and Boussinesq type equations with arbitrary positive initial energy

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NTADES 2014

1– 5 July, 2014, Sofia, Bulgaria

Supported by National Science Fund Grant DDVU 02/71

Presentation outline

Generalized Boussinesq type equation

$$\alpha_1 U_{tt} - \alpha_2 \Delta U - \alpha_3 \Delta U_{tt} + \alpha_4 \Delta^2 U + \alpha_5 \Delta^2 U_{tt} + Mu + \Delta f(u) = 0,$$

$$X \in \mathbb{R}^n, \ t \in [0, T_m), T_m \leq \infty,$$

$$U(X,0) = U_0(X), \quad U_t(X,0) = U_1(X), \quad X \in \mathbb{R}^n$$

Nonlinearities

$$f(U) = \alpha |U|^{p-1} U, \quad \alpha = const \neq 0, \quad 1 < p < \begin{cases} \infty & \text{for } n = 1,2 \\ \frac{n+2}{n-2} & \text{for } n \geq 3 \end{cases}$$

Boussinesq type equation with linear restoring force $(\mathbf{BE_{rf}})$

$$\alpha_5 = 0, \ \ \alpha_1 > 0, \ \ \alpha_2 > 0, \ \ \alpha_4 > 0, \ \ M > 0, \ \ \alpha > 0,$$

$$x = \sqrt{\frac{\alpha_2}{\alpha_4}} X, \qquad u(x,t) = \frac{\alpha_2^2}{\alpha_4} U(X,t), \qquad \beta_1 \geq 0, \beta_2 > 0, m > 0, a > 0$$

$$\beta_2 u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \Delta^2 u + mu + a\Delta(|u|^{p-1} u) = 0$$

Boussinesq type equation with linear restoring force **(BE_{lrf})**

$$\beta_2 u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \Delta^2 u + mu + a\Delta(|u|^{p-1}u) = 0, \qquad x \in \mathbb{R}^n,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad t \in [0, T_m), T_m \leq \infty$$

$$u_0 \in \mathrm{H}^1(\mathbb{R}^n),\ (-\Delta)^{-1/2}u_0 \in \mathrm{L}^2(\mathbb{R}^n), \qquad u_1 \in \mathrm{L}^2(\mathbb{R}^n),\ (-\Delta)^{-1/2}u_1 \in \mathrm{L}^2(\mathbb{R}^n)$$

- transverse deflections of an elastic rod on elastic foundation

Modeling

- A.M. Samsonov, Nonlinear waves in elastic waveguides, Springer, 1994
- A.D. Mishkis, P.M. Belotserkovskiy, *ZAMM Z. Angew. Math. Mech.* 79 (1999) 645–647
- A.V. Porubov, Amplification of nonlinear strain waives in solids, World Sci., 2003

Numerical Study

- C.I. Christov, T.T. Marinov, R.S. Marinova, *Math. Comp. Simulation* 80 (2009) 56–65
- M.A. Cristou, *AIP Conf. Proc.* 1404 (2011) 41–48

Theoretical Investigations

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Nonlinear Klein-Gordon equation (**KG**)

$\alpha_4 = 0, \ \ \alpha_2 = -m < 0, \ \ \alpha_3 = \alpha_4 = 1, \ \ M = 0, \ \ \alpha < 0, \ \ a > 0???$

$$u_{tt} - \Delta u + mu - a\Delta(|u|^{p-1}u) = 0, \qquad x \in \mathbb{R}^n, \ t \in [0, T_m), T_m \leq \infty$$
$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \qquad u_0 \in \mathrm{H}^1(\mathbb{R}^n), u_1 \in \mathrm{L}^2(\mathbb{R}^n)$$

- Pesher 1976 (local existence)
- Strauss 1989 (global existence with small data)
- H. Levine 1974 (finite time blow up with negative initial energy)

Finite time blow up for subcritical initial positive initial energy (Potential well method)

- Payne, Sattinger 1975
- Ball 1978
- Strauss 1989

Finite time blow up for arbitrary high positive initial energy

- F. Gazzola, M.Squassina, *Ann. Inst. Henri Poincaré, Analyse non Linéaire* 23 (2006) 185–207 (nonlinear wave equation)
- R. Xu, Y.Ding, *Acta Mathematica Scienta* 33B(3) (2013) 643–652 (Klein-Gordon equation)

Short notations: $\|u\|_{\mathrm{L}^2(\mathbb{R}^n)} = \|u\|_{\mathrm{L}^2}, \quad \|u\|_{\mathrm{H}^1(\mathbb{R}^n)} = \|u\|_{\mathrm{H}^1}$

$$\langle u, v\rangle = \begin{cases} \int_{\mathbb{R}^n} u\, v\, dx & \textbf{KG} \\ \beta_1 \int_{\mathbb{R}^n} u\, v\, dx + \beta_2 \int_{\mathbb{R}^n} (-\Delta)^{-1/2} u . (-\Delta)^{-1/2} v\, dx & \textbf{BE}_{\textbf{lrf}} \end{cases}$$

Conservation low: $E(t) = E(0) \;\; \forall\, t \in [0, T_m),$

$E(t) := E(u(\cdot, t), u_t(\cdot, t))$

$$E(t) = \begin{cases} \frac{1}{2}\langle u_t(\cdot,t), u_t(\cdot,t)\rangle^2 + \frac{1}{2}\|u(\cdot,t)\|^2_{\mathrm{H}^1} - \frac{a}{p+1}\int_{\mathbb{R}^n} |u(x,t)|^{p+1}\, dx & \textbf{KG} \\ \frac{1}{2}\langle u_t(\cdot,t), u_t(\cdot,t)\rangle^2 + \frac{1}{2}\|u(\cdot,t)\|^2_{\mathrm{H}^1} - \frac{a}{p+1}\int_{\mathbb{R}^n} |u(x,t)|^{p+1}\, dx & \\ \qquad + \frac{m}{2}\left\|(-\Delta)^{-1/2} u(\cdot,t)\right\|^2_{\mathrm{L}^2} & \textbf{BE}_{\textbf{lrf}} \end{cases}$$

Important functionals

Potential energy functional $J(u)$:

$$J(u) = \begin{cases} \frac{1}{2}\|u\|^2_{\mathrm{H}^1} - \frac{a}{p+1}\int_{\mathbb{R}^n} |u(x,t)|^{p+1}\,dx & \textbf{KG} \\ \frac{1}{2}\|u\|^2_{\mathrm{H}^1} - \frac{a}{p+1}\int_{\mathbb{R}^n} |u(x,t)|^{p+1}\,dx + \frac{m}{2}\left\|(-\Delta)^{-1/2}u\right\|^2_{\mathrm{L}^2} & \textbf{BE}_{\mathbf{lrf}} \end{cases}$$

Nehari functional $I(u)$:

$$I(u) = J'(u)u = \begin{cases} \|u\|^2_{\mathrm{H}^1} - a\int_{\mathbb{R}^n} |u(x,t)|^{p+1}\,dx & \textbf{KG} \\ \|u\|^2_{\mathrm{H}^1} - a\int_{\mathbb{R}^n} |u(x,t)|^{p+1}\,dx + m\left\|(-\Delta)^{-1/2}u\right\|^2_{\mathrm{L}^2} & \textbf{BE}_{\mathbf{lrf}} \end{cases}$$

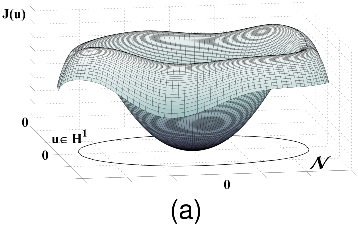
Potential well method

Nehari manifold $\mathcal{N}$: $\mathcal{N} = \{u \in \mathrm{H}^1 : \; I(u) = 0, \; \|u\|_{\mathrm{H}^1} \neq 0\}$

Critical energy constant d: $d = \inf_{u \in \mathcal{N}} J(u)$

Depth D of the potential well: $D = \inf_{u \in \mathrm{H}^1 \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u) > 0$

$d = D$



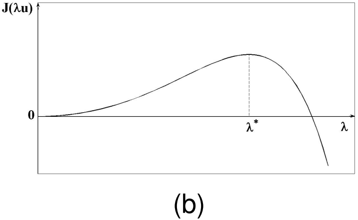


Figure: (a) Schematic illustration of $J(u)$ as a function of $u \in \mathrm{H}^1$; (b) Cross section of $J(\lambda u)$ as a function of λ for a fixed $u \in \mathrm{H}^1$.

Potential well method

Theorem (Global existence – KG)

Suppose $u_0 \in \mathrm{H}^1$ and $u_1 \in \mathrm{L}^2$. If $0 < E(0) < d$ and $I(u_0) > 0$ or $\|u_0\|_{\mathrm{H}^1} = 0$ then problem **KG** has a unique global solution defined for every $t \in [0, \infty)$.

Theorem (Finite time blow up – KG)

Suppose $u_0 \in \mathrm{H}^1$ and $u_1 \in \mathrm{L}^2$. If $0 < E(0) < d$ and $I(u_0) < 0$ then the weak solution of **KG** blows up in a finite time.

Theorem (Global existence – $\mathbf{BE_{rf}}$)

Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2} u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$, and $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$. If $0 < E(0) < d$ and $I(u_0) > 0$ or $\|u_0\|_{\mathrm{H}^1} = 0$ then problem $\mathbf{BE_{rf}}$ has a unique global solution defined for every $t \in [0, \infty)$.

Theorem (Finite time blow up – $\mathbf{BE_{rf}}$)

Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2} u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$, and $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$. If $0 < E(0) < d$ and $I(u_0) < 0$ then the weak solution of $\mathbf{BE_{rf}}$ blows up

Theorem (Finite time blow up with arbitrary positive energy – **KG)**

Suppose $u_0 \in \mathrm{H}^1$ and $u_1 \in \mathrm{L}^2$. If $\|u_0\|_{\mathrm{L}^2} \neq 0$,

$$\int_{\mathbb{R}^n} u_0(x)u_1(x)\,dx \geq 0, \qquad \frac{p-1}{2(p+1)}\|u_0\|_{\mathrm{L}^2} > E(0) > 0$$

then the weak solution of **KG** blows up in a finite time.

High energy blow up method

Theorem (Sign preserving properties of $I(u(t))$)

Suppose $u_0 \in \mathrm{H}^1$, $u_1 \in \mathrm{L}^2$ (and additionally $(-\Delta)^{-1/2} u_0 \in \mathrm{L}^2$ and $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$ for...). If $\|u_0\|_{\mathrm{L}^2} \neq 0$, $m_0 = \min\left(\frac{m}{\beta_2}, \frac{1}{\beta_1}\right)^{1/2}$ and either

(i) $\langle u_0, u_1 \rangle \geq 0, \quad m_0 \sqrt{\frac{p-1}{p+1}} \langle u_0, u_1 \rangle \geq E(0) > 0$

or $\qquad m_0 = \min\left(\frac{m}{\beta_2}, \frac{1}{\beta_1}\right)^{1/2}$

(ii) $0 \leq \langle u_0, u_1 \rangle \leq m_0 \sqrt{\frac{p-1}{p+1}} \langle u_0, u_0 \rangle,$

$$\frac{m_0^2(p-1)}{2(p+1)} \langle u_0, u_0 \rangle + \frac{\langle u_0, u_1 \rangle^2}{2\langle u_0, u_0 \rangle} \geq E(0) > 0$$

is satisfied then $I(u(t))$ has a negative constant sign under the flow of **KG** or **BE**.

Moreover if $\widetilde{t} < T_m$, where $\widetilde{t} = \begin{cases} \frac{1}{m_0}\sqrt{\frac{p-1}{p+1}} & \text{in case (i);} \\ \frac{p+1}{2(p-1)m_0^2} \frac{\langle u_0, u_1 \rangle}{\langle u_0, u_0 \rangle} & \text{in case (ii),} \end{cases}$

then $I(u(t)) \leq -\frac{p+1}{2} \langle u_t(t), u_t(t) \rangle$ for $t \in [\widetilde{t}, T_m)$.

High energy blow up method

Theorem (Finite time blow up)

Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$, $(-\Delta)^{-1/2}u_1 \in \mathrm{L}^2$, and $\|u_0\|_{\mathrm{H}^1} \neq 0$. If either

(i) $\langle u_0, u_1\rangle \geq 0, \quad m_0\sqrt{\frac{p-1}{p+1}}\langle u_0, u_1\rangle \geq E(0) > 0$

or

$$m_0 = \min\left(\frac{m}{\beta_2}, \frac{1}{\beta_1}\right)^{1/2}$$

(ii) $0 \leq \langle u_0, u_1\rangle \leq m_0\sqrt{\frac{p-1}{p+1}}\langle u_0, u_0\rangle,$

$$\frac{m_0^2(p-1)}{2(p+1}\langle u_0, u_0\rangle + \frac{\langle u_0, u_1\rangle^2}{2\langle u_0, u_0\rangle} \geq E(0) > 0$$

is satisfied then every weak solution of $\mathbf{BE_{rf}}$ blows up for a finite time $t_* < \infty$, where either $t_* < \tilde{t}$ or

$$t_* \leq \frac{2\langle u(\tilde{t}), u(\tilde{t})\rangle}{(p-1)\langle u(\tilde{t}), u_t(\tilde{t})\rangle}, \quad \tilde{t} = \begin{cases} \frac{1}{m_0}\sqrt{\frac{p-1}{p+1}} & \text{in case (i);} \\ \frac{p+1}{2(p-1)m_0^2}\frac{\langle u_0, u_1\rangle}{\langle u_0, u_0\rangle} & \text{in case (ii).} \end{cases}$$

Comparison between Potential well method (**PWM**) and High energy blow up method (**HEBM**)

method	u_0, u_1	$I(u_0)$	$E(0)$
PWM	no conditions	$I(u_0) < 0$	$0 < E(0) < d$
HEBM (i)	$\langle u_0, u_1 \rangle \geq 0$ $m_0 \sqrt{\frac{p-1}{p+1}} \langle u_0, u_1 \rangle \geq E(0) > 0$	no conditions	arbitrary high positive
HEBM (ii)	$0 \leq \langle u_0, u_1 \rangle \leq m_0 \sqrt{\frac{p-1}{p+1}} \langle u_0, u_0 \rangle$ $\frac{m_0^2 (p-1)}{2(p+1} \langle u_0, u_0 \rangle + \frac{\langle u_0, u_1 \rangle^2}{2 \langle u_0, u_0 \rangle} \geq E(0) > 0$	no conditions	arbitrary high positive

Conclusions:

- Subcritical initial energy: $0 < E(0) < d$ – **PWM**
- Supercritical initial energy: $E(0) > d$ – **HEBM**

Choice of initial data

$$u_0(x) = r(w(\theta x))'_x, \quad u_1(x) = rq(w(\theta x))'_x,$$

$w: \quad w \in \mathrm{H}^2, \ \|w\|_{\mathrm{H}^2} \neq 0;$

θ, r, q – constants

parameters:

$\beta_1 = \beta_2 = 1$, $m = 1$, $m_0 = 1$, $a = 2$, $p = 2$ $(f(u) = 2|u|u)$

$$d_0 = \frac{6}{5a^2} \leq d = \inf_{u \in \mathcal{N}} J(u) \leq J(v_0), \forall v_0 \in \mathcal{N}$$

$$0.3 \leq d \leq 9.59601581$$

Example 1 : $w(x) = -\dfrac{2}{e^{2x}+1} + \dfrac{1}{e^{(0.4x-2)}+1} + \dfrac{1}{e^{(0.4x+2)}+1}$

Example 2 : $w(x) = \dfrac{1}{\cosh(x)}$

Family of finite difference schemes

$$B\left(\frac{v_i^{n+1}-2v_i^n+v_i^{n-1}}{\tau^2}\right)-\Lambda^{xx}v_i^n+\beta_2\Lambda^{xxxx}v_i^n-mv_i^n = \frac{a}{3}\Lambda^{xx}\left(\frac{|v_i^{n+1}|^3-|v_i^{n-1}|^3}{v_i^{n+1}-v_i^{n-1}}\right),$$

$$B=(1+\frac{m}{2}\tau^2)I-(\beta_1+\sigma\tau^2)\Lambda^{xx}+\sigma\tau^2\beta_2\Lambda^{xxxx}$$

- v_i^n – a discrete approximation to u at (x_i, t_n), τ is a time-step
- $\Lambda^{xx}v_i=\frac{v_{i+1}-2vu_i+v_{i-1}}{h^2}$, $\quad \Lambda^{xxxx}v_i=\frac{v_{i+2}-4v_{i+1}+6v_i-4v_{i-1}+v_{i-2}}{h^4}$, $\quad \sigma$ - parameter

Properties:

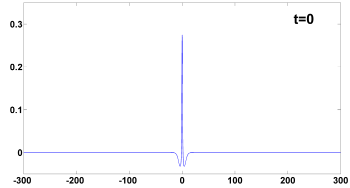
- **Convergence:** The schemes have second order of convergence in space and time $O(|h|^2+\tau^2)$.
- **Stability:** The schemes are unconditionally stable for $\sigma > 1/4$.
- **Conservativeness:** The discrete energy is conserved in time, i.e. $E_h(v^{(n)})=E_h(v^{(0)}), \quad n=1,2,\dots$

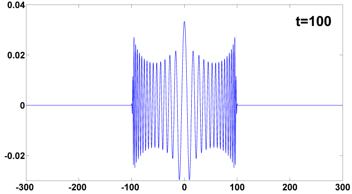
Potential well method, numerical experiments

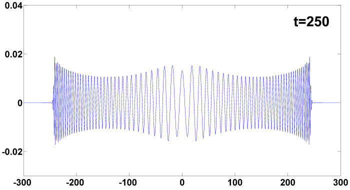
Example 1: $\beta_1 = \beta_2 = 1$, $m = 1$, $m_0 = 1$, $a = 2$, $p = 2$, $d \geq d_0 = 0.3$

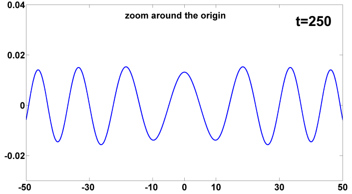
$$\theta = 1.2, \quad r = 0.3, \quad q = 0.0057$$

$$\widetilde{E}(0) \approx 0.22850225 < d_0 \leq d, \quad \widetilde{I}(u_0) \approx 0.44530787 > 0$$









Example 1: $\beta_1 = \beta_2 = 1$, $m = 1$, $m_0 = 1$, $a = 2$, $p = 2$,
$\widetilde{t} = \sqrt{3}/3 \approx 0.57735027$, $d < 9.56015811$

$S = 10 > d$ – fixed, $\theta = 1.2$, $r = 5.7789$, $q = 0.27$:

$$\widetilde{E}(0) \approx 10.50341845 > S > d, \qquad \widetilde{t} \approx 0.58, \ t_* \le \frac{2\langle u(\widetilde{t}), u(\widetilde{t})\rangle}{\langle u(\widetilde{t}), u_t(\widetilde{t})\rangle} \approx 1.91$$

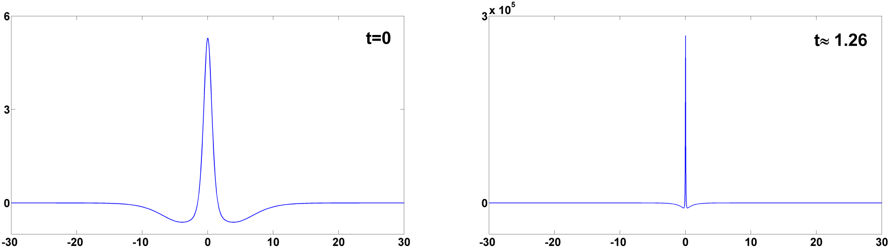


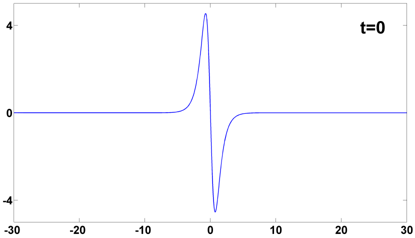
Figure: Profiles of the numerical solution $u(x, t)$ at evolution time $t = 0$ and t=1.26; $t_* \approx 1.27 < 1.91$ – blow up time.

High energy blow up method, numerical experiments

Example 2: $\beta_1 = \beta_2 = 1$, $m = 1$, $m_0 = 1$, $a = 2$, $p = 2$,
$\widetilde{t} = \sqrt{3}/3 \approx 0.57735027$, $d < 9.56015811$

$S = 10 > d$ – fixed, $\theta = 1.2$, $r = 9.0881$, $q = 0.25$:

$\widetilde{E}(0) \approx 10.09279508 > S > d$, $\widetilde{t} \approx 0.58$, $t_* \leq \dfrac{2\langle u(\widetilde{t}), u(\widetilde{t})\rangle}{\langle u(\widetilde{t}), u_t(\widetilde{t})\rangle} \approx 1.60$



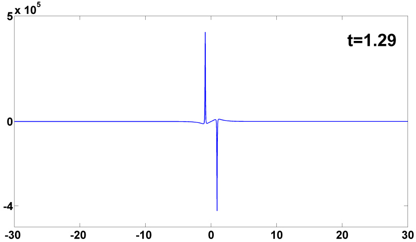


Figure: Profiles of the numerical solution $u(x, t)$ at evolution time $t = 0$ and t=1.29; $t^* \approx 1.30 < 1.60$ – blow up time.

Thank you
for your attention!