Finite Difference Schemes for Generalized Boussinesq Equation

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Mathematical Modeling and Computational Physics,
DUBNA, 2013

Supported by Bulgarian National Science Fund Grant DDVU 02/71

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Introduction

We study the Cauchy problem for

Boussinesq Equation (BE)

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \Delta f(u), \quad x \in \mathbb{R}^n, \ t > 0,$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

on the unbounded region $\mathbb{R}^n$ with asymptotic boundary conditions

$u(x,t) \to 0$, $\Delta u(x,t) \to 0$ as $|x| \to \infty$,

where Δ is the Laplace operator, β_1 and β_2 are positive constants.

This is a 4-th order differential equation in x and 2-nd order in t with non-linearity contained in the term $f(u)$.
f is a polynomial of u. Examples: $f(u) = \alpha u^2$; $f(u) = au^3 + bu^5$.

Numerical methods for BE, references

- finite difference methods (Ortega, Sanz Serna, 1990; Christov, 1994);
- finite element methods (Pani, 1997);
- spectral method with Christov functions (Christou, 2010);
- Godunov-type central-upwind scheme (Chertock, Christov, Kurganov, 2011)

- theoretical analysis, numerical implementation, comparison of several FDS (Kolkovska, 2010; Christov, Vasileva, Kolkovska, 2010; Kolkovska, Dimova, 2011, 2012);
- vector additive schemes (multicomponent alternating direction method) (Kolkovska, Angelow, 2013) - $O(h^2 + \tau)$, non conservative method;

We assume that the initial data satisfy such regularity conditions that BE has a unique smooth enough solution.

Properties to the Boussinesq equation

Let $\|\cdot\|$ denote the standard norm in $L_2(R^n)$.
Define the energy functional

$$E\left(u(t)\right)=\left\|(-\Delta)^{-1/2}\frac{\partial u}{\partial t}\right\|^2+\beta_1\left\|\frac{\partial u}{\partial t}\right\|^2+\|u\|^2+\beta_2\left\|\nabla u\right\|^2+\int_{R^n}F(u)du$$

with

$$F(u)=\int_0^u f(s)ds$$

Theorem (Conservation law)

The solution u to Boussinesq problem satisfies the following energy identity

$$E\left(u(t)\right)=E\left(u(0)\right)\quad\forall t\in[0,T].$$

We obtain similar energy identities for the solutions of the FDS employed in the discretization of BPE.

Notations for case $n = 2$:

- Domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$, L_1, L_2 – sufficiently large;
- a uniform mesh with steps h_1, h_2 in Ω:
  $x_i = ih_1, i = -M_1, M_1;\ y_j = jh_2, j = -M_2, M_2;$
- τ - the time step, $t_k = k\tau, k = 0, 1, 2, \ldots;$
- mesh points (x_i, y_j, t_k);
- $v^{(k)}_{(i,j)}$ denotes the discrete approximation $u(x_i, y_j, t_k)$;
- notations for some discrete derivatives of mesh functions:
  - $v^{(k)}_{t,(i,j)} = (v^{(k+1)}_{(i,j)} - v^{(k)}_{(i,j)})/\tau;$
  - $v^{(k)}_{\bar{x}x,(i,j)} = \left(v^{(k)}_{(i+1,j)} - 2v^{(k)}_{(i,j)} + v^{(k)}_{(i-1,j)} \right) / h_1^2;$
  - $v^{(k)}_{\bar{t}t,(i,j)} = \left(v^{(k+1)}_{(i,j)} - 2v^{(k)}_{(i,j)} + v^{(k-1)}_{(i,j)} \right) / \tau^2;$
  - $\Delta_h v = v_{\bar{x}x} + v_{\bar{y}y}$ – the 5-point discrete Laplacian.
  - $(\Delta_h)^2 v = v_{\bar{x}x\bar{x}x} + v_{\bar{y}y\bar{y}y} + 2v_{\bar{x}x\bar{y}y}$ – the discrete biLaplacian

In approximation of $\Delta_h v$ and $(\Delta_h)^2 v$ we use v^θ – the symmetric θ-weighted approximation to $v^k_{(i,j)}$:

$v^{\theta,k}_{(i,j)} = \theta v^{k+1}_{(i,j)} + (1-2\theta) v^k_{(i,j)} + \theta v^{k-1}_{(i,j)}$, $\theta \in R$.

Three level FDS

$$v^k_{\bar{t}t} - \Delta_h v^k_{\bar{t}t} - \Delta_h v^{\theta,k} + (\Delta_h)^2 v^{\theta,k} = \Delta_h \frac{F(v^{k+1}) - F(v^{k-1})}{v^{k+1} - v^{k-1}}$$

$$\begin{aligned} v^0_{(i,j)} &= u_0(x_i, y_j), \\ v^1_{(i,j)} &= u_0(x_i, y_j) + \tau u_1(x_i, y_j) \\ &\quad + 0.5\tau^2 (I - \Delta_h)^{-1} \left(\Delta_h u_0 - (\Delta_h)^2 u_0 + \Delta_h g(u_0) \right) (x_i, y_j). \end{aligned}$$

The equations, boundary and initial conditions form a family of 3-levels finite difference schemes.

The second time derivative at the time level $t^k + \tau/2$ is approximated with error $O(\tau^2)$ using four consecutive time levels $(k+2)$, $(k+1)$, (k) and $(k-1)$ as

$$v_{\hat{t}\hat{t}}^{(k)} = 0.5(v^{(k+2)} - v^{(k+1)} - v^{(k)} + v^{(k-1)})\tau^{-2}.$$

For the approximation of $\Delta_h v$ and $(\Delta_h)^2 v$ we introduce two symmetric approximations to $u(\cdot, t^k + \tau/2)$ with real parameters θ and μ:

$$v^{\theta(k)} = \theta v^{(k+2)} + (0.5-\theta)v^{(k+1)} + (0.5-\theta)v^{(k)} + \theta v^{(k-1)},$$
$$v^{\mu(k)} = \mu v^{(k+2)} + (0.5-\mu)v^{(k+1)} + (0.5-\mu)v^{(k)} + \mu v^{(k-1)}$$

For the approximation of non-linear term we use

$$\frac{F(v^{(k+1)}) - F(v^{(k)})}{v^{(k+1)} - v^{(k)}}.$$

Note that function $f(v)$ is a polynomial of v, thus the integrals $F(v)$ could be explicitly evaluated!

Four level FDS:

$$
\begin{aligned}
&(I-\beta_1\Delta_h)(v^{(k+2)}-v^{(k+1)}-v^{(k)}+v^{(k-1)})/(2\tau^2)\\
&-\Delta_h v^{\theta(k)}+\beta_2(\Delta_h)^2 v^{\mu(k)}=\Delta_h\frac{F(v^{(k+1)})-F(v^{(k)})}{v^{(k+1)}-v^{(k)}}
\end{aligned}
$$

Here I stands for the identity operator.
Initial values $v^{(0)}$, $v^{(1)}$ and $v^{(-1)}$ on time levels $t=0$, $t=\tau$ and $t=-\tau$ are evaluated by formulas

$$
\begin{aligned}
v_{i,j}^{(0)} &= u_0(x_i,y_j),\\
v_{i,j}^{(1)} &= u_0(x_i,y_j)+\tau u_1(x_i,y_j)\\
&\quad +0.5\tau^2(I-\beta_1\Delta_h)^{-1}\left(\Delta_h u_0-\beta_2(\Delta_h)^2u_0+\Delta_h f(u_0)\right)(x_i,y_j),\\
v_{\bar{t}t(i,j)}^{(0)} &= \left(v_{(i,j)}^{(1)}-2v_{(i,j)}^{(0)}+v_{(i,j)}^{(-1)}\right)\tau^{-2}\\
&= (I-\beta_1\Delta_h)^{-1}\left(\Delta_h u_0-\beta_2\Delta_h^2u_0+\Delta_h f(u_0)\right)(x_i,y_j).
\end{aligned}
$$

Consider the space of functions, which vanish on the boundary of Ω_h, with the scalar product

$$\langle v, w \rangle = \sum_{i,j} h_1 h_2 v_{(i,j)}^{(k)} w_{(i,j)}^{(k)};$$

We define operators

$$A = -\Delta_h$$

$$B = I - \beta_1 \Delta_h - 2\tau^2 \theta \Delta_h + 2\tau^2 \beta_2 \mu (\Delta_h)^2$$

A , B - self-adjoint and positive definite operators for $\theta \geq 0$ and $\mu \geq 0$

The energy functional E_h^L (obtained from the linear part of the equation) at the k-th time level is

$$E_h^L(v^{(k)})$$
$$= \left\langle A^{-1} v_t^{(k)}, v_t^{(k)} \right\rangle + \left\langle v_t^{(k)}, v_t^{(k)} \right\rangle + \tau^2(\theta - 1/4) \left\langle (I + A) v_t^{(k)}, v_t^{(k)} \right\rangle$$
$$+ 1/4 \left\langle v^{(k)} + v^{(k+1)} + A(v^{(k)} + v^{(k+1)}), v^{(k)} + v^{(k+1)} \right\rangle$$

The *full discrete energy functional* is (including the non-linearity)

$$E_h(v^{(k)}) = E_h^L(v^{(k)}) + \left\langle F(v^{(k+1)}), 1 \right\rangle + \left\langle F(v^{(k)}), 1 \right\rangle$$

Theorem (Discrete conservation law, Three level FDS)

The solution to the iterative scheme (IM) satisfies the energy equalities

$$E_h(v^{(k)}) = E_h(v^{(0)}), \qquad k = 1, 2, \dots .$$

i.e. the discrete energy is conserved in time.

We introduce the linear functional $E_{h,L}(v^{(k)})$ as

$$E_{h,L}(v^{(k)}) = 0.5 \left\langle A^{-1} B v_t^{(k)}, v_t^{(k-1)} \right\rangle + 0.5 \left\langle v^{(k)} + \beta_2 A v^{(k)}, v^{(k)} \right\rangle$$

and the full discrete energy functional $E_h(v^{(k)})$ as

$$E_h(v^{(k)}) = E_{h,L}(v^{(k)}) + \left\langle F(v^{(k)}), 1 \right\rangle .$$

Theorem (Discrete conservation law, Four level FDS)

The solution to the considered FDS satisfies the energy equalities

$$E_h(v^{(k)}) = E_h(v^{(0)}), \qquad k = 1, 2, \dots .$$

i.e. the discrete energy is conserved in time.

Our calculations confirm that the discrete energy functional $E_h(v^{(k)})$ is preserved in time with a high accuracy (for $t \in (0, 20]$ - with 10^{-8} error)

Theorem (Convergence of the 3-level FDS)

Assume that the parameter θ satisfies

$$\theta > \frac{1+\epsilon}{4} - \frac{1}{\tau^2||A||}, \epsilon > 0$$

Assume that the solution u is in $C^{4,4}\left(\mathbb{R}^2 \times (0,T)\right)$ and the solution v to the 3-level FDS is bounded in the maximal norm. Then v converges to the exact solution u as $|h|, \tau \to 0$ and the following estimate holds for the error $z = y - u$:

$$\max_i |z_i^{(k)}| < Ce^{Mt_k}\sqrt{\frac{1+\epsilon}{\epsilon}}\left(|h|^2+\tau^2\right), \qquad n=1;$$

$$\max_{i,j} |z_{i,j}^{(k)}| < Ce^{Mt_k}\sqrt{\ln N}\sqrt{\frac{1+\epsilon}{\epsilon}}\left(|h|^2+\tau^2\right), \qquad n=2.$$

Theorem (**Convergence of the 4-level FDS**)

Assume that f is a polynomial of u and that:

(i) *parameters θ and μ satisfy the operator inequality*

$$A^{-1} + \beta_1 I + \tau^2(2\theta - 0.5)I + \tau^2\beta_2(2\mu - 0.5)A > \epsilon I, \ \epsilon > 0$$

with some positive real number ϵ independent on h, τ, u;

(ii) $u \in C^{4,4}(\mathbb{R}^2 \times [0, T))$;

(iii) *the discrete solution v is bounded in the maximal norm.*

Then the discrete solution v converges to the exact solution u as $|h|, \tau \to 0$ and the following estimate holds for the error $z = u - v$:

$$\max_i |z_i^{(k)} + z_i^{(k+1)}| \leq Ce^{Mt^k} \left(|h|^2 + \tau^2\right), n = 1;$$

$$\max_{i,j} |z_i^{(k)} + z_i^{(k+1)}| \leq Ce^{Mt^k} \sqrt{\ln(\max\{N_1, N_2\})} \left(|h|^2 + \tau^2\right), n = 2.$$

Table: Restrictions on parameters θ, μ for validity of condition (i) in the convergence Theorem

μ	θ	sufficient conditions
$\mu \geq 0.25$	$\theta \geq 0.25$	no restrictions
$\mu \geq 0.25$	$\theta < 0.25$	$\tau^2 < \frac{\beta_1 - \epsilon + \tau^2(2\mu - 0.5)\beta_2 4/L^2}{(0.5 - 2\theta)}$
$\mu < 0.25$	$\theta \geq 0.25$	$\tau^2 < h^2 \frac{\beta_1 - \epsilon}{4n(0.5 - 2\mu)\beta_2}$
$\mu < 0.25$	$\theta < 0.25$	$\tau^2 < h^2 \frac{\beta_1 - \epsilon + \tau^2(2\theta - 0.5)}{4n(0.5 - 2\mu)\beta_2}$

Here $L = \max(L_1, L_2)$ is the semi-length of the computational domain and $n = 1, 2$ is the dimension.

Algorithm for 3 level FDS

A. Evaluate $v^{(0)}$, $v^{(1)}$ from the initial conditions

B. For $k = 1, 2, \ldots$ do ($v^{(k-1)}$, $v^{(k)}$ are known)

1. take $v^{(k+1)[0]} = v^{(k)}$
2. for $s = 1, 2, \ldots$ repeat steps (a), (b) below until $|v^{(k+1)[s+1]} - v^{(k+1)[s]}| < \epsilon |v^{(k+1)[s]}|$
   - (a) find w by standard elliptic solver
     $$(I - \Delta_h) w = \Delta_h \frac{F(v^{(k+1)[s]}) - F(v^{(k-1)})}{v^{(k+1)[s]} - v^{(k-1)}},$$
     $w = 0$ (BC),
   - (b) obtain $v^{(k+1)[s+1]}$ from
     $$\left(I - \theta\tau^2\Delta_h\right) v_{\bar{t}t}^{(k)[s+1]} - \Delta_h v^{(k)[s+1]} = w,$$
     $v^{(k+1)[s+1]} = 0$ (BC)
3. set $v^{(k+1)} = v^{(k+1)[s+1]}$

Numerical algorithm, 4-level FDS

1. Evaluate $v^{(0)}$, $v^{(1)}$, $v^{(-1)}$ from the initial conditions;
2. For $k = 0, 1, 2, \ldots$ do $(v^{(k-1)}, v^{(k)}, v^{(k+1)}$ are known):

$$(I - \beta_1 \Delta_h)(v^{(k+2)})/(2\tau^2) - \theta \Delta_h v^{(k+2)} + \mu \beta_2 \Delta_h^2 v^{(k+2)}$$
$$= \Delta_h \frac{F(v^{(k+1)}) - F(v^{(k)})}{v^{(k+1)} - v^{(k)}} - (0.5 - \mu)\beta_2 (\Delta_h)^2 (v^{(k+1)} + v^{(k)})$$
$$- \mu \beta_2 \Delta_h^2 v^{(k-1)} + (0.5 - \theta)\Delta_h (v^{(k+1)} + v^{(k)}) + \theta \Delta_h v^{(k-1)}$$
$$+ (I - \beta_1 \Delta_h)(v^{(k+1)} + v^{(k)} - v^{(k-1)})/(2\tau^2)$$

Remarks:
if $\mu \neq 0$ - 4-th order elliptic equation for $v^{(k+2)} \Rightarrow$ choose $\mu = 0$!
for $\mu = 0$ - second order elliptic equation for $v^{(k+2)}$ - the numerical method is efficient!
No inner iterations are needed for evaluation of $v^{(k+2)}$.
Despite this fact, this method is conservative!

Preliminaries

- An analytical solution of the 1D equation (one solitary wave):

$$u(x,t;x_0,c) = \frac{3}{2}\frac{c^2-1}{\alpha}\operatorname{sech}^2\left(\frac{x-x_0-ct}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}\right),$$

  where x_0 is the initial position of the peak of the solitary wave,
- Parameters: $\alpha = 3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, c is the wave speed.
- Initial conditions for one solitary wave or two solitary waves:

$$u(x,0) = u(x,0;-40,2) + u(x,0;50,-1.5)$$

$$\frac{du}{dt}(x,0) = u(x,0;-40,2)_t + u(x,0;50,-1.5)_t$$

- schemes with $\mu = 0$ and several θ: $\theta = 0.25$, $\theta = 0.5$, $\theta = 0$.

One solitary wave

Errors in uniform norm and rate of convergence for $t \in [0, 20]$, $\theta = 0.5$

	c=2		c=0.5	
h	Error	Rate	Error	Rate
0.1	0.0011424		0.0094145	
0.05	0.00028569	1.99954544	0.0022174	2.08601543
0.025	7.1534 e-005	1.99783019	0.0005475	2.01793817
0.0125	1.9402 e-005	1.88234306	0.0001359	2.01031351

- $\tau = h\sqrt{(\beta_1/(8\beta_2))}$, $\epsilon = 0.5\beta_1$, $\tau^2 < 0.5\beta_1$
- The error is the difference between the calculated and the exact solution in uniform norm for $t = 20$.
- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.

Interaction of two solitary waves with different speeds

Errors in uniform norm and rate of convergence for $t \in [0, 40]$

h	$\theta = 0.5$		$\theta = 0$	
	error	rate	error	rate
0.08				
0.04	0.00231463		0.00034355	
0.02	0.00057865	2.00002063	8.55658155e-005	2.31582697
0.01	0.00013966	2.05076806	1.71856875e-005	2.00541487

- For every h the error is calculated by Runge method as $E_1^2/(E_1 - E_2)$ with $E_1 = \|u_{[h]} - u_{[h/2]}\|$, $E_2 = \|u_{[h/2]} - u_{[h/4]}\|$, where $u_{[h]}$ is the calculated solution with step h for $t = 40$.
- The numerical rate of convergence is $(\log E_1 - \log E_2)/\log 2$.
- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.
- For two solitary waves the scheme with $\theta = 0$ is 6 to 7 times more precise than the scheme with $\theta = 0.5$!

Comparison, 3-level and 4-level conservative schemes

Errors in uniform norm for one and two solitary waves

	1 soliton, T=40			2 solitons, T=80	
h	4-level $\theta = 0.25$	3-level Con.FDS	h	4-level $\theta = 0.25$	3-level Con.FDS
0.2	0.01288	0.14412	0.2		
0.1	0.00324	0.03753	0.1	0.04019	
0.05	0.00081	0.00948	0.05	0.01907	0.102754
0.025	0.00020	0.00238	0.025	0.009212	0.026027
0.0125	5.25e-05	0.00059	0.0125	0.004010	0.006528

- for one solitary wave: the 4-level FDS is approximately 10 times more precise than the 3-level FDS;
- for two solitary waves: the 4-level FDS is approximately 2 times more precise than the 3-level FDS.

With respect to the error magnitude the 'new' four-level scheme performs much better than the 'old' three-level schemes!

Justification: Consider both FDS. We expand all terms in Taylor series about the point $(x_i, t^{(k)} + \tau/2)$ or (x_i, t^k) and get for the leading terms

$$R_{4-lev} = \frac{1}{8}\alpha\tau^2\Delta_h\frac{\partial f}{\partial u}(x_i, t^{(k)} + \tau/2)\frac{\partial^2 u}{\partial t^2}(x_i, t^{(k)} + \tau/2),$$

$$R_{3-lev} = \frac{1}{4}\alpha\tau^2\Delta_h\frac{\partial f}{\partial u}(x_i, t^k)\frac{\partial^2 u}{\partial t^2}(x_i, t^k).$$

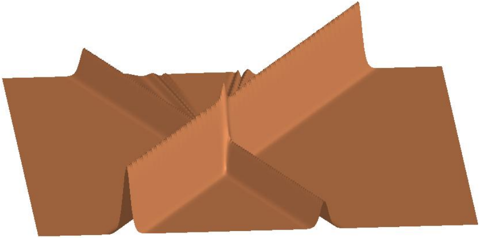
Thus, $R_{3-lev} \approx 2 * R_{4-lev}$. This has essential impact on the total error, when the solution has large derivatives!

Movie

Interaction of two solitary waves with different speeds

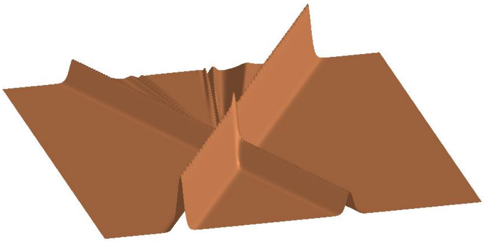
$x \in [-120, 120],\ t \in [0, 35],\ c_1 = 2,\ c_2 = -1.5$

Graphics, 1D



Interaction of two solitary waves with different speeds
$x \in [-80, 120], \ t \in [0, 35], \ c_1 = 2, \ c_2 = -1.5$

Graphics, 1D



Interaction of two solitary waves with different speeds

$x \in [-80, 120],\ t \in [0, 35],\ c_1 = 2,\ c_2 = -1.5$

Graphics, 2D

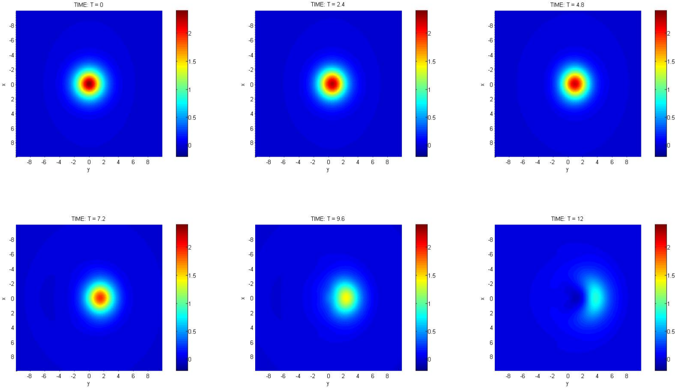


Figure: Evolution of the numerical solution in time

For $t < 5$ the shape of the numerical solution is similar to the initial solution. For larger times the numerical solution changes its initial form and transforms into a diverging propagating wave.

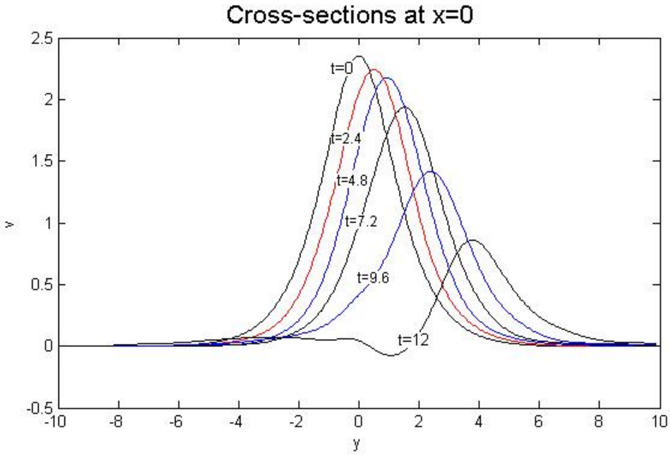


Figure: Cross section $x = 0$ of the solution v with $c = 0.2$ at times $t = 0, 2.4, 4.8, 7.2, 9.6, 12$

Concluding remarks:

- We compare a three level FDS and a four level FDS for BE.
- Both schemes are conservative, i.e. the corresponding discrete energy of the numerical solution is preserved in time.
- Both schemes are second order accurate with respect to space and time steps in the uniform norm and in the first Sobolev norm.
- The numerical algorithm for evaluation of the discrete solution to 3 level FDS needs inner iterations and it is efficient.
- For $\mu = 0$ the numerical algorithm for evaluation of the discrete solution to 4 level FDS is efficient.
- The numerical experiments show good agreement with the theoretical results in 1D and 2D cases.

Thank you for your attention!