Modeling Interactions of Soliton Trains. Effects of External Potentials. Part II

Michail Todorov

Department of Applied Mathematics and Computer Science
Technical University of Sofia, Bulgaria

Work done in collaboration with V. Gerdjikov and A. Kyuldjiev from INRNE-BAS, Sofia

6th International Conference AMiTaNS, Albena, Bulgaria, June 26-July 1, 2014

Outline

- NLS Adiabatic Approximation
- Vector NLS/Manakov System
- Variational Approach
- Perturbed CTC
- Effects of Polarization Vectors
- Lax Representation GCTC
- RTC CTC Asymptotic Regimes
- Conservative Numerical Method vs Variational Approach
- Results and Discussion
- References

NLS Soliton Trains

The idea of the adiabatic approximation to the soliton interactions (Karpman&Solov'ev (1981)) led to effective modeling of the N-soliton trains of the perturbed scalar NLS eq.:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u(x,t) = iR[u]. \qquad (1)$$

By N-soliton train we mean a solution of the NLSE (1) with initial condition

$$u(x, t=0) = \sum_{k=1}^{N} u_k(x, t=0), \qquad (2)$$

$$u_k(x,t) = 2\nu_k \mathrm{e}^{i\phi_k} \mathrm{sech} z_k, \quad z_k = 2\nu_k(x - \xi_k(t)), \quad \xi_k(t) = 2\mu_k t + \xi_{k,0},$$

$$\phi_k = \frac{\mu_k}{\nu_k} z_k + \delta_k(t), \quad \delta_k(t) = 2(\mu_k^2 + \nu_k^2)t + \delta_{k,0}.$$

Here μ_k are the amplitudes, ν_k – the velocities, δ_k – the phase shifts, ξ_k - the centers of solitons.

The adiabatic approximation holds if the soliton parameters satisfy the restrictions

$$|\nu_k - \nu_0| \ll \nu_0, \ |\mu_k - \mu_0| \ll \mu_0, \ |\nu_k - \nu_0||\xi_{k+1,0} - \xi_{k,0}| \gg 1, \quad (3)$$

where ν_0 and μ_0 are the average amplitude and velocity respectively. In fact we have two different scales:

$$|\nu_k - \nu_0| \simeq \varepsilon_0^{1/2}, \quad |\mu_k - \mu_0| \simeq \varepsilon_0^{1/2}, \quad |\xi_{k+1,0} - \xi_{k,0}| \simeq \varepsilon_0^{-1}.$$

In this approximation the dynamics of the N-soliton train is described by a dynamical system for the $4N$ soliton parameters. In our previous works we investigate it in presence of periodic and polynomial potentials. Now, we are interested in what follow perturbation(s) by external sech-potentials:

$$iR[u] \equiv V(x)u(x,t), \qquad V(x) = \sum_s c_s \operatorname{sech}^2(2\nu_0 x - y_s). \quad (4)$$

The latter allows us to realize the idea about localized potential wells(depressions) and humps.

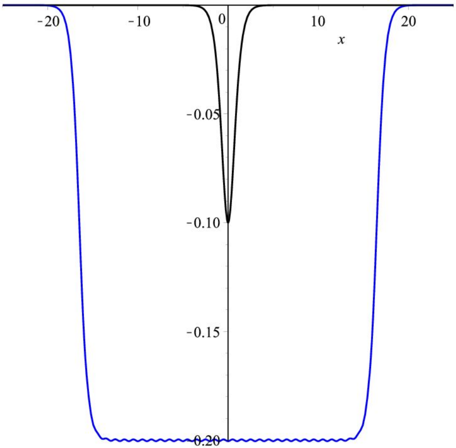

Figure: 1. Single sech-potential vs. composite potential well $V(x) = \sum_{s=0}^{32} c_s \operatorname{sech}^2(x - x_s)$, $c_s = -10^{-1}$, $x_s = -16 + sh$, $h = 1$, $s = 0, ..., 32$.

Perturbed Vector NLS

Here we generalize to the perturbed vector NLS

$$i\vec{u}_t + \frac{1}{2}\vec{u}_{xx} + (\vec{u}^\dagger, \vec{u})\vec{u}(x,t) = iR[\vec{u}]. \qquad (5)$$

The corresponding vector N-soliton train is determined by the initial condition

$$\vec{u}(x, t=0) = \sum_{k=1}^{N} \vec{u}_k(x, t=0), \quad \vec{u}_k(x,t) = 2\nu_k e^{i\phi_k} \operatorname{sech} z_k \vec{n}_k, \qquad (6)$$

and the amplitudes, the velocities, the phase shifts, and the centers of solitons are as in Eq.(2). The phenomenology, however, is enriched by introducing a constant polarization vectors $\vec{n}_k$ that are normalized by the conditions

$$(\vec{n}_k^\dagger, \vec{n}_k) = 1, \qquad \sum_{s=1}^{n} \arg \vec{n}_{k;s} = 0.$$

Variational approach and evolution

We use the variational approach (Anderson and Lisak (1986)) and derive the GCTC model. Like the (unperturbed) CTC, GCTC is a finite dimensional completely integrable model allowing Lax representation.

The Lagrangian of the vector NLS perturbed by external potential is:

$$
\begin{aligned}
\mathcal{L}[\vec{u}] &= \int_{-\infty}^{\infty} dt\, \frac{i}{2} \left[(\vec{u}^\dagger, \vec{u}_t) - (\vec{u}_t^\dagger, \vec{u}) \right] - H, \\
H[\vec{u}] &= \int_{-\infty}^{\infty} dx\, \left[-\frac{1}{2}(\vec{u}_x^\dagger, \vec{u}_x) + \frac{1}{2}(\vec{u}^\dagger, \vec{u})^2 - (\vec{u}^\dagger, \vec{u}) V(x) \right].
\end{aligned} \tag{7}
$$

Then the Lagrange equations of motion:

$$
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \vec{u}_t^\dagger} - \frac{\delta \mathcal{L}}{\delta \vec{u}^\dagger} = 0, \tag{8}
$$

coincide with the vector NLS with external potential $V(x)$.

Variational approach and evolution

Next we insert $\vec{u}(x,t) = \sum_{k=1}^{N} \vec{u}_k(x,t)$ (see eq. (6)) and integrate over x neglecting all terms of order ϵ and higher.
Thus after long calculations we obtain:

$$\mathcal{L} = \sum_{k=1}^{N} \mathcal{L}_k + \sum_{k=1}^{N} \sum_{n=k\pm 1} \widetilde{\mathcal{L}}_{k,n}, \qquad \mathcal{L}_{k,n} = 16\nu_0^3 e^{-\Delta_{k,n}} (R_{k,n} + R_{k,n}^*),$$
$$R_{k,n} = e^{i(\widetilde{\delta}_n - \widetilde{\delta}_k)} (\vec{n}_k^\dagger \vec{n}_n), \qquad \widetilde{\delta}_k = \delta_k - 2\mu_0 \xi_k,$$
$$\Delta_{k,n} = 2 s_{k,n} \nu_0 (\xi_k - \xi_n) \gg 1, \qquad s_{k,k+1} = -1, \qquad s_{k,k-1} = 1. \tag{9}$$

where

$$\mathcal{L}_k = -2i\nu_k \left((\vec{n}_{k,t}^\dagger, \vec{n}_k) - (\vec{n}_k^\dagger, \vec{n}_{k,t}) \right) + 8\mu_k \nu_k \frac{d\xi_k}{dt}$$
$$- 4\nu_k \frac{d\delta_k}{dt} + \dots \tag{10}$$

Variational approach and evolution

The equations of motion are given by:

$$\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta p_{k,t}} - \frac{\delta \mathcal{L}}{\delta p_k} = 0, \qquad (11)$$

where $p_k = \{\delta_k, \xi_k, \mu_k, \nu_k, \vec{n}_k^\dagger\}$.
Let $\lambda_k = \mu_k + i\nu_k, \quad X_k = 2\mu_k \Xi_k + D_k$ and

$$Q_k = -2\nu_0\xi_k + k \ln 4\nu_0^2 - i(\delta_k + \delta_0 + k\pi - 2\mu_0\xi_k). \qquad (12)$$

Then:

$$\frac{d\lambda_k}{dt} = -4\nu_0 \left(e^{Q_{k+1}-Q_k} (\vec{n}^\dagger_{k+1}, \vec{n}_k) - e^{Q_k - Q_{k-1}} (\vec{n}^\dagger_k, \vec{n}_{k-1}) \right) + M_k + iN_k,$$
$$\frac{dQ_k}{dt} = -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \tag{13}$$

Potential's impact

where

$$N_k[u] = \frac{1}{2}\mathfrak{Re}\int_{-\infty}^{\infty} R[u_k]\,\mathrm{sech}\, z_k\, \mathrm{e}^{-i\phi_k}\mathrm{d}z_k,$$

$$M_k[u] = \frac{1}{2}\mathfrak{Im}\int_{-\infty}^{\infty} R[u_k]\tanh z_k\,\mathrm{sech}\, z_k\, \mathrm{e}^{-i\phi_k}\mathrm{d}z_k,$$

$$\Xi_k[u] = \frac{1}{4}\mathfrak{Re}\int_{-\infty}^{\infty} R[u_k] z_k\,\mathrm{sech}\, z_k\, \mathrm{e}^{-i\phi_k}\mathrm{d}z_k,$$

$$D_k[u] = \frac{1}{2\nu_k}\mathfrak{Im}\int_{-\infty}^{\infty} R[u_k](1 - z_k\tanh z_k)\,\mathrm{sech}\, z_k\, \mathrm{e}^{-i\phi_k}\mathrm{d}z_k,$$

Connection with PCTC

So, we have a generalization of CTC:

$$\frac{d\lambda_k}{dt} = -4\nu_0 \left(e^{Q_{k+1}-Q_k}(\vec{n}^\dagger_{k+1}, \vec{n}_k) - e^{Q_k - Q_{k-1}}(\vec{n}^\dagger_k, \vec{n}_{k-1}) \right) + M_k + iN_k,$$
$$\frac{dQ_k}{dt} = -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \qquad \frac{d\vec{n}_k}{dt} = \mathcal{O}(\epsilon), \tag{14}$$

The explicit form of M_k, N_k, Ξ_k and D_k for the potential chosen is given by

$$M_k = \sum_s 2c_s\nu_k P(\Delta_{k,s}), \qquad N_k = 0,$$
$$\Xi_k = 0, \qquad D_k = \sum_s c_s R(\Delta_{k,s}). \tag{15}$$

where $\Delta_{k,s} = 2\nu_0\xi_k - y_s$ and the functions $P(\Delta)$ and $R(\Delta)$ are known explicitly.

Integrals

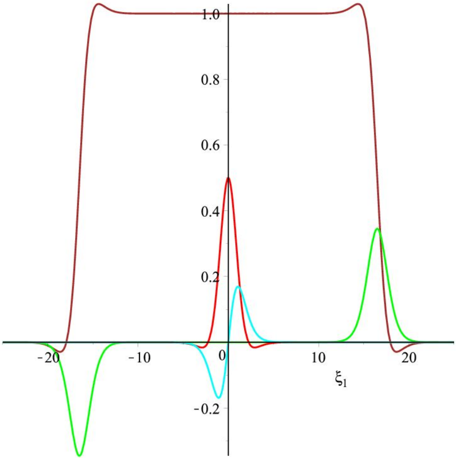

Figure: 2. *P* and *R* functions: for a single sech-potential and for the composite potential.

Integrals

For the potential well

$$V(x, y_s) = c_s[\tanh(2\nu_0 x - y_{\rm i}) - \tanh(2\nu_0 x - y_{\rm f})]$$

functions

$$P(\Delta) = \frac{\sinh\Delta - \Delta\cosh\Delta}{\sinh^3\Delta}$$

and

$$R(\Delta) = \frac{\exp(-3\Delta) + (4\Delta^2 - 1)\cosh\Delta + (3 - 8\Delta)\sinh\Delta}{8\sinh^3\Delta}.$$

Polarization Vectors

Now we have additional equations describing the evolution of the polarization vectors. But note, that their evolution is slow, and in addition the products $(\vec{n}^\dagger_{k+1}, \vec{n}_k)$ multiply the exponents $e^{Q_{k+1}-Q_k}$ which are also of the order of ϵ. Since we are keeping only terms of the order of ϵ we can replace $(\vec{n}^\dagger_{k+1}, \vec{n}_k)$ by their initial values

$$(\vec{n}^\dagger_{k+1}, \vec{n}_k)\Big|_{t=0} = m^2_{0k} e^{2i\phi_{0k}}, \qquad k = 1, \ldots, N-1 \qquad (16)$$

Lax representation of CTC

The CTC is completely integrable model; it allows Lax representation $L_t = [A.L]$, where:

$$
\begin{aligned}
L &= \sum_{s=1}^{N} \left(b_s E_{ss} + a_s (E_{s,s+1} + E_{s+1,s}) \right), \\
A &= \sum_{s=1}^{N} \left(a_s (E_{s,s+1} - E_{s+1,s}) \right),
\end{aligned}
\qquad (17)
$$

where $a_s = \exp((Q_{s+1} - Q_s)/2)$, $b_s = \frac{1}{2}(\mu_{s,t} + i\nu_{s,t})$ and the matrices E_{ks} are determined by $(E_{ks})_{pj} = \delta_{kp}\delta_{sj}$. The eigenvalues of L are integrals of motion and determine the asymptotic velocities.

Lax representation of GCTC

The GCTC is also a completely integrable model because it allows Lax representation $\tilde{L}_t = [\tilde{A}.\tilde{L}]$, where:

$$\begin{aligned} \tilde{L} &= \sum_{s=1}^{N} \left(b_s E_{ss} + \tilde{a}_s (E_{s,s+1} + E_{s+1,s}) \right), \\ \tilde{A} &= \sum_{s=1}^{N} \left(\tilde{a}_s (E_{s,s+1} - E_{s+1,s}) \right), \end{aligned} \tag{18}$$

where $\tilde{a}_s = m_{0k} e^{i\phi_{0k}} a_s$, $b_s = \mu_{s,t} + i\nu_{s,t}$. Like for the scalar case, the eigenvalues of $\tilde{L}$ are integrals of motion. If we denote by $\zeta_s = \kappa_s + i\eta_s$ (resp. $\tilde{\zeta}_s = \tilde{\kappa} + i\tilde{\eta}_s$) the set of eigenvalues of L (resp. $\tilde{L}$) then their real parts κ_s (resp. $\tilde{\kappa}_s$) determine the asymptotic velocities for the soliton train described by CTC (resp. GCTC).

While for the RTC the set of eigenvalues ζ_s of the Lax matrix are all real, for the CTC they generically take complex values, e.g., $\zeta_s = \kappa_s + \mathrm{i}\eta_s$.
Hence, the only possible asymptotic behavior in the RTC is an asymptotically separating, free motion of the solitons. In opposite, for the CTC the real parts $\kappa_s \equiv \mathfrak{Re}\zeta_s$ of eigenvalues of the Lax matrix ζ_s determines the asymptotic velocity of the sth soliton.

Thus, starting from the set of initial soliton parameters we can calculate $L|_{t=0}$ (resp. $\tilde{L}|_{t=0}$), evaluate the real parts of their eigenvalues and thus determine the asymptotic regime of the soliton train.

- Regime (i) $\kappa_k \neq \kappa_j$ ($\tilde{\kappa}_k \neq \tilde{\kappa}_j$) for $k \neq j$ – asymptotically separating, free solitons;
- Regime (ii) $\kappa_1 = \kappa_2 = \cdots = \kappa_N = 0$ ($\tilde{\kappa}_1 = \tilde{\kappa}_2 = \cdots = \tilde{\kappa}_N = 0$) – a "bound state;"
- Regime (iii) group of particles move with the same mean asymptotic velocity and the rest of the particles will have free asymptotic motion.

Varying only the polarization vectors one can change the asymptotic regime of the soliton train.

Problem Formulation: Conservation Laws

Define "mass", M, (pseudo)momentum, P, and energy, E:

$$M \stackrel{\text{def}}{=} \frac{1}{2\beta}\int_{-L_1}^{L_2}(|\psi|^2+|\phi|^2)\mathrm{d}x, \quad P \stackrel{\text{def}}{=} -\int_{-L_1}^{L_2}\mathcal{I}(\psi\bar{\psi}_x+\phi\bar{\phi}_x)\mathrm{d}x,$$

$$E \stackrel{\text{def}}{=} \int_{-L_1}^{L_2}\mathcal{H}\mathrm{d}x, \quad \text{where} \tag{19}$$

$$\mathcal{H} \stackrel{\text{def}}{=} \beta\left(|\psi_x|^2+|\phi_x|^2\right)-\tfrac{1}{2}\alpha_1(|\psi|^4+|\phi|^4) -(\alpha_1+2\alpha_2)\left(|\phi|^2|\psi|^2\right)-2\Gamma[\Re(\bar{\psi}\bar{\phi})]$$

is the Hamiltonian density of the system. Here $-L_1$ and L_2 are the left end and the right end of the interval under consideration. The following conservation/balance laws hold, namely

$$\frac{dM}{dt}=0, \qquad \frac{dP}{dt}=\mathcal{H}\Big|_{x=L_2}-\mathcal{H}\Big|_{x=-L_1}, \qquad \frac{dE}{dt}=0, \tag{20}$$

Numerical Method

To solve the main problem numerically, we use an implicit conservative scheme in complex arithmetic.

$$\mathrm{i}\frac{\psi_i^{n+1}-\psi_i^n}{\tau} = \frac{\beta}{2h^2}\left(\psi_{i-1}^{n+1} - 2\psi_i^{n+1} + \psi_{i+1}^{n+1} + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n\right)$$
$$+ \frac{\psi_i^{n+1}+\psi_i^n}{4}\left[\alpha_1\left(|\psi_i^{n+1}|^2 + |\psi_i^n|^2\right) + (\alpha_1+2\alpha_2)\left(|\phi_i^{n+1}|^2 + |\phi_i^n|^2\right)\right]$$
$$-\tfrac{1}{2}\Gamma\left(\phi_i^{n+1}+\phi_i^n\right),$$

$$\mathrm{i}\frac{\phi_i^{n+1}-\phi_i^n}{\tau} = \frac{\beta}{2h^2}\left(\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1} + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n\right)$$
$$+ \frac{\phi_i^{n+1}+\phi_i^n}{4}\left[\alpha_1\left(|\phi_i^{n+1}|^2 + |\phi_i^n|^2\right) + (\alpha_1+2\alpha_2)\left(|\psi_i^{n+1}|^2 + |\psi_i^n|^2\right)\right]$$
$$-\tfrac{1}{2}\Gamma\left(\psi_i^{n+1}+\psi_i^n\right).$$

Numerical Method: Internal Iterations

$$
\begin{aligned}
\mathrm{i}\frac{\psi_i^{n+1,k+1}-\psi_i^n}{\tau} =& \frac{\beta}{2h^2}\Big(\psi_{i-1}^{n+1,k+1}-2\psi_i^{n+1,k+1}+\psi_{i+1}^{n+1,k+1} \\
&+\psi_{i-1}^n-2\psi_i^n+\psi_{i+1}^n\Big) \\
+\frac{\psi_i^{n+1,k}+\psi_i^n}{4}&\Big[\alpha_1\big(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}|+|\psi_i^n|^2\big) \\
&+(\alpha_1+2\alpha_2)\big(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}|+|\phi_i^n|^2\big)\Big] \\
\mathrm{i}\frac{\phi_i^{n+1,k+1}-\phi_i^n}{\tau} =& \frac{\beta}{2h^2}\big(\phi_{i-1}^{n+1,k+1}-2\phi_i^{n+1,k+1}+\phi_{i+1}^{n+1,k+1} \\
&+\phi_{i-1}^n-2\phi_i^n+\phi_{i+1}^n\big) \\
+\frac{\phi_i^{n+1,k}+\phi_i^n}{4}&\Big[\alpha_1\big(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}|+|\phi_i^n|^2\big) \\
&+(\alpha_1+2\alpha_2)\big(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}|+|\psi_i^n|^2\big)\Big].
\end{aligned}
$$

Numerical Method: Conservation Properties

It is not only convergent (consistent and stable), but also conserves mass and energy, i.e., there exist discrete analogs for (20), which arise from the scheme.

$$M^n = \sum_{i=2}^{N-1} \left(|\psi_i^n|^2 + |\phi_i^n|^2 \right) = \mathrm{const},$$

$$E^n = \sum_{i=2}^{N-1} \frac{-\beta}{2h^2} \left(|\psi_{i+1}^n - \psi_i{}^n|^2 + |\phi_{i+1}^n - \phi_i{}^n|^2 \right) + \frac{\alpha_1}{4} \left(|\psi_i^n|^4 + |\phi_i^n|^4 \right)$$
$$+ \tfrac{1}{2}(\alpha_1 + 2\alpha_2) \left(|\psi_i^n|^2 |\phi_i^n|^2 \right) - \Gamma \Re[\bar{\phi}_i^n \psi_i^n] = \mathrm{const},$$
$$\text{for} \quad \text{all} \quad n \geq 0.$$

These values are kept constant during the time stepping. The above scheme is of Crank-Nicolson type for the linear terms and we employ internal iterations to achieve implicit approximation of the nonlinear terms, i.e., we use its linearized implementation.

Effects of the external potentials. Numeric checks vs Variational approach

The predictions and validity of the CTC and GCTC are compared and verified with the numerical solutions of the corresponding CNSE using fully implicit difference scheme of Crank-Nicolson type, which conserves the energy, the mass, and the pseudomomentum. The scheme is implemented in a complex arithmetics. Such comparison is conducted for all dynamical regimes considered.

- First we study the soliton interaction of the pure Manakov model (without perturbations, $V(x) \equiv 0$) and with vanishing cross-modulation $\alpha_2 = 0$;
- 2- and 3-soliton configurations and transitions between different asymptotic regimes under the effect of well- and hump-like external potential.

Three-soliton configuration in free asymptotic regime corresponding to real parts of eigenvalues of the Lax pair $\mathrm{Re}\zeta_1 = -0.0116$, $\mathrm{Re}\zeta_2 = 0$, $\mathrm{Re}\zeta_3 = 0.0116$

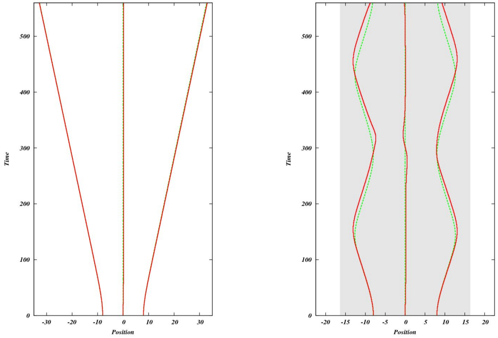

Figure: 3. Free potential behavior (left); External potential well $V(x) = \sum_{s=0}^{32} c_s \mathrm{sech}^2(x - x_s)$, $c_s = -10^{-1}$, $x_s = -16 + sh$, $h = 1$, $s = 0, \ldots, 32$(right).

Three-soliton configuration in bound state regime corresponding to real parts of eigenvalues of the Lax pair $\mathrm{Re}\zeta_1 = \mathrm{Re}\zeta_2 = \mathrm{Re}\zeta_3 = 0$

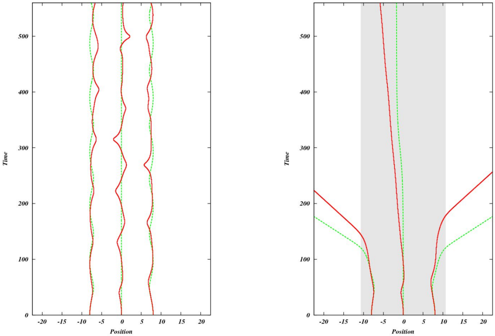

Figure: 4. Free potential behavior (left); External potential hump $V(x) = \sum_{s=0}^{12} c_s \mathrm{sech}^2(x - x_s)$, $c_s = 10^{-2}$, $x_s = -10 + sh$, $h = 5/3$, $s = 0, ..., 12$ (right).

Three-soliton configuration in mixed asymptotic regime corresponding to real parts of eigenvalues of the Lax pair $\mathrm{Re}\zeta_1 = \mathrm{Re}\zeta_2 = -0.00321$, $\mathrm{Re}\zeta_3 = 0.00642$

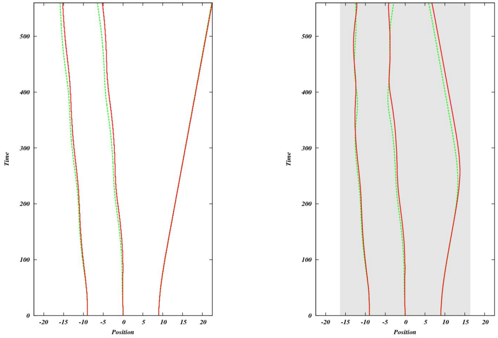

Figure: 5. Free potential behavior (left); External potential well $V(x) = \sum_{s=0}^{32} c_s \mathrm{sech}^2(x - x_s)$, $c_s = -10^{-2}$, $x_s = -16 + sh$, $h = 1$, $s = 0, \ldots, 32$.

Nine-soliton asymptotic regime: $\xi_k = -45 + 9k$, $\nu_k = 0.4625 + 0.0075(k-1)$, $\mu_k = 0$, $\delta_k = k\pi$, $k = 1, \ldots, 9$, $\theta_{k+1} = \theta_k - \frac{\pi}{10}$, $k = 1, \ldots, 8$, $\theta_1 = \frac{9\pi}{10}$

Calculated eigenvalues of the potentialfree Lax pair are

$$\zeta_1 = -0.005720 + 0.239562\mathrm{i}$$
$$\zeta_2 = 0.005720 + 0.239562\mathrm{i}$$
$$\zeta_3 = -0.001564 + 0.245551\mathrm{i}$$
$$\zeta_4 = 0.001564 + 0.245551\mathrm{i}$$
$$\zeta_5 = -0.005720 + 0.260438\mathrm{i}$$
$$\zeta_6 = -2.939394 \times 10^{-19} + 0.250000\mathrm{i}$$
$$\zeta_7 = -0.001564 + 0.254449\mathrm{i}$$
$$\zeta_8 = 0.001564 + 0.254449\mathrm{i}$$
$$\zeta_9 = 0.005720 + 0.260438\mathrm{i}$$

Nine-soliton configuration

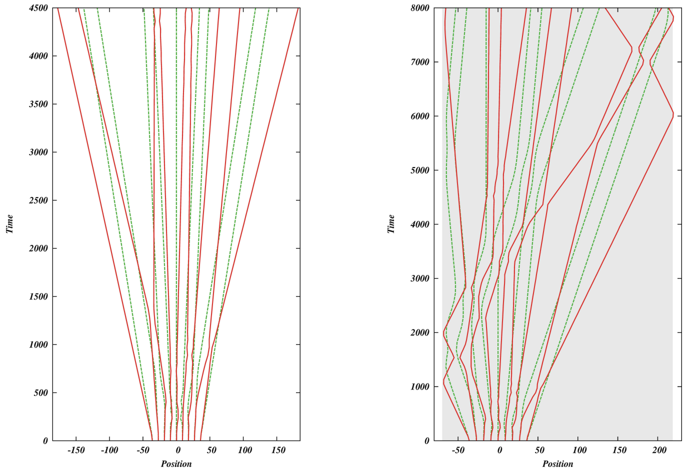

Figure: 6. Free potential behavior (left); External potential well $V(x) = c_s[\tanh(2\nu_0 x - y_\mathrm{i}) - \tanh(2\nu_0 x - y_\mathrm{f})]$, $c_s = -0.004$, $y_\mathrm{i} = -69.5$, $y_\mathrm{f} = 218.5$ (right).

Nine-soliton asymptotic regime: $\xi_k = -40 + 8k$, $\nu_k = 0.4625 + 0.0075(k-1)$, $\mu_k = 0$, $\delta_k = k\pi$, $k = 1, \dots, 9$, $\theta_{k+1} = \theta_k - \frac{\pi}{10}$, $k = 1, \dots, 8$, $\theta_1 = \frac{9\pi}{10}$

Calculated eigenvalues of the potentialfree Lax pair are

$$\zeta_1 = -0.011877 + 0.241165\mathrm{i}$$
$$\zeta_2 = 0.011877 + 0.241165\mathrm{i}$$
$$\zeta_3 = -0.011877 + 0.258835\mathrm{i}$$
$$\zeta_4 = -0.006926 + 0.248312\mathrm{i}$$
$$\zeta_5 = -0.006926 + 0.251688\mathrm{i}$$
$$\zeta_6 = -8.679330 \times 10^{-20} + 0.250000\mathrm{i}$$
$$\zeta_7 = 0.006926 + 0.248312\mathrm{i}$$
$$\zeta_8 = 0.006926 + 0.251688\mathrm{i}$$
$$\zeta_9 = 0.011877 + 0.258835\mathrm{i}$$

Nine-soliton configuration

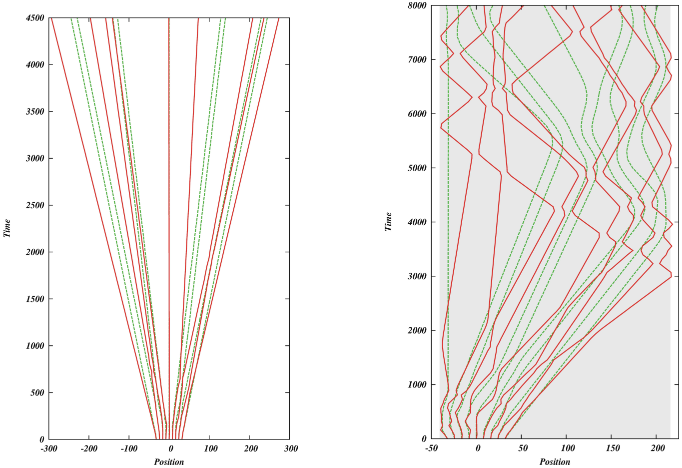

Figure: 7. Free potential behavior (left); External potential well $V(x) = c_s[\tanh(2\nu_0 x - y_\mathrm{i}) - \tanh(2\nu_0 x - y_\mathrm{f})]$, $c_s = -0.004$, $y_\mathrm{i} = -40.5$, $y_\mathrm{f} = 215.5$ (right).

Nine-soliton configuration

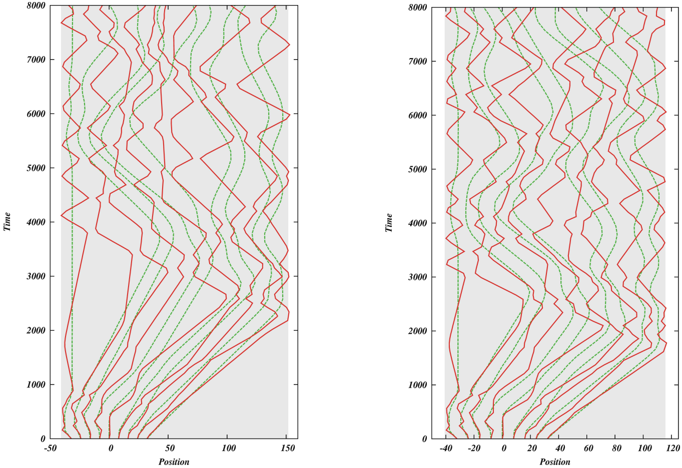

Figure: 8. External potential well $V(x) = c_s[\tanh(2\nu_0 x - y_{\rm i}) - \tanh(2\nu_0 x - y_{\rm f})]$, $c_s = -0.004$: $y_{\rm i} = -40.5$, $y_{\rm f} = 151.5$ (left); $y_{\rm i} = -40.5$, $y_{\rm f} = 115.5$ (right).

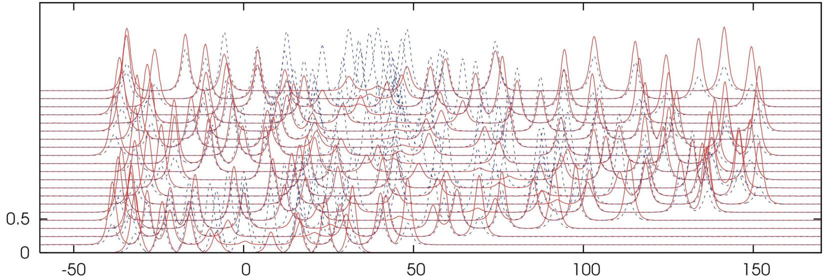

Figure: 9. External potential well
$V(x) = c_s[\tanh(2\nu_0 x - y_{\rm i}) - \tanh(2\nu_0 x - y_{\rm f})]$, $c_s = -0.004$: $y_{\rm i} = -40.5$, $y_{\rm f} = 151.5$.

Bibliography

J. Moser. Finitely many mass points on the line under the influence of an exponential potential – an integrable system. in *Dynamical Systems, Theory and Applications*. Lecture Notes in Physics, vol. 38, Springer, 1975, p.467–497.

V.I. Karpman, V.V. Solov'ev. A perturbational approach to the two-soliton systems. *Physica D* 3:487–502, 1981.

T. R. Taha and M. J. Ablowitz. Analytical and numerical aspects of certain nonlinear evolution equations. ii. numerical, Schrödinger equation. *J. Comp. Phys.*, 55:203–230, 1984.

C. I. Christov, S. Dost, and G. A. Maugin. Inelasticity of soliton collisions in system of coupled NLS equations. *Physica Scripta*, 50:449–454, 1994.

V.S. Gerdjikov, E.G. Dyankov, D.J. Kaup, G.L. Diankov, I.M. Uzunov. Stability and quasi-equidistant propagation of NLS soliton trains. *Physics Letters A* 241:323–328, 1998.

Bibliography

V. S. Gerdjikov. *N*-soliton interactions, the Complex Toda Chain and stability of NLS soliton trains. In E. Kriezis (Ed), Proceedings of the International Symposium on Electromagnetic Theory, vol. 1, pp. 307–309, Aristotle University of Thessaloniki, Greece, 1998.

V.S. Gerdjikov, E.V. Doktorov, N.P. Matsuka. *N*-soliton train and Generalized Complex Toda Chain for Manakov System. *Theor. Math. Phys.* 151:762–773, 2007.

M. D. Todorov and C. I. Christov. Conservative scheme in complex arithmetic for coupled nonlinear Schrödinger equations. *Discrete and Continuous Dynamical Systems*, Supplement:982–992, 2007

M. D. Todorov and C. I. Christov. Impact of the Large Cross-Modulation Parameter on the Collision Dynamics of Quasi-Particles Governed by Vector NLSE. *Journal of Mathematics and Computers in Simulation*, 80: 46–55, 2009.

Bibliography

R. Goodman. Hamiltonian Hopf bifurcations and dynamics of NLS/GP standing-wave modes. *J. Phys. A: Math. Theor.*, 44: 425101 (28pp), 2011.

V.S. Gerdjikov and M.D. Todorov. On the Effects of sech-like Potentials on Manakov Solitons. AIP Conf. Proc. **1561**, pp. 75–83, 2013, Melville, NY, http://dx.doi.org/10.1063/1.4827216.

V.S. Gerdjikov and M.D. Todorov. *N*-soliton interactions for the Manakov system. Effects of external potentials. In: R. Carretero-Gonzalez et al. (eds.), Localized Excitations in Nonlinear Complex Systems, Nonlinear Systems and Complexity **7**, 147–169, 2014. DOI 10.1007/978-3-319-02057-0__7, Springer International Publishing Switzerland.

Bibliography

V.S. Gerdjikov, M.D. Todorov,and A.V. Kyuldjiev. Asymptotic Behavior of Manakov Solitons: Effects of Potential Wells and Humps. submitted to *Math. Comp. Simul.*, MATCOM-D-13-00465, 2014.

Thanks

Thank you for your attention!