# Multisoliton Interaction of Perturbed Manakov System

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# Outline

- Idea of Adiabatic Approximation
- Variational Approach and Perturbed CTC
- Non-perturbed and Perturbed CNSE. Manakov System.
- Choice of Initial Conditions and Potential Perturbations
- Effects of Polarization Vectors
- Conservative Numerical Method vs Variational Approach
- Main Results and Discussion
- References

# Idea of Adiabatic Approximation

The idea of the adiabatic approximation to the soliton interactions (Karpman&Solov'ev (1981)) led to effective modeling of the $N$-soliton trains of the perturbed scalar NLS eq.:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u(x,t) = iR[u]. \qquad (1)$$

By $N$-soliton train we mean a solution of the NLSE (1) with initial condition

$$u(x, t=0) = \sum_{k=1}^{N} u_k(x, t=0), \qquad (2)$$

$$u_k(x,t) = 2\nu_k \mathrm{e}^{i\phi_k} \operatorname{sech} z_k, \quad z_k = 2\nu_k(x - \xi_k(t)), \quad \xi_k(t) = 2\mu_k t + \xi_{k,0},$$

$$\phi_k = \frac{\mu_k}{\nu_k} z_k + \delta_k(t), \quad \delta_k(t) = 2(\mu_k^2 + \nu_k^2)t + \delta_{k,0}.$$

Here $\mu_k$ are the amplitudes, $\nu_k$ – the velocities, $\delta_k$ – the phase shifts, $\xi_k$ - the centers of solitons.

# Idea of Adiabatic Approximation

The adiabatic approximation holds if the soliton parameters satisfy the restrictions

$$|\nu_k - \nu_0| \ll \nu_0, \; |\mu_k - \mu_0| \ll \mu_0, \; |\nu_k - \nu_0||\xi_{k+1,0} - \xi_{k,0}| \gg 1, \quad (3)$$

where $\nu_0$ and $\mu_0$ are the average amplitude and velocity respectively. In fact we have two different scales:

$$|\nu_k - \nu_0| \simeq \varepsilon_0^{1/2}, \quad |\mu_k - \mu_0| \simeq \varepsilon_0^{1/2}, \quad |\xi_{k+1,0} - \xi_{k,0}| \simeq \varepsilon_0^{-1}.$$

In this approximation the dynamics of the $N$-soliton train is described by a dynamical system for the $4N$ soliton parameters. In our previous works we investigate it in presence of periodic and polynomial potentials. Now, we are interested in what follow perturbation(s) by external sech-potentials:

$$iR[u] \equiv V(x)u(x,t), \qquad V(x) = \sum_s c_s \mathrm{sech}^2(2\nu_0 x - y_s). \quad (4)$$

The latter allows us to realize the idea about located potential wells(depressions) and humps.

# Potential Perturbations

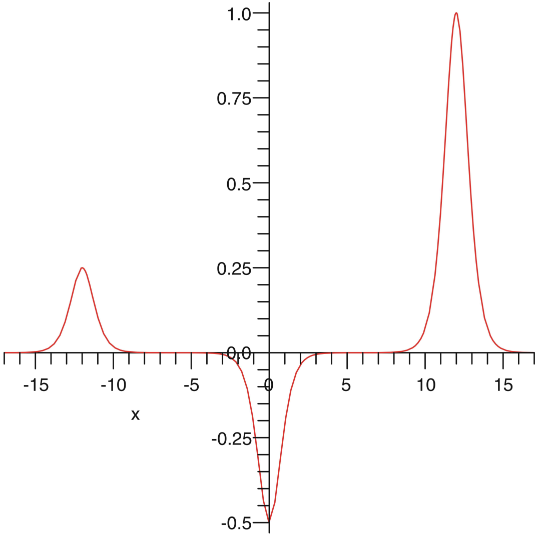


Figure: 1. Sketch of potential sech-like wells and humps used.

# Perturbed Vector Complex Toda Chain

In the present paper we generalize the above results to the perturbed vector NLS

$$i\vec{u}_t + \frac{1}{2}\vec{u}_{xx} + (\vec{u}^\dagger, \vec{u})\vec{u}(x,t) = iR[\vec{u}]. \quad (5)$$

The corresponding vector $N$-soliton train is determined by the initial condition

$$\vec{u}(x, t=0) = \sum_{k=1}^{N} \vec{u}_k(x, t=0), \quad \vec{u}_k(x,t) = 2\nu_k e^{i\phi_k} \mathrm{sech} z_k \vec{n}_k, \quad (6)$$

and the amplitudes, the velocities, the phase shifts, and the centers of solitons are as in Eq.(2). The phenomenology, however, is enriched by introducing a constant polarization vectors $\vec{n}_k$ that are normalized by the conditions

$$(\vec{n}_k^\dagger, \vec{n}_k) = 1, \qquad \sum_{s=1}^{n} \arg \vec{n}_{k;s} = 0.$$

More precisely after involving these vectors we derive a generalized version of the CTC (GCTC) model, which allows to have in mind the polarization effects in the $N$-soliton train of the vector NLS.

# Perturbed Vector Complex Toda Chain Model. Initial Conditions

The corresponding model is known as the perturbed CTC model which can be written down in the form

$$\begin{aligned} \frac{d\lambda_k}{dt} &= -4\nu_0 \left( e^{Q_{k+1}-Q_k} - e^{Q_k - Q_{k-1}} \right) + M_k + iN_k, \\ \frac{dQ_k}{dt} &= -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \end{aligned} \tag{7}$$

where $\lambda_k = \mu_k + i\nu_k$ and $X_k = 2\mu_k\Xi_k + D_k$ and

$$\begin{aligned} &Q_k = -2\nu_0\xi_k + k\ln 4\nu_0^2 - i(\delta_k + \delta_0 + k\pi - 2\mu_0\xi_k), \\ &\nu_0 = \frac{1}{N}\sum_{s=1}^{N}\nu_s, \qquad \mu_0 = \frac{1}{N}\sum_{s=1}^{N}\mu_s, \qquad \delta_0 = \frac{1}{N}\sum_{s=1}^{N}\delta_s. \end{aligned} \tag{8}$$

# Variational approach and PCTC

We use the variational approach (Anderson and Lisak (1986)) and derive the GCTC model. Like the (unperturbed) CTC, GCTC is a finite dimensional completely integrable model allowing Lax representation.

The Lagrangian of the vector NLS perturbed by external potential is:

$$\begin{aligned} \mathcal{L}[\vec{u}] &= \int_{-\infty}^{\infty} dt \, \frac{i}{2} \left[ (\vec{u}^\dagger, \vec{u}_t) - (\vec{u}_t^\dagger, \vec{u}) \right] - H, \\ H[\vec{u}] &= \int_{-\infty}^{\infty} dx \, \left[ -\frac{1}{2} (\vec{u}_x^\dagger, \vec{u}_x) + \frac{1}{2} (\vec{u}^\dagger, \vec{u})^2 - (\vec{u}^\dagger, \vec{u}) V(x) \right]. \end{aligned} \tag{9}$$

Then the Lagrange equations of motion:

$$\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \vec{u}_t^\dagger} - \frac{\delta \mathcal{L}}{\delta \vec{u}^\dagger} = 0, \tag{10}$$

coincide with the vector NLS with external potential $V(x)$.

Next we insert $\vec{u}(x,t) = \sum_{k=1}^{N} \vec{u}_k(x,t)$ (see eq. (6)) and integrate over $x$ neglecting all terms of order $\epsilon$ and higher.
Thus after long calculations we obtain:

$$\mathcal{L} = \sum_{k=1}^{N} \mathcal{L}_k + \sum_{k=1}^{N} \sum_{n=k\pm 1} \widetilde{\mathcal{L}}_{k,n}, \qquad \mathcal{L}_{k,n} = 16\nu_0^3 e^{-\Delta_{k,n}} (R_{k,n} + R_{k,n}^*),$$
$$R_{k,n} = e^{i(\widetilde{\delta}_n - \widetilde{\delta}_k)} (\vec{n}_k^\dagger \vec{n}_n), \qquad \widetilde{\delta}_k = \delta_k - 2\mu_0 \xi_k,$$
$$\Delta_{k,n} = 2 s_{k,n} \nu_0 (\xi_k - \xi_n) \gg 1, \qquad s_{k,k+1} = -1, \qquad s_{k,k-1} = 1. \tag{11}$$

where

$$\mathcal{L}_k = -2i\nu_k \left( (\vec{n}_{k,t}^\dagger, \vec{n}_k) - (\vec{n}_k^\dagger, \vec{n}_{k,t}) \right) + 8\mu_k \nu_k \frac{d\xi_k}{dt} - 4\nu_k \frac{d\delta_k}{dt} + \dots \tag{12}$$

# Variational approach and PCTC

The equations of motion are given by:

$$\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta p_{k,t}} - \frac{\delta \mathcal{L}}{\delta p_k} = 0, \tag{13}$$

where $p_k = \{\delta_k, \xi_k, \mu_k, \nu_k, \vec{n}_k^{\dagger}\}$.

$$\begin{aligned} \frac{\partial \nu_k}{\partial t} &= N[u_k], & \frac{\partial \mu_k}{\partial t} &= M[u_k], \\ \frac{\partial \xi_k}{\partial t} &= -\frac{1}{2\nu_k}\mathfrak{Im} h(\zeta) + \Xi[u_k], & \frac{\partial \delta_k}{\partial t} &= 2\mu_k \frac{\partial \xi_k}{\partial t} + \mathfrak{Re} h(\zeta) + D[u_k], \end{aligned} \tag{14}$$

where $h(\zeta) = -2\zeta^2$ and

$$N_k[u] = \frac{1}{2}\mathfrak{Re}\int_{-\infty}^{\infty} R[u_k]\,\mathrm{sech}\, z_k\, \mathrm{e}^{-i\phi_k} \mathrm{d}z_k,$$

$$M_k[u] = \frac{1}{2}\mathfrak{Im}\int_{-\infty}^{\infty} R[u_k]\,\tanh z_k\,\mathrm{sech}\, z_k\, \mathrm{e}^{-i\phi_k} \mathrm{d}z_k,$$

$$\Xi_k[u] = \frac{1}{4}\mathfrak{Re}\int_{-\infty}^{\infty} R[u_k] z_k\,\mathrm{sech}\, z_k\, \mathrm{e}^{-i\phi_k} \mathrm{d}z_k,$$

$$D_k[u] = \frac{1}{2\nu_k}\mathfrak{Im}\int_{-\infty}^{\infty} R[u_k](1 - z_k \tanh z_k)\,\mathrm{sech}\, z_k\, \mathrm{e}^{-i\phi_k} \mathrm{d}z_k,$$

The corresponding system is a generalization of CTC:

$$\frac{d\lambda_k}{dt} = -4\nu_0 \left( e^{Q_{k+1}-Q_k} (\vec{n}^\dagger_{k+1}, \vec{n}_k) - e^{Q_k - Q_{k-1}} (\vec{n}^\dagger_k, \vec{n}_{k-1}) \right) + M_k + iN_k,$$

$$\frac{dQ_k}{dt} = -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \qquad \frac{d\vec{n}_k}{dt} = \mathcal{O}(\epsilon), \tag{15}$$

where again $\lambda_k = \mu_k + i\nu_k$ and the other variables are given by (8). The explicit form of $M_k$, $N_k$, $\Xi_k$ and $D_k$ is given by

$$M_k = \sum_s 2c_s\nu_k P(\Delta_{k,s}), \qquad N_k = 0,$$

$$\Xi_k = 0, \qquad D_k = \sum_s c_s R(\Delta_{k,s}). \tag{16}$$

where $\Delta_{k,s} = 2\nu_0\xi_k - y_s$ and the functions $P(\Delta)$ and $R(\Delta)$ are known explicitly.

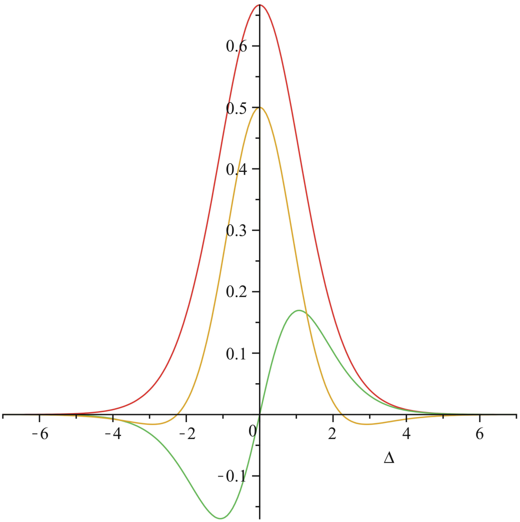


Figure: 2. The integrals $N(\Delta)$, $R(\Delta)$, and $P(\Delta)$.

Now we have additional equations describing the evolution of the polarization vectors. But note, that their evolution is slow, and in addition the products $(\vec{n}^{\dagger}_{k+1}, \vec{n}_k)$ multiply the exponents $e^{Q_{k+1}-Q_k}$ which are also of the order of $\epsilon$. Since we are keeping only terms of the order of $\epsilon$ we can replace $(\vec{n}^{\dagger}_{k+1}, \vec{n}_k)$ by their initial values

$$(\vec{n}^{\dagger}_{k+1}, \vec{n}_k)\Big|_{t=0} = m_{0k}^2 e^{2i\phi_{0k}}, \qquad k = 1, \dots, N-1 \qquad (17)$$

We formulate a condition on $\vec{n}_s$ that is compatible with the adiabatic approximation. We also formulate the conditions on the initial vector soliton parameters responsible for the different asymptotic regimes.
The CTC is completely integrable model; it allows Lax representation $L_t = [A.L]$, where:

$$\begin{aligned} L &= \sum_{s=1}^{N} \left( b_s E_{ss} + a_s (E_{s,s+1} + E_{s+1,s}) \right), \\ A &= \sum_{s=1}^{N} \left( a_s (E_{s,s+1} - E_{s+1,s}) \right), \end{aligned} \tag{18}$$

where $a_s = \exp((Q_{s+1} - Q_s)/2)$, $b_s = \mu_{s,t} + i\nu_{s,t}$ and the matrices $E_{ks}$ are determined by $(E_{ks})_{pj} = \delta_{kp}\delta_{sj}$. The eigenvalues of $L$ are integrals of motion and determine the asymptotic velocities.

# Effects of the polarization vectors on the soliton interaction

The GCTC is also a completely integrable model because it allows Lax representation $\tilde{L}_t = [\tilde{A}.\tilde{L}]$, where:

$$\begin{aligned} \tilde{L} &= \sum_{s=1}^{N} \left( \tilde{b}_s E_{ss} + \tilde{a}_s (E_{s,s+1} + E_{s+1,s}) \right), \\ \tilde{A} &= \sum_{s=1}^{N} \left( \tilde{a}_s (E_{s,s+1} - E_{s+1,s}) \right), \end{aligned} \tag{19}$$

where $\tilde{a}_s = m_{0k}^2 e^{2i\phi_{0k}} a_s$, $b_s = \mu_{s,t} + i\nu_{s,t}$. Like for the scalar case, the eigenvalues of $\tilde{L}$ are integrals of motion. If we denote by $\zeta_s = \kappa_s + i\eta_s$ (resp. $\tilde{\zeta}_s = \tilde{\kappa} + i\tilde{\eta}_s$) the set of eigenvalues of $L$ (resp. $\tilde{L}$) then their real parts $\kappa_s$ (resp. $\tilde{\kappa}_s$) determine the asymptotic velocities for the soliton train described by CTC (resp. GCTC).

# RTC and CTC. Asymptotic regimes

While for the RTC the set of eigenvalues $\zeta_s$ of the Lax matrix are all real, for the CTC they generically take complex values, e.g., $\zeta_s = \kappa_s + \mathrm{i}\eta_s$.
Hence, the only possible asymptotic behavior in the RTC is an asymptotically separating, free motion of the solitons. In opposite, for the CTC the real parts $\kappa_s \equiv \mathfrak{Re}\zeta_s$ of eigenvalues of the Lax matrix $\zeta_s$ determines the asymptotic velocity of the $s$th soliton.

Thus, starting from the set of initial soliton parameters we can calculate $L|_{t=0}$ (resp. $\tilde{L}|_{t=0}$), evaluate the real parts of their eigenvalues and thus determine the asymptotic regime of the soliton train.

**Regime (i)** $\kappa_k \neq \kappa_j$ $(\tilde{\kappa}_k \neq \tilde{\kappa}_j)$ for $k \neq j$ – asymptotically separating, free solitons;

**Regime (ii)** $\kappa_1 = \kappa_2 = \cdots = \kappa_N = 0$
$(\tilde{\kappa}_1 = \tilde{\kappa}_2 = \cdots = \tilde{\kappa}_N = 0)$ – a "bound state;"

**Regime (iii)** group of particles move with the same mean asymptotic velocity and the rest of the particles will have free asymptotic motion.

Varying only the polarization vectors one can change the asymptotic regime of the soliton train.

# Effects of the external potentials on the GCTC. Numeric checks vs Variational approach

The predictions and validity of the CTC and GCTC are compared and verified with the numerical solutions of the corresponding CNSE using fully explicit difference scheme of Crank-Nicolson type, which conserves the energy, the mass, and the pseudomomentum. The scheme is implemented in a complex arithmetics. Such comparison is conducted for all dynamical regimes considered.

- First we study the soliton interaction of the pure Manakov model (without perturbations, $V(x) \equiv 0$) and with vanishing cross-modulation $\alpha_2 = 0$;
- 2-soliton configurations and transitions between different asymptotic regimes;
- 3-soliton configurations and transitions between different asymptotic regimes;
- 2- and 3-soliton configurations and transitions under the effect of well- and hump-like external potential.

# Two-soliton configurations and transitions between different asymptotic regimes: free asymptotic regime

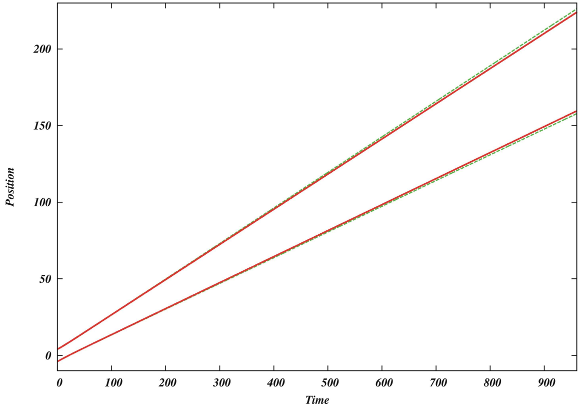


Figure: 3. $\Delta\nu = |\nu_2 - \nu_1| < \nu_{\rm cr} = 0.01786$.
$\nu_{\rm cr} = 2\sqrt{2\cos(\theta_1 - \theta_2)} n u_0 \exp\left(-u_0 r_0\right)$, $\mu_{k0} = 0.1$, $\nu_{10} = 0.49$, $\nu_{20} = 0.51$, $\xi_{10,20} = \pm 4$, $\delta_{10} = 0$, $\delta_{20} = \pi + 2\mu_0 r_0$, $\theta_{10} = 2\pi/10$, $\theta_{20} = \theta_{10} - \pi/10$.

# Three-soliton configurations in mixed asymptotic regimes: two-bound state + free soliton

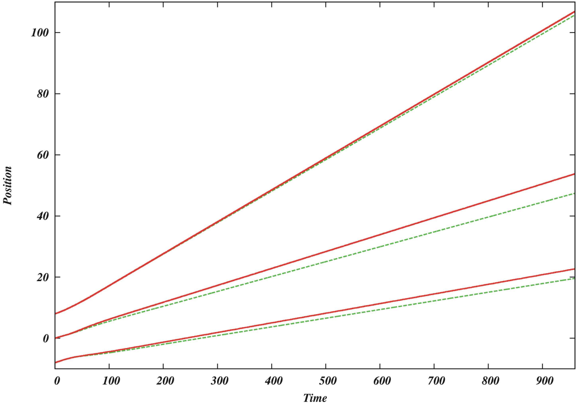


Figure: 4. $\Delta\nu = 0.01 < \nu_{\rm cr}$. $\nu_{\rm cr} = 2\sqrt{2\cos(\theta_1 - \theta_2)} nu_0 \exp\left(-nu_0 r_0\right)$, $\mu_{k0} = 0.03$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{k0} = 0$, $\delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$.

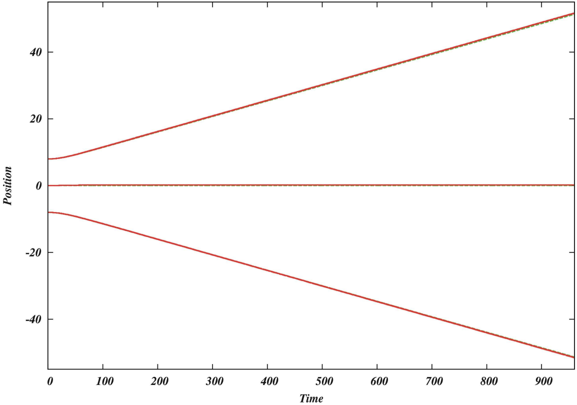


Figure: 5. $\Delta\nu = 0.01 < \nu_{\mathrm{cr}}$. $\nu_{\mathrm{cr}} = 2\sqrt{2\cos(\theta_1 - \theta_2)} nu_0 \exp\left(-nu_0 r_0\right)$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{k0} = 0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$.

# Three-soliton configurations and transitions in mixed regimes – two-soliton bound state + free soliton

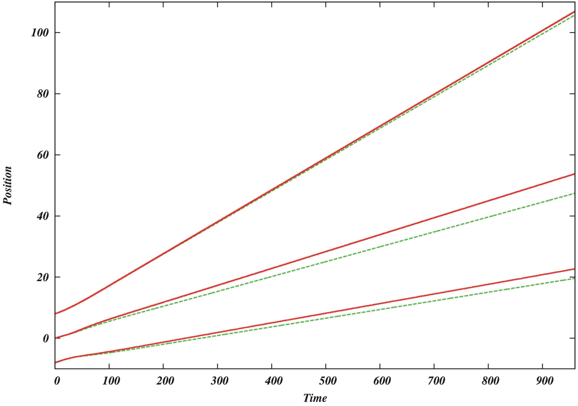


Figure: 6. $\Delta\nu = 0.01 < \nu_{\mathrm{cr}} = 0.02526$.
$\nu_{\mathrm{cr}} = 2\sqrt{2\cos(\theta_1 - \theta_2)} nu_0 \exp\left(-nu_0 r_0\right)$, $\mu_{k0} = 0.03$, $\nu_{20} = 0.5$,
$\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$,
$\Delta_{20,30} = \pm pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$.

# Effects of the external potentials on the GCTC – 3-soliton configurations

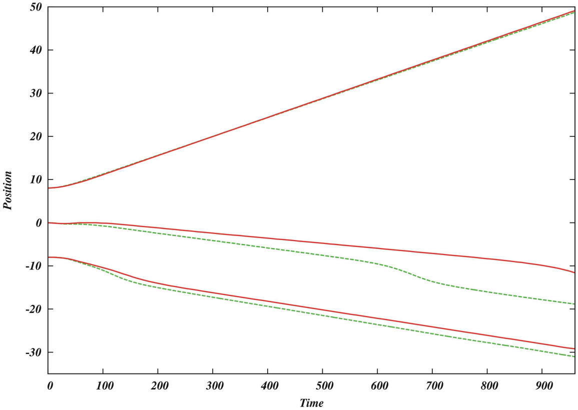


Figure: 7. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\Delta_{20,30} = \pm pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, two potential wells $y_1 = -12$, $y_2 = 4$, $c_s = 0.001$.

# Effects of the external potentials on the GCTC – 3-soliton configurations

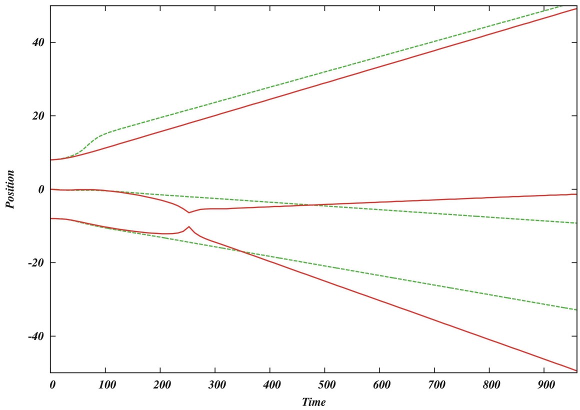


Figure: 8. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\Delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, one potential hump at $y = 12$, $c_s = 0.01$.

# Effects of the external potentials on the GCTC – 3-soliton configurations

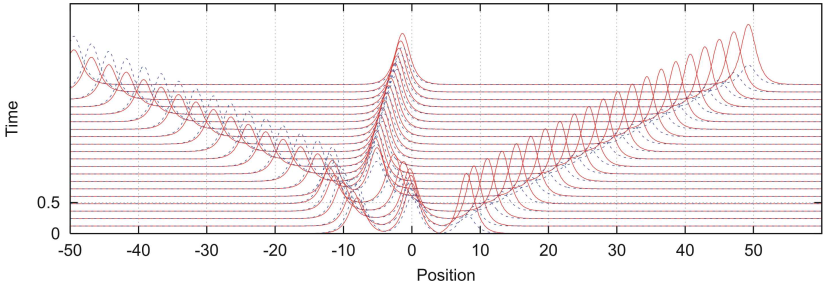


Figure: 9. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\Delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, one potential hump at $y = 12$, $c_s = 0.01$.

# Effects of the external potentials on the GCTC – 3-soliton configurations

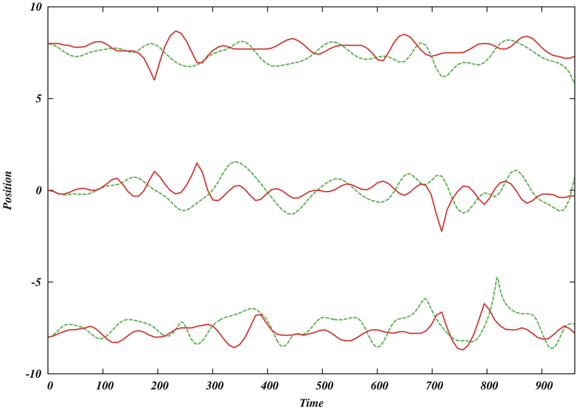


Figure: 10. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\Delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, two potential humps at $y = \pm 12$, $c_s = 0.1$.

# Effects of the external potentials on the GCTC – 3-soliton configurations

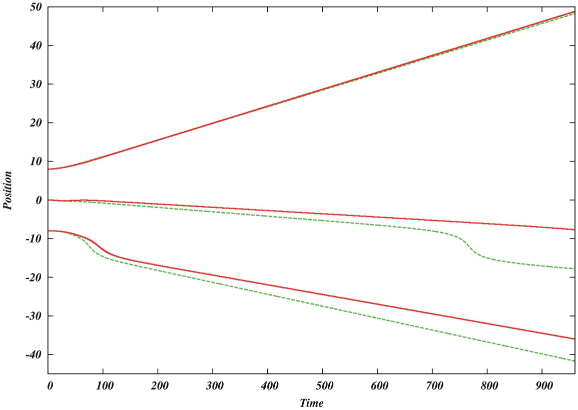


Figure: 11. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\Delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, one potential well at $y = -12$, $c_s = 0.01$.

# Effects of the external potentials on the GCTC – 3-soliton configurations

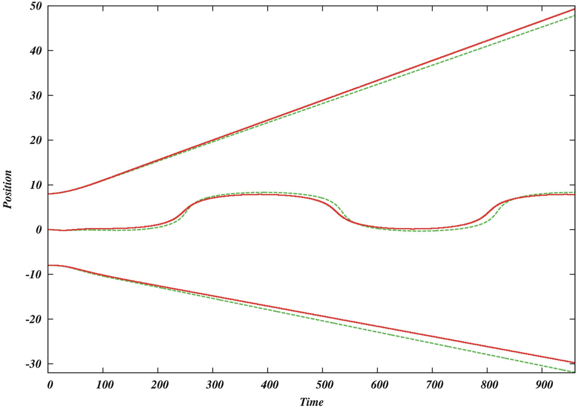


Figure: 12. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\Delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, one potential well at $y = 4$, $c_s = 0.01$.

# Effects of the external potentials on the GCTC – 3-soliton configurations

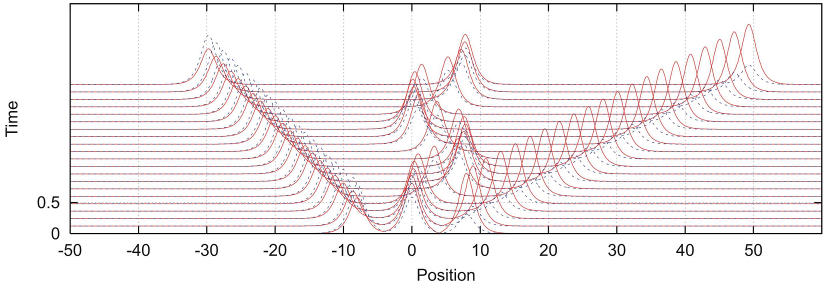


Figure: 13. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\Delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, one potential well at $y = 4$, $c_s = 0.01$.

# Effects of the external potentials on the GCTC. Numeric checks vs Variational approach

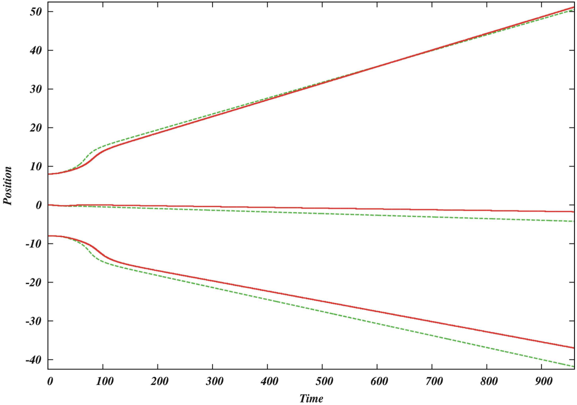


Figure: 14. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\Delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, two potential wells at $y = \pm 12$, $c_s = 0.01$.

# Effects of the external potentials on the GCTC – 2-soliton configurations

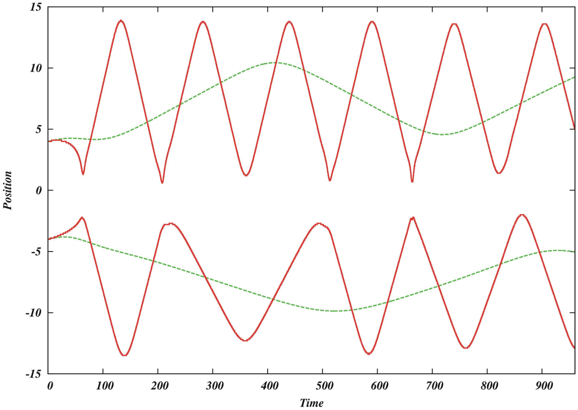


Figure: 15. $\Delta\nu = 0$, $\mu_{k0} = 0.005$, $\nu_{k0} = 0.5$, $\xi_{10,20} = \pm 4$, $\xi_{10,20} = \pm 4$, $\delta_{10} = 0$, $\Delta_{20} = 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{20} = \theta_{10} - \pi/10$, three potential humps at $y = \pm 15$, $y = 0$, $c_s = 0.0485$.

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# Thanks

Thank you for your kind attention !