Numerical Implementation of Fourier Integral-Transform Method for the Wave Equations

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Outline

- Problem Formulation
- Numerical Method
- Numerical Tests and Verification
- Discussion and Further Activity

Problem Formulation. Boussinesq Paradigm Equation

Consider the Boussinesq equation in two dimensions

$$u_{tt} = \Delta(u - \alpha u^2 + \beta_1 u_{tt} - \beta_2 \Delta u) \quad (1)$$

where w is the surface elevation, $\beta_1, \beta_2 > 0$ are two dispersion coefficients and α is an amplitude parameter.
The initial conditions can be prepared by a single soliton (computed numerically and semi-analytically) or as a superposition of two solitons.
Possible ways to solve numerically the above problem are

- by using a semi-implicit scheme
- by using a fully implicit difference scheme
- by using Fourier integral-transform method

Instead of using a multigrid solver we can use a 2D Fourier transform. Applying it to the original equation (1) we get

$$[1 + 4\pi\beta_1(\xi^2 + \eta^2)]\hat{u}_{tt} = -4\pi^2(\xi^2 + \eta^2)\hat{u} \\ -16\beta_2\pi^4(\xi^2 + \eta^2)^2\hat{u} + 4\pi^2\alpha(\xi^2 + \eta^2)\hat{N} \qquad (2)$$

where $\hat{u} = \hat{u}(\xi, \eta, t)$ and $\hat{N} := \mathcal{F}[u^2]$.
Solving the last ODE is very easy and requires very few operations per time step for given $\hat{N}$ but the lion's share of the computational resources are consumed by the computation of the convolution integral that represents the Fourier transform of the nonlinear term u^2.

Fourier Integral-Transform Method. Numerical Implementation

We introduce an uniform grid in the Fourier space and discretize the Fourier integral. Suppose that we know $\hat{u}^p$, $\hat{u}^{p-1}$,..., $\hat{u}^0$. Then the next $(p+1)$-st stage is computed from the following three-stage difference scheme

$$[1+4\pi\beta_1(\xi_m^2+\eta_n^2)]\frac{\hat{u}_{mn}^{p+1}-2\hat{u}_{mn}^{p}+\hat{u}_{mn}^{p-1}}{\tau^2}$$
$$=[2\pi^2(\xi_m^2+\eta_n^2)+8\beta_2\pi^4(\xi_m^2+\eta_n^2)^2][\hat{u}_{mn}^{p+1}+\hat{u}_{mn}^{p-1}]$$
$$+4\pi^2\alpha(\xi_m^2+\eta_n^2)\hat{N}_{mn}^p \qquad (3)$$

where $\hat{N}_{mn}^p := \mathcal{DF}[(u_{kl}^p)^2]$ is the DFT of u^2. The problem is that u_{kl}^p is not known. It has to be found from the inverse Fourier transform of the known function $\hat{u}_{mn}^p$, namely $u_{kl}^p := \mathcal{F}^{-1}[\hat{u}_{mn}^p]$.

Fourier Integral-Transform Method. Numerical Implementation

After computing the grid function on the grid in the configurational space one inverses it at each collocation point and then takes the square of it. From the obtained grid function one computes $\hat{N}^p$.

Numerical Test with the 1D string equation and D'Alembert's formula

Let us consider the following Cauchy problem

$$u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty, \qquad t > 0 \tag{4}$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \tag{5}$$

with exact solution $u = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g d\theta$
The respective equation in the Fourier space looks like

$$\hat{u}_{tt} = -c^2 p^2 \hat{u} \tag{6}$$

and the initial conditions

$$\hat{u} = \hat{f}(p) \qquad \hat{u}_t = \hat{g}(p) \tag{7}$$

with exact solution $\hat{u} = \hat{f}(p)\cos cpt + \frac{\hat{g}(p)}{cp}\sin cpt$.
$\mathcal{F}^{-1}[\hat{u}] = u$.

Numerical Test with the 1D string equation and D'Alembert's formula

We solve by usual three-stage explicit scheme (6)-(7)

$$\frac{\hat{u}_{k+1} - 2\hat{u}_k + \hat{u}_{k-1}}{\tau^2} = -c^2 p^2 \hat{u}_k \qquad (8)$$

and set $c = 1$, $f(x) = \exp(-x^2)$, $g(x) = 2x \exp(-x^2)$. The Fourier integral is approximated by Filon's quadrature, which seems to be generalized trapezoidal formula when $\omega h < 1$. When $\omega h > 1$ the error $\sim O(N\omega^{-3} y_{xx})$.

Comparison of the numerical solution with the D'Alembert formula

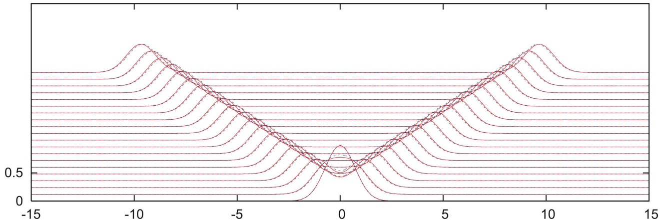

Figure: 1.

Right going single Gaussian pulse

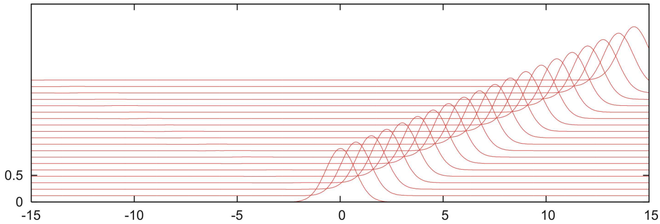

Figure: 2.

Left going single Gaussian pulse

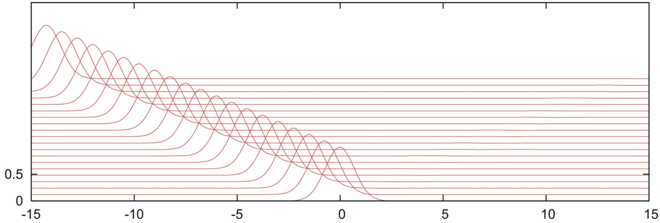

Figure: 3.

Superposition and elastic interaction of two Gaussian pulses

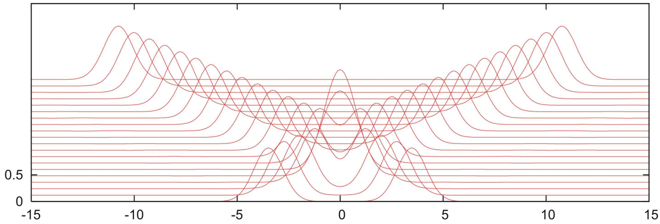

Figure: 4.

D'Alembert solution with sech-like initial condition (10)

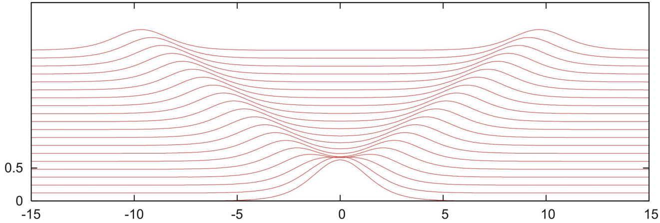

Figure: 5.

Numerical Test with the Regularized Long Wave Equation

In this case the Boussinesq equation has the form

$$u_{tt} = (u - \alpha u^2 + \beta u_{tt})_{xx} \tag{9}$$

and admits the following exact solution

$$u = -\frac{3}{2}\frac{c^2-1}{\alpha}\operatorname{sech}^2\left(\frac{x-ct}{2c}\sqrt{\frac{c^2-1}{\beta}}\right) \tag{10}$$

Skipping the mechanical sense of this equation we transform it in Fourier space together with the initial conditions which we build based on the exact solution in the time moment $t = 0$. The transformed equation

$$\hat{u}_{tt} = -p^2\hat{u} + \alpha p^2\hat{N} - \beta p^2\hat{u}_{tt} \tag{11}$$

is discretized by usual three-stage difference scheme and treat following the algorithm described earlier.

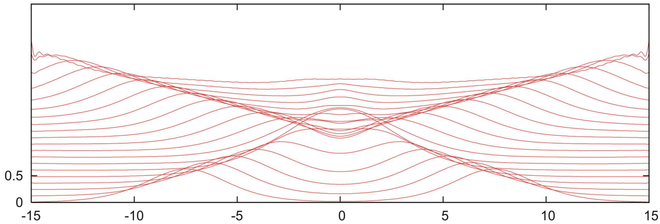

Figure: 6. The inelastic interaction in RLWE near to the threshold of nonlinear blow-up, $c_l = -c_r = 1.5$, $\alpha = -3$, $\beta = 1$.

There is an excellent comparison with Fig.9 in Christov & Velarde, *Int. Journal of Bifurcation and Chaos*, **4** (1994) 1095-1112, where a difference method is used.

RLWE numerical result

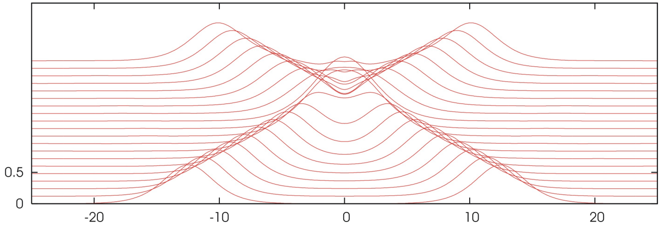

Figure: 7. The inelastic interaction in RLWE near to the threshold of nonlinear blow-up, $c_l = -c_r = 1.5$, $\alpha = -3$, $\beta = 1$.

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RLWE numerical result

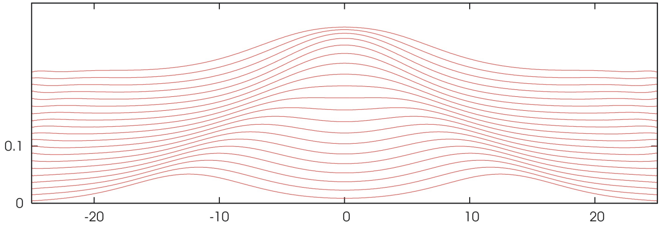

Figure: 8. The inelastic interaction in RLWE for slightly supersonic phase velocities, $c_l = -c_r = 1.05$, $\alpha = -3$, $\beta = 1$.

See for comparison Fig.7 in Christov & Velarde, *Int. Journal of Bifurcation and Chaos*, **4** (1994) 1095-1112

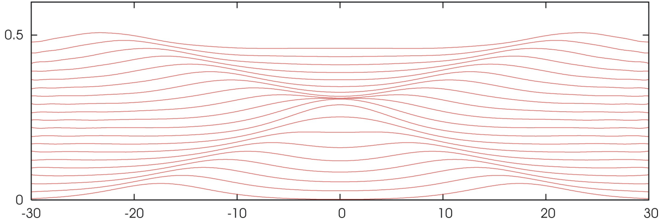

Figure: 9. The inelastic interaction in RLWE for slightly supersonic phase velocities, $c_l = -c_r = 1.05$, $\alpha = -3$, $\beta = 1$.

See for comparison Fig.7 in Christov & Velarde, *Int. Journal of Bifurcation and Chaos*, **4** (1994) 1095-1112

RLWE numerical result

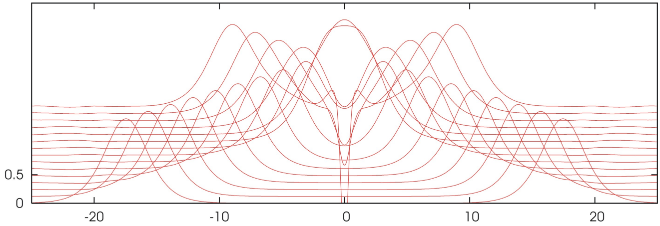

Figure: 10. The inelastic interaction in RLWE for moderate supersonic phase velocities, $c_l = -c_r = 2$, $\alpha = -3$, $\beta = 1$.

See for comparison Fig.12 in Christov & Velarde, *Int. Journal of Bifurcation and Chaos*, **4** (1994) 1095-1112

Thanks

Thank you for your kind attention !