Boussinesq Paradigm and Long-Time Evolution and Interaction of Localized Solutions of Nonlinear Wave Systems. Consistency versus Integrability

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Outline

- Problem Formulation. Bousinesq Paradigm. The Waves as quasi-particles
- Integrability and Nonintegrability. Nonlinearity and Dispersion
- Numerical methods
- Hamiltonian Structure, Initial and Boundary Conditions. Conservation Laws
- Dynamics of Quasi-particles. Self-similar behavior
- Net Polarization Law
- References

Outline of the Problem

- Boussinesq's equation (BE) was the first model for the propagation of surface waves over shallow inviscid fluid layer. He found an analytical solution of his equation and thus proved that the balance between the steepening effect of the nonlinearity and the flattening effect of the dispersion maintains the shape of the wave. This discovery can be properly termed 'Boussinesq Paradigm.'
- Apart from the significance for the shallow water flows, this paradigm is very important for understanding the particle-like behavior of nonlinear localized waves. This behavior was discovered by Zabusky & Kruskal in 1965 (the so-called 'collision property') and the localized waves were called solitons.

- The localized solutions (with finite energy in infinite region) which can retain their identity during the interaction are called quasi-particles if some mechanical properties (such as mass, energy momentum) are conserved by the governing equations. Special interest are the generalized wave equations containing both a nonlinearity and a dispersion (for example, Boussinesq equation and Schrodinger equation) as well as the nonlinear evolution equations (for example, Korteweg-de Vries equation).
- As it should have been expected, most of the physical systems are not fully integrable (even in one spatial dimension) and only a numerical approach can lead to unearthing the pertinent physical mechanisms of the interactions.

Generalized Wave Equation

The overwhelming majority of the analytical and numerical results obtained so far are for one spatial dimension, while in multidimension, much less is possible to achieve analytically, and almost nothing is known about the unsteady solutions that involve interactions, especially when the full-fledged Boussinesq equations are involved. As shown in Christov (2001), the consistent implementation of the Boussinesq method yields the following Generalized Wave Equation (GWE) for $f = \phi(x, y, z = 0; t)$:

$$f_{tt} + 2\beta\nabla f \cdot \nabla f_t + \beta f_t \Delta f$$
$$+ \tfrac{3\beta^2}{2}(\nabla f)^2\Delta f - \Delta f + \frac{\beta}{6}\Delta^2 f - \frac{\beta}{2}(\Delta f)_{tt} = 0$$

with Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\left[f_t^2 + (\nabla f)^2 - \tfrac{1}{4}\beta^2(\nabla f)^4 + \tfrac{1}{6}\beta(\nabla f)^2 + \tfrac{1}{2}\beta(\nabla f_t)^2\right].$$

This equation is the most rigorous amplitude equation that can be derived for the surface waves over an inviscid shallow layer, when the length of the wave is considered large in comparison with the depth of the layer. It was derived only in 2001. Besides it a plethora of different inconsistent Boussinesq equations are still vigorously investigated. The most popular are the versions that contain a quadratic nonlinearity which are useful from the paradigmatic point of view.

Outline the Problem

The equation derived by Boussinesq in 1871 (called "Original Boussinesq Equation")

$$\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[(gH)h + \frac{3g}{2}h^2 + \frac{gH^3}{3}\frac{\partial^2 h}{\partial x^2}\right] \qquad (1)$$

is fully integrable but incorrect in sense of Hadamard when $\beta > 0$. During the years, it was 'improved' in a number of works changing the sign of β. The mere change of the incorrect sign of the fourth derivative in BBE yields the so-called 'good' or 'proper' Boussinesq equation (BE)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - \alpha u^2 + \beta\frac{\partial^2 u}{\partial x^2}\right] \qquad (2)$$

Outline the problem

A different approach to removing the incorrectness is by changing the spatial fourth derivative to a mixed fourth derivative, which resulted into an equation know nowadays as the Regularized Long Wave Equation (RLWE) or Benjamin–Bona–Mahony equation (BBME):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - \alpha u^2 + \beta \frac{\partial^2 u}{\partial t^2}\right]. \tag{3}$$

In fact, the mixed derivative occurs naturally in Boussinesq derivation (see Eq. (1)), and was changed by Boussinesq to a fourth spatial derivative under an assumption that $\frac{\partial}{\partial t} \approx \frac{\partial}{\partial x}$, which is currently known as the 'Linear Impedance Relation' (or LIA). The LIA has produced innumerable instances of unphysical results. The newly derived equation is called "improper" or RLWE (Benjamin-Bona-Mahony equation).

Outline the problem

Let us emphasize that the above considered equations being versions of the Boussisesq equation admit localized *sech* solutions. For example, the explicit solution of the Original is

$$u = -\frac{3}{2}\frac{c^2-1}{\alpha}\operatorname{sech}^2\left(\frac{x-ct}{2}\sqrt{\frac{c^2-1}{\beta}}\right).$$

Generalized Wave Equation. One-dimensional Case

Consider the equation when the velocity potential and surface elevation do not depend on the coordinate y. Having in mind that f is the velocity potential on the bottom of the layer we introduce a vertical velocity $u = f_x$ and an auxiliary function q. Then we obtain so-called Dispersive Wave System is a progenitor of the different 1d Boussinesq equations:

$$u_t + \frac{\alpha}{2}(u^2)_x = q_{xx},$$
$$q_t + \alpha u q_x = u - \beta_2 u_{xx} + \beta_1 u_{tt},$$

where: β_1 and β_2 are two dispersion coefficients, $\beta_1 = 3\beta_2 = \beta$; $\alpha = \beta$ is an amplitude parameter;

Cubic-Quintic Boussinesq Paradigm

$$u_{tt} = \Delta[u - \alpha(u^3 - \sigma u^5) + \beta_1 u_{tt} - \beta_2 \Delta u] \qquad (4)$$

where u is again surface deviation, $\beta_1, \beta_1 > 0$ are dispersion factors. The energy law is

$$\frac{dE}{dt} = 0 \quad \text{with}$$

$$E = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(\nabla u_t)^2 + u^2 - \frac{1}{2}u^4 + \frac{\sigma}{3}u^6 + \beta_1 u^2 + \beta_2(\nabla u)^2]dxdy.$$

In contrast to the Paradigmatic equation with a quadratic nonlinearity, when the amplitude increases for $\sigma > \frac{3}{16}$ the fifth degree dominates and one cannot expect a blow-up of the solution becaus $E > 0$.

Problem Formulation: Boundary Conditions

The boundary conditions are

$$u|_{x=-L_1,L_2} = 0 \qquad q_x|_{x=-L_1,L_2} = 0.$$

When the interval $[-L_1, L_2]$ is finite they provide the conservation of the total energy.

Problem Formulation: Choice of Initial Conditions

The initial condition is a superposition of two solitary waves traveling in opposite directions with a phase velocities c_l and c_r

$$u(x, t=0) = \frac{a\,\mathrm{sgn}(c)}{\frac{|c|-1}{2} + \cosh^2[b(x - X - ct)]}$$

where $a = \frac{c^2-1}{\alpha}$, $b = \sqrt{\frac{c^2-1}{2(\beta_1 c^2 - \beta_2)}}$. The sech-like solutions exist in two domains – subcritical (subsonic): $0 < c < 1/\sqrt{3}$ and supercritical (supersonic): $c > 1$. The physically relevant are only supercritical modes because the subcritical do not admit long waves for small β and are out of our interest.

If f_t is replaced by f_x in the quadratic nonlinear term one arrives

$$u_{tt} = (u + \frac{3\beta}{2}u^2 + \frac{\beta}{2}u_{tt} - \frac{\beta}{6}u_{xx})_{xx},$$

which was called "Boussinesq Paradigm Equation" (BPE). Note that it is not the equation derived by Boussinesq himself. The above simplification, however, destroys the Galilean invariance of the system.

Problem Formulation. Conservation Laws. Energy Consistent Boussinesq Paradigm

Define "mass", M, (pseudo)momentum, P, and (pseudo)energy,

$$M = \int_{-L_1}^{L^2} u \mathrm{d}x, \quad P = \int_{-L_1}^{L^2} (uq_x - \frac{\beta}{2} u_t u_x) dx,$$

$$E = \frac{1}{2} \int_{-L_1}^{L^2} \left(u^2 + q_x^2 + \frac{\beta}{2} u^3 + \frac{\beta}{2} u_t^2 + \frac{\beta}{6} u_x^2 \right) \mathrm{d}x,$$

Here $-L_1$ and L_2 are the left end and the right end of the interval under consideration. The following conservation/balance laws hold, namely

$$\frac{dM}{dt} = 0, \quad \frac{dP}{dt} = -\frac{\beta}{2} u_x^2 \Big|_{x=-L_1}^{x=L_2}, \quad \frac{dE}{dt} = 0.$$

We construct a conservative scheme for the Galilean invariant case treated here. We introduce a regular mesh in the interval $[-L_1, L_2]$, $x_i = -L_1 + (i-1)h$, $h = (L_1 + L_2)/(N-1)$, N is the total number of grid points. We use a simplest linearization combined with an internal iteration (referred to by the composite superscript k). It appears to be robust enough and economical.

$$\frac{u_i^{n+1,k}-u_i^n}{\tau}=\frac{q_{i+1}^{n+1/2,k}-2q_i^{n+1/2,k}+q_{i-1}^{n+1/2,k}}{h^2}$$

$$-\frac{\beta}{8h}\left[(u_{i+1}^{n+1,k-1})^2-(u_{i-1}^{n+1,k-1})^2+(u_{i+1}^n)^2-(u_{i-1}^n)^2\right]$$

$$\frac{q_i^{n+1/2,k}-q_i^{n-1/2}}{\tau}=-\frac{\beta}{8h}\left(q_{i+1}^{n+1/2,k-1}-q_{i-1}^{n+1/2,k}+q_{i+1}^{n-1/2}-q_{i-1}^{n-1/2}\right)$$

$$\times\,(u_i^{n+1,k}+u_i^{n-1})$$

$$-\frac{\beta}{12h^2}\left[(u_{i+1}^{n+1,k}-2u_i^{n+1,k}+u_{i-1}^{n+1,k})+(u_{i+1}^{n-1}-2u_i^{n-1}+u_{i-1}^{n-1})\right]$$

$$+\frac{\beta}{2}\frac{u_i^{n+1,k}-2u_i^n+u_i^{n-1}}{\tau^2}+\frac{u_i^{n+1,k}+u_i^{n-1}}{2}.$$

We prove that the above approximation secures the conservation of energy on difference level for arbitrary potential $U(u)$, namely the difference approximations of the mass and energy are conserved by the difference scheme in the sense that $M^{n+1} = M^n$ and $E^{n+1/2} = E^{n-1/2}$.

Airy function

The Airy function is defined by the improper integral

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos(\tfrac{t^3}{3} + xt)\mathrm{d}t.$$

It is oscillatory in the negative part of x and decays exponentially in the positive part of x. The asymptotic behavior in the negative direction is

$$\mathrm{Ai}(-x) \sim \frac{\sin(\frac{2}{3}x^{3/2} + \frac{\pi}{4})}{\sqrt{\pi}x^{1/4}}.$$

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

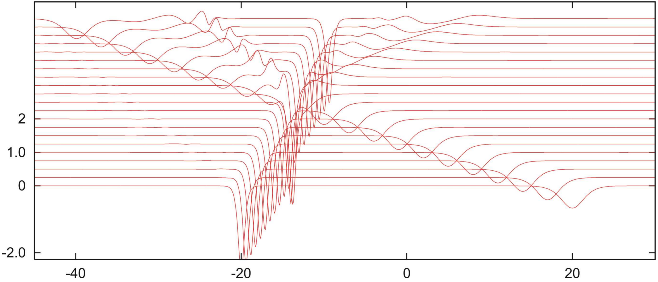


Рис.: 1. Short-times evolution. Forming of accompanying excitations

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

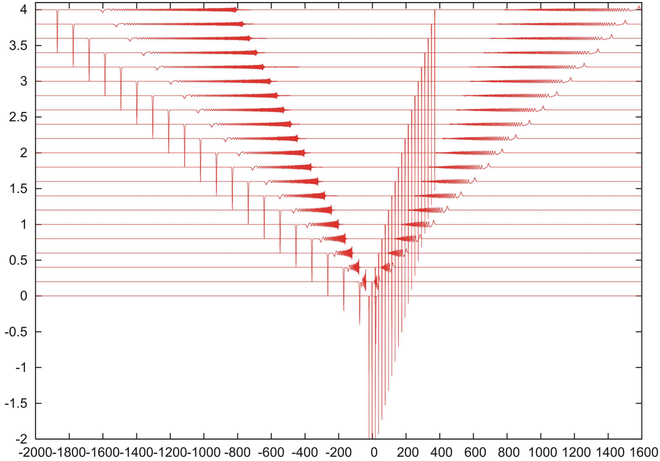


Рис.: 2. Long-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

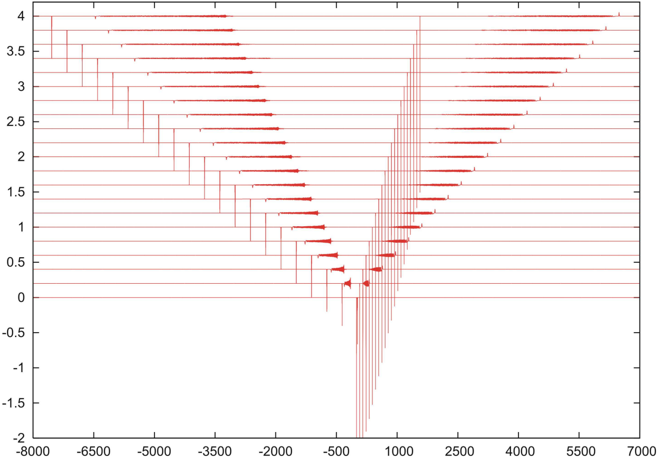


Рис.: 3. Very long-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

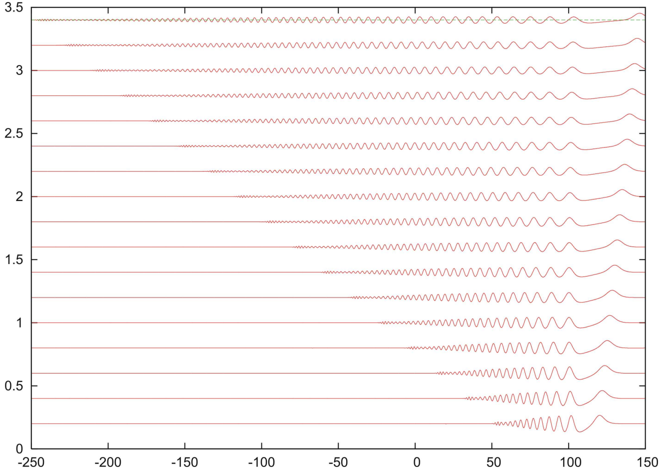


Рис.: 4. The evolution of the left excitation (shifted).

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

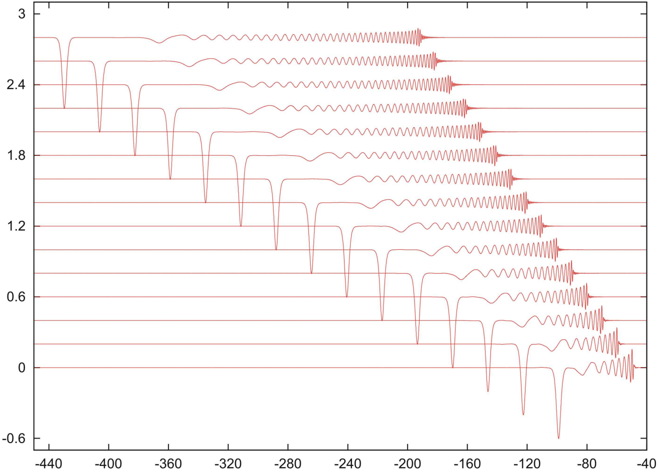


Рис.: 5. The right left-going soliton with its trail. Middle-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

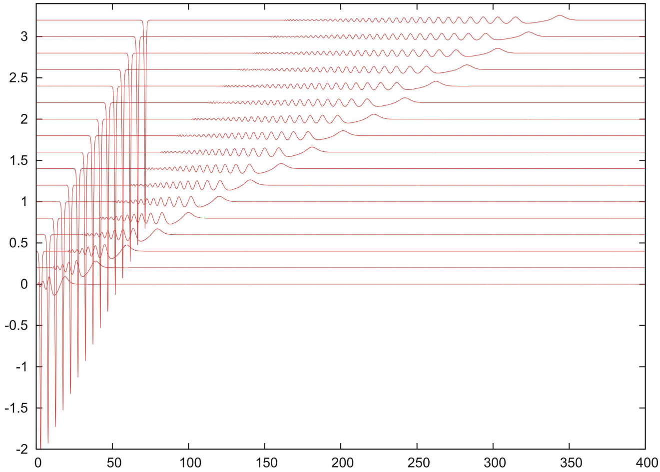


Рис.: 6. The left right-going soliton with its trail. Middle-times evolution.

Discussion

The main solitary waves appear virtually non-deformed from the interaction, but additional oscillations are excited at the trailing edge of each one of them. We extract the perturbations and track their evolution for very long times when they tend to adopt a self-similar shape: their amplitudes decrease with the time while the length scales increase. We test a hypothesis about the dependence on time of the amplitude and the support of Airy-function shaped coherent structures which gives a very good quantitative agreement with the numerically obtained solutions.

CNLSE is system of nonlinearly coupled Schrödinger equations (called the Gross-Pitaevskii or Manakov-type system):

$$\begin{aligned} \mathrm{i}\psi_t &= \beta\psi_{xx} + \left[\alpha_1|\psi|^2 + (\alpha_1 + 2\alpha_2)|\phi|^2\right]\psi(+\Gamma\phi), \\ \mathrm{i}\phi_t &= \beta\phi_{xx} + \left[\alpha_1|\phi|^2 + (\alpha_1 + 2\alpha_2)|\psi|^2\right]\phi(+\Gamma\psi), \end{aligned} \tag{5}$$

where:
β is the dispersion coefficient;
α_1 describes the self-focusing of a signal for pulses in birefringent media;
$\Gamma = \Gamma_r + \mathrm{i}\Gamma_i$ is the magnitude of linear coupling. Γ_r governs the oscillations between states termed as breathing solitons, while Γ_i describes the gain behavior of soliton solutions.
α_2 (called cross-modulation parameter) governs the nonlinear coupling between the equations.

Problem Formulation: Equations

When $\alpha_2 = 0$, no nonlinear coupling is present despite the fact that "cross-terms" proportional to α_1 appear in the equations. For $\alpha_2 = 0$, the solutions of the two equations are identical, $\psi \equiv \phi$, and equal to the solution of single NLSE with nonlinearity coefficient $\alpha = 2\alpha_1$.

For $\Gamma = 0$, CNSE is alternately called the Gross-Pitaevskii equation or an equation of Manakov type. It was derived independently by Gross and Pitaevskii to describe the behavior of Bose-Einstein condensates as well as optic pulse propagation. It was solved analytically for the case $\alpha_2 = 0$, $\beta = \frac{1}{2}$ by Manakov via Inverse Scattering Transform who generalized an earlier result by Zakharov & Shabat for the scalar cubic NLSE (i.e. Eq.(2-ψ) with $\varphi(x, t) = 0$). Recently, Chow, Nakkeeran, and Malomed studied periodic waves in optic fibers using a version of CNSE with $\Gamma \neq 0$.

The single NLSE has the form

$$i\psi_t + \beta\psi_{xx} + \alpha|\psi|^2\psi = 0 \qquad (6)$$

As far as the applications in nonlinear optics are concerned, the above equation describes the single-mode wave propagation in a fiber. Depending on the sign of (6) admits single and multiple sech-solutions (*bright solitons*), as well as tanh-profile, or *dark soliton* solutions. In this paper we concentrate on the case of bright solitons.

A dynamical system with infinite number of conservation laws (integrals) is called integrable. NSE is an integrable dynamical system. CNSE is a nonintegrable generalisation of NSE.

Define "mass", M, (pseudo)momentum, P, and energy, E:

$$M \stackrel{\text{def}}{=} \frac{1}{2\beta}\int_{-L_1}^{L_2}(|\psi|^2+|\phi|^2)\mathrm{d}x, \quad P \stackrel{\text{def}}{=} -\int_{-L_1}^{L_2}\mathcal{I}(\psi\bar{\psi}_x+\phi\bar{\phi}_x)\mathrm{d}x,$$

$$E \stackrel{\text{def}}{=} \int_{-L_1}^{L_2}\mathcal{H}\mathrm{d}x, \quad \text{where} \tag{7}$$

$$\mathcal{H} \stackrel{\text{def}}{=} \beta\left(|\psi_x|^2+|\phi_x|^2\right)-\tfrac{1}{2}\alpha_1(|\psi|^4+|\phi|^4) -(\alpha_1+2\alpha_2)\left(|\phi|^2|\psi|^2\right)-2\Gamma[\Re(\bar{\psi}\bar{\phi})]$$

is the Hamiltonian density of the system. Here $-L_1$ and L_2 are the left end and the right end of the interval under consideration. The following conservation/balance laws hold, namely

$$\frac{dM}{dt}=0, \qquad \frac{dP}{dt}=\mathcal{H}\big|_{x=L_2}-\mathcal{H}\big|_{x=-L_1}, \qquad \frac{dE}{dt}=0, \tag{8}$$

Problem Formulation: Choice of Initial Conditions

We concern ourselves with the soliton solutions whose modulation amplitude is of general form (non-sech) and which are localized envelops on a propagating carrier wave. This allows us to play various scenario of initial polarization. Unfortunately in sech-case the initial polarization can be only linear. Then we assume that for each of the functions ϕ, ψ the initial condition is of the form of a single propagating soliton, namely

$$\begin{Bmatrix} \psi(x,t) \\ \phi(x,t) \end{Bmatrix} = \begin{Bmatrix} A^{\psi} \\ A^{\phi} \end{Bmatrix} \operatorname{sech}\left[b(x-X-ct)\right] \exp\left\{ \mathrm{i}\left[\frac{c}{2\beta}(x-X)-nt\right]\right\}.$$

$$b^2 = \frac{1}{\beta}\left(n+\frac{c^2}{4\beta}\right), \quad A = b\sqrt{\frac{2\beta}{\alpha_1}}, \quad u_c = \frac{2n\beta}{c}, \qquad (9)$$

where X is the spatial position (center of soliton), c is the phase speed, n is the carrier frequency, and b^{-1} – a measure of the support of the localized wave.

We assume that for each of the functions ϕ, ψ the initial condition has the general type

$$\begin{aligned} \psi &= A_\psi(x + X - c_\psi t) \exp\left\{\mathrm{i}\left[n_\psi t - \tfrac{1}{2} c_\psi (x - X - c_\psi t) + \delta_\psi\right]\right\} \\ \phi &= A_\phi(x + X - c_\phi t) \exp\left\{\mathrm{i}\left[n_\phi t - \tfrac{1}{2} c_\phi (x - X - c_\phi t) + \delta_\phi\right]\right\}, \end{aligned} \tag{10}$$

where c_ψ, c_ϕ are the phase speeds and X's are the initial positions of the centers of the solitons; n_ψ, n_ϕ are the carrier frequencies for the two components; δ_ψ and δ_ϕ are the phases of the two components. Note that the phase speed must be the same for the two components ψ and ϕ. If they propagate with different phase speeds, after some time the two components will be in two different positions in space, and will no longer form a single structure. For the envelopes (A_ψ, A_ϕ), $\theta \equiv \arctan(\max|\phi|/\max|\psi|)$ is a polarization angle.

Generally the carrier frequencies for the two components $n_\psi \neq n_\phi$ – elliptic polarization. When $n_\psi = n_\phi$ – circular polarization. If one of them vanishes – linear polarization (sech soliton, $\theta = 0; 90^\circ$).
In general case the initial condition is solution of the following system of nonlinear conjugated equations

$$\begin{aligned} A''_\psi + \left(n_\psi + \tfrac{1}{4}c_\psi^2\right) A_\psi + \left[\alpha_1 A_\psi^2 + (\alpha_1 + 2\alpha_2) A_\phi^2\right] A_\psi &= 0 \\ A''_\phi + \left(n_\phi + \tfrac{1}{4}c_\phi^2\right) A_\phi + \left[\alpha_1 A_\phi^2 + (\alpha_1 + 2\alpha_2) A_\psi^2\right] A_\phi &= 0. \end{aligned} \quad (11)$$

The system admits bifurcation solutions since the trivial solution obviously is always present.

Initial Conditions and Initial Polarization

We solve the auxiliary conjugated system (11) with asymptotic boundary conditions using Newton method and the initial approximation of sought nontrivial solution is sech-function. The final solution, however, is not obligatory sech-function. It is a two-component polarized soliton solution.

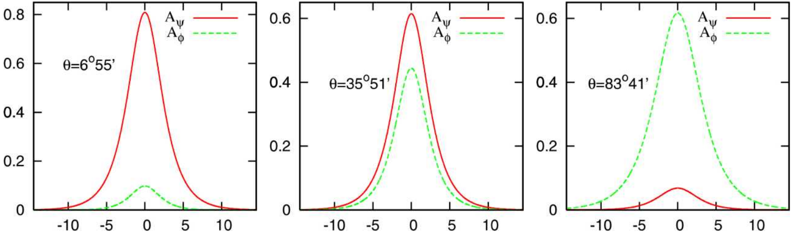


Рис.: 7. Amplitudes A_ψ and A_ϕ for $c_l = -c_r = 1$, $\alpha_1 = 0.75$, $\alpha_2 = 0.2$. Left:$n_\psi = -0.68$; middle: $n_\psi = -0.55$; right: $n_\psi = -0.395$.

Initial Conditions and Initial Phase Difference

Another dimension of complexity is introduced by the phases of the different components. The initial difference in phases can have a profound influence on the polarizations of the solitons after the interaction and the magnitude of the full energy. The relative shift of real and imaginary parts is what matters in this case.

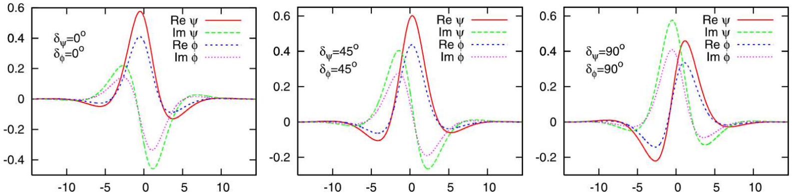


Рис.: 8. Real and imaginary parts of the amplitudes from the case shown in the middle panel of Figure 7 and the dependence on phase angle.

Problem Formulation: Initial Conditions

After completing the initial conditions our aim is to understand better the influence of the initial polarization and initial phase difference on the particle-like behavior of the localized waves. We call a localized wave a quasi-particle (QP) if it survives the collision with other QPs (or some other kind of interactions) without losing its identity.

Linearly Coupled Problem Formulation: Equations and Initial Conditions

For the linearly coupled system of NLSE the magnitude of linear coupling Γ_r generates breathing the solitons although noninteracting The initial conditions must be

$$\Psi = \psi \cos(\Gamma t) + \mathrm{i}\phi \sin(\Gamma t), \quad \Phi = \phi \cos(\Gamma t) + \mathrm{i}\psi \sin(\Gamma t), \qquad (12)$$

where ϕ and ψ are assumed to be sech-solutions of (5) for $\alpha_2 = 0$. Hence (5) posses solutions, which are combinations of interacting solitons oscillating with frequency Γ_r and their motion gives rise to the so-called rotational polarization.

In all considered cases we found that a conservation of the total polarization is present. Only for the linearly CNLSE ($\alpha_2 = 0$) the total polarizations breathe with an amplitude evidently depending on the initial phase difference but is conserved within one full period of the breathing.

$\delta_r - \delta_l$	θ_l^i	θ_r^i	$\theta_l^i + \theta_r^i$	θ_l^f	θ_r^f	$\theta_l^f + \theta_r^f$
45°	45°	45°	90°	$33^\circ 48'$	$56^\circ 12'$	90°
90°	45°	45°	90°	$24^\circ 06'$	$65^\circ 54'$	90°
0°	20°	20°	40°	$20^\circ 00'$	$20^\circ 00'$	40°
90°	20°	20°	40°	$28^\circ 48'$	$2^\circ 02'$	$30^\circ 50'$
0°	36°	36°	72°	$36^\circ 00'$	$36^\circ 00'$	72°
90°	36°	36°	72°	$53^\circ 00'$	$13^\circ 20'$	$66^\circ 20'$
0°	10°	80°	90°	$21^\circ 05'$	$68^\circ 54'$	$89^\circ 59'$
90°	10°	80°	90°	$9^\circ 27'$	$80^\circ 30'$	$89^\circ 57'$

To solve the main problem numerically, we use an implicit conservative scheme in complex arithmetic.

$$\mathrm{i}\frac{\psi_i^{n+1}-\psi_i^n}{\tau}=\frac{\beta}{2h^2}\left(\psi_{i-1}^{n+1}-2\psi_i^{n+1}+\psi_{i+1}^{n+1}+\psi_{i-1}^{n}-2\psi_i^{n}+\psi_{i+1}^{n}\right)$$
$$+\frac{\psi_i^{n+1}+\psi_i^n}{4}\left[\alpha_1\left(|\psi_i^{n+1}|^2+|\psi_i^n|^2\right)+(\alpha_1+2\alpha_2)\left(|\phi_i^{n+1}|^2+|\phi_i^n|^2\right)\right]$$
$$-\frac{1}{2}\Gamma\left(\phi_i^{n+1}+\phi_i^n\right),$$

$$\mathrm{i}\frac{\phi_i^{n+1}-\phi_i^n}{\tau}=\frac{\beta}{2h^2}\left(\phi_{i-1}^{n+1}-2\phi_i^{n+1}+\phi_{i+1}^{n+1}+\phi_{i-1}^{n}-2\phi_i^{n}+\phi_{i+1}^{n}\right)$$
$$+\frac{\phi_i^{n+1}+\phi_i^n}{4}\left[\alpha_1\left(|\phi_i^{n+1}|^2+|\phi_i^n|^2\right)+(\alpha_1+2\alpha_2)\left(|\psi_i^{n+1}|^2+|\psi_i^n|^2\right)\right]$$
$$-\frac{1}{2}\Gamma\left(\psi_i^{n+1}+\psi_i^n\right).$$

Numerical Method: Internal Iterations

$$
\begin{aligned}
\mathrm{i}\frac{\psi_i^{n+1,k+1}-\psi_i^n}{\tau} =& \frac{\beta}{2h^2}\Big(\psi_{i-1}^{n+1,k+1}-2\psi_i^{n+1,k+1}+\psi_{i+1}^{n+1,k+1} \\
&+\psi_{i-1}^n-2\psi_i^n+\psi_{i+1}^n\Big) \\
+\frac{\psi_i^{n+1,k}+\psi_i^n}{4}&\Big[\alpha_1\big(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}|+|\psi_i^n|^2\big) \\
&+(\alpha_1+2\alpha_2)\big(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}|+|\phi_i^n|^2\big)\Big] \\
\mathrm{i}\frac{\phi_i^{n+1,k+1}-\phi_i^n}{\tau} =& \frac{\beta}{2h^2}\big(\phi_{i-1}^{n+1,k+1}-2\phi_i^{n+1,k+1}+\phi_{i+1}^{n+1,k+1} \\
&+\phi_{i-1}^n-2\phi_i^n+\phi_{i+1}^n\big) \\
+\frac{\phi_i^{n+1,k}+\phi_i^n}{4}&\Big[\alpha_1\big(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}|+|\phi_i^n|^2\big) \\
&+(\alpha_1+2\alpha_2)\big(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}|+|\psi_i^n|^2\big)\Big].
\end{aligned}
$$

It is not only convergent (consistent and stable), but also conserves mass and energy, i.e., there exist discrete analogs for (8), which arise from the scheme.

$$M^n = \sum_{i=2}^{N-1} \left(|\psi_i^n|^2 + |\phi_i^n|^2\right) = \text{const},$$

$$E^n = \sum_{i=2}^{N-1} \frac{-\beta}{2h^2} \left(|\psi_{i+1}^n - \psi_i{}^n|^2 + |\phi_{i+1}^n - \phi_i{}^n|^2\right) + \frac{\alpha_1}{4}\left(|\psi_i^n|^4 + |\phi_i^n|^4\right)$$
$$+ \tfrac{1}{2}(\alpha_1 + 2\alpha_2)\left(|\psi_i^n|^2|\phi_i^n|^2\right) - \Gamma\Re[\bar{\phi}_i^n\psi_i^n] = \text{const},$$
$$\text{for} \quad \text{all} \quad n \geq 0.$$

These values are kept constant during the time stepping. The above scheme is of Crank-Nicolson type for the linear terms and we employ internal iterations to achieve implicit approximation of the nonlinear terms, i.e., we use its linearized implementation.

Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

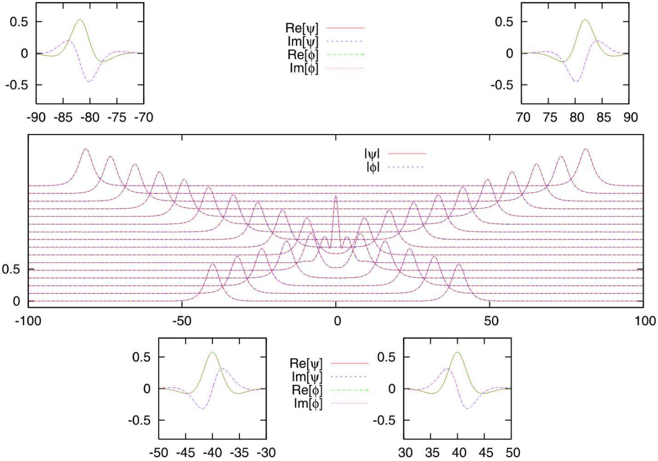


Рис.: 9. $\delta_l = 0^\circ$, $\delta_r = 0^\circ$

Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

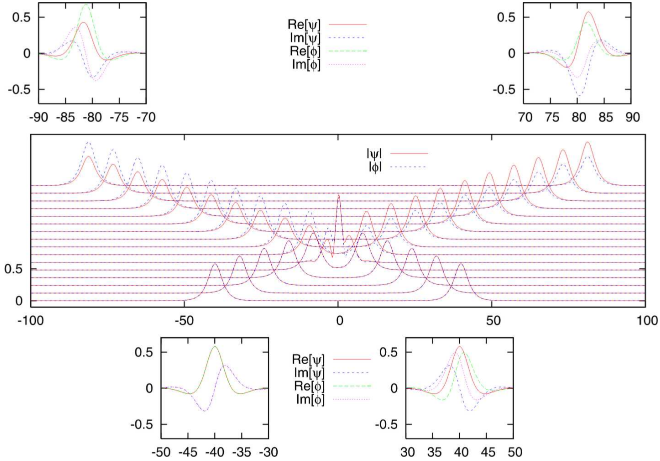


Рис.: 10. $\delta_l = 0^\circ$, $\delta_r = 45^\circ$

Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

—When both of QPs have zero phases (Fig. 9), the interaction perfectly follows the analytical Manakov two-soliton solution.
—The surprise comes in Fig. 10 where is presented an interaction of two QPs, the right one of which has a nonzero phase $\delta_r = 45^\circ$. After the interaction, the two QPs become different Manakov solitons than the original two that entered the collision. The outgoing QPs have polarizations $33^\circ 48'$ and $56^\circ 12'$. Something that can be called a 'shock in polarization' takes place. All the solutions are perfectly smooth, but because the property called polarization cannot be defined in the cross-section of interaction and for this reason, it appears as undergoing a shock.

Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

Here is to be mentioned that when rescaled the moduli of ψ and ϕ from Fig. 10 perfectly match each other which means that the resulting solitons have circular polarization (see left panel of Fig. 11 below). The Manakov solution is not unique. There exists a class of Manakov solution and in the place of interaction becomes a bifurcation between them.

Results and Discussion: Initial Circular Polarization and Nonuniqueness of the Manakov Solution

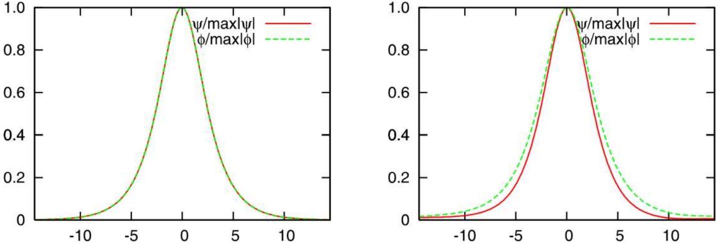


Рис.: 11. Circular polarization (left); Elliptic Polarization (right).

Equal Elliptic Initial Polarizations of $50°08'$ for $\alpha_2 = 2$

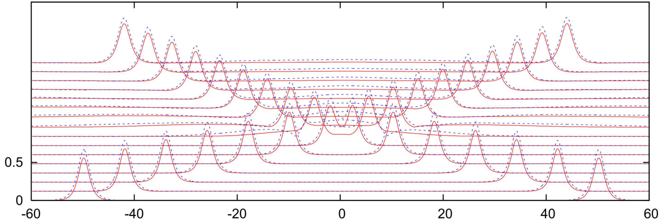


Рис.: 12. $\delta_l = 0°, \delta_r = 0°$

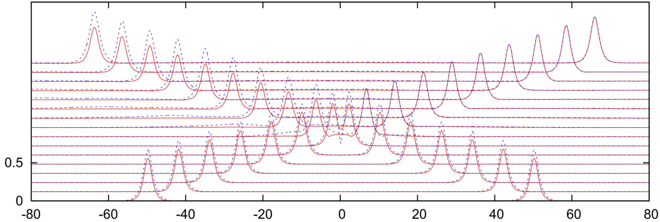


Рис.: 13. $\delta_l = 0°, \delta_r = 180°$

Equal Elliptic Initial Polarizations of $50^\circ 08'$ for $\alpha_2 = 2$

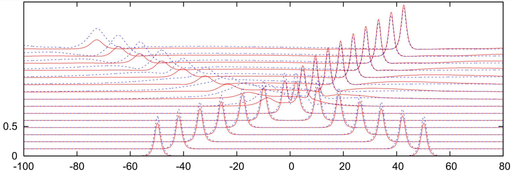


Рис.: 14. $\delta_l = 0^\circ, \delta_r = 130^\circ$

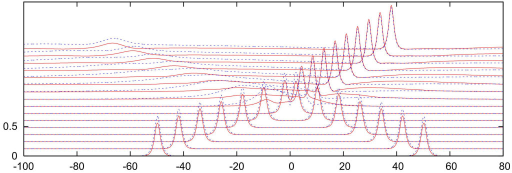


Рис.: 15. $\delta_l = 0^\circ, \delta_r = 135^\circ$

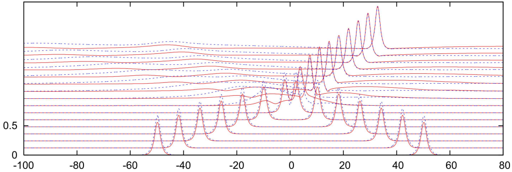


Рис.: 16. $\delta_l = 0^\circ, \delta_r = 140^\circ$

Equal Elliptic Initial Polarizations of $50^\circ 08'$ for $\alpha_2 = 2$.

We choose $n_{l\psi} = n_{r\psi} = -1.5$, $n_{l\phi} = n_{r\phi} = -1.1$, $c_l = -c_r = 1$, $\alpha_1 = 0.75$, and focus on the effects of α_2 and $\vec{\delta}$.

One sees that the desynchronisations of the phases leads in the final stage to a superposition of two one-soliton solutions but with different polarizations from the initial polarization. Yet, for $\delta_r = 130^\circ \div 140^\circ$ one of the QPs loses its energy contributing it to the other QP during the collision and then virtually disappears: kind of energy trapping (Figs.14, 15, 16).

For $\delta_r = 180^\circ$ another interesting effect is seen, when the right outgoing QP is circularly polarized (Fig. 13).

All these interactions are accompanied by changes of phase speeds. The total polarization exhibits some kind of conservation.

Strong Nonlinear Interaction: $\alpha_2 = 10$

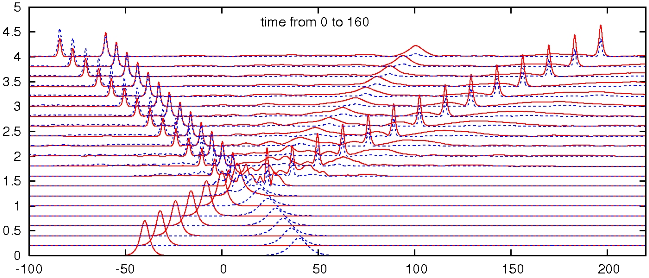


Рис.: 17. $\alpha_2 = 10$, $c_l = 1$, $c_r = -0.5$.

Strong Nonlinear Interaction: $\alpha_2 = 10$

- Two new solitons are born after the collision.
- The kinetic energies of the newly created solitons correspond their phase speeds and masses, but the internal energy is very different for the different QP.
- the total energy of the QPs is radically different from the total energy of the initial wave profile. The differences are so drastic that the sum of QPs energies can even become negative. This means that the energy was carried away by the radiation.
- The predominant part of the energy is concentrated in the left and right forerunners because of the kinetic energies of the latter are very large. This is due to the fact that the forerunners propagate with very large phase speeds, and span large portions of the region.
- All four QPs have elliptic polarizations.
- Energy transformation is a specific trait of the coupled system considered here.

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23°44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

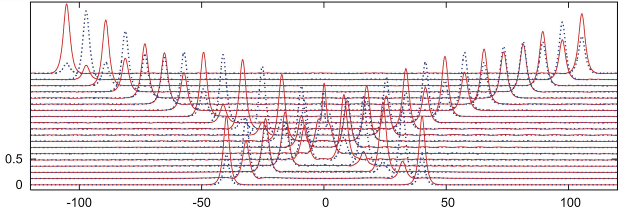


Рис.: 18. $\delta_l = 0°$, $\delta_r = 0°$

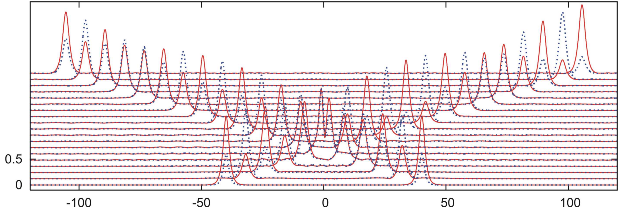


Рис.: 19. $\delta_l = 0°$, $\delta_r = 90°$

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

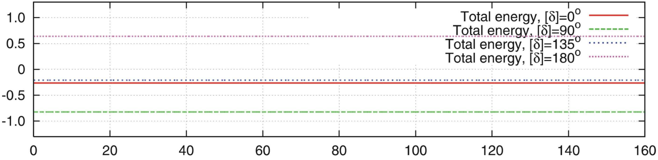


Рис.: 20. Influence of the initial phase difference on the total energy:
$\delta_l = 0^\circ$, $\delta_r = 0^\circ$ – $E = -0.262$;
$\delta_l = 0^\circ$, $\delta_r = 90^\circ$ – $E = -0.821$;
$\delta_l = 0^\circ$, $\delta_r = 135^\circ$ – $E = -0.206$;
$\delta_l = 0^\circ$, $\delta_r = 180^\circ$ – $E = 0.640$

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

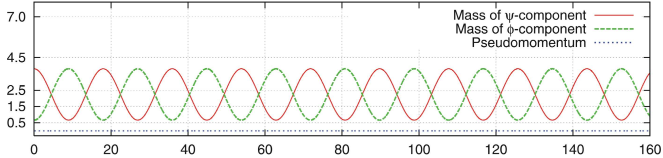


Рис.: 21. $\delta_l = 0^\circ$, $\delta_r = 90^\circ$, $P = 10^{-3} \div 10^{-5}$

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23°44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

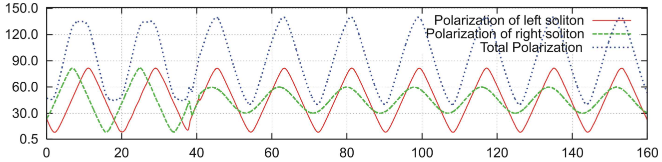


Рис.: 22. $\delta_l = 0°$, $\delta_r = 0°$

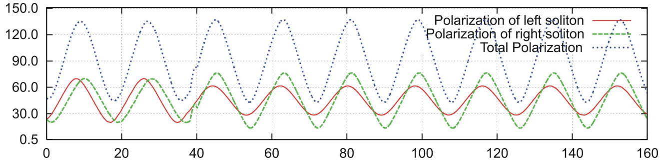
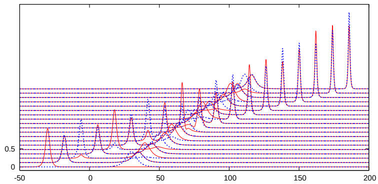
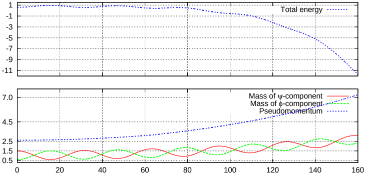


Рис.: 23. $\delta_l = 0°$, $\delta_r = 90°$

Linear Coupling: Initial Linear Polarizations: $\theta_l = 0^\circ$, $\theta_r = 90^\circ$, $\alpha_2 = 0$, $c_l = 1.5$, $c_r = 0.6$, $\Gamma = 0.175 + 0.005i$



a) Profiles of the components



b) Masses, psedomomentum, and total energy as functions of time.

Рис.: 24. $\delta_l = 0^\circ$, $\delta_r = 90^\circ$

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

- We have found that the phases of the components play an essential role on the full energy of QPs. The magnitude of the latter essentially depends on the choice of initial phase difference (Figure 20);
- The pseudomomentum is also conserved and it is trivial due to the symmetry (Figure 21);
- The individual masses, however, breathe together with the individual (rotational) polarizations. Their amplitude and period do not influenced from the initial phase difference (Figure 21);

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

- The total mass is constant while the total polarization oscillates and suffers a 'shock in polarization ' when QPs enter the collision. The polarization amplitude evidently depends on the initial phase difference (Figures 22,23);
- Due to the real linear coupling the polarization angle of QPs can change independently of the collision.
- Complex parameter of linear coupling: Along with the oscillations of the energy and masses the (negative) energy decreases very fast, while the masses M_ψ and M_ϕ increase all of them oscillating. The pseudomomentum P increases without appreciable oscillation (Figure 24).

References

J. V. Boussinesq. Comp. Rend. Hebd. des Seances de l'Acad. des Sci. 73:256—260 (1871).

J. V. Boussinesq. Journal de Mathйmatiques Pures et Appliquйes 17: 55-–108, 1872.

G. A. Maugin and S. Cadet. Existence of solitary waves in martensitic alloys, Int. J. Engng. Sci., 29:243-55, 1991.

C. I. Christov, and M. G. Velarde. Inelastic interaction of Boussinesq solitons. Intern. J. of Bifurc. and Chaos, 4(5):1095—1112, 1994.

C. I. Christov. Conservative difference scheme for Boussinesq model of surface waves, in Proc. ICFDV, Oxford University Press, W.K. Morton and M.J. Baines (Eds), pp. 343—349, 1995.

C. I. Christov and G. A. Maugin. An implicit difference scheme for the long-time evolution of localized solutions of a generalized Boussinesq system, J. Comp. Phys. 116: 39–51, 1995.

References

C. I. Christov. An energy-consistent dispersive shallow-water model. Wave Motion, 4(5):161—174, 2001.

A. Kurganov and E. Tadmor. Solution of two-dimensional Riemann problems for gas dynamics without Riemann problem solvers, Numerical Methods for Partial Differential Equations, 18:548–608, 2002.

A. Chertock, C. I. Christov, and A. Kurganov, "Central-Upwind Schemes for the Boussinesq Paradigm Equation," in Computational Sci., & High performance Computing IV, NNFM, edited by E. Krause et al, 2011, pp. 267—281.

C. I. Christov and J. Choudhury. Perturbation solution for the 2D Boussinesq equation, Mech. Res. Commun., 38:274–281, 2011.

C. I. Christov, S. Dost, and G. A. Maugin. Inelasticity of soliton collisions in system of coupled NLS equations. *Physica Scripta*, 50:449–454, 1994.

References

S. V. Manakov. On the theory of two-dimensional stationary self-focusing of electromagnetic waves. *Soviet Physics JETP*, 38:248–253, 1974.

W. J. Sonnier and C. I. Christov. Strong coupling of Schrödinger equations: Conservative scheme approach. *Mathematics and Computers in Simulation*, 69:514–525, 2005.

C. Sophocleous and D. F. Parker. Pulse colliasions and polarization conversion for optical fibers. *Opt. Commun.*, 112:214–224, 1994.

T. R. Taha and M. J. Ablovitz. Analytical and numerical aspects of certain nonlinear evolution equations. ii. numerical, Schrödinger equation. *J. Comp. Phys.*, 55:203–230, 1984.

M. D. Todorov and C. I. Christov. Conservative scheme in complex arithmetic for coupled nonlinear Schrödinger equations. *Discrete and Continuous Dynamical Systems*", Supplement:982–992, 2007.

References

M. D. Todorov and C. I. Christov. Collision Dynamics of Elliptically Polarized Solitons in Coupled Nonlinear Schrodinger Equations. *Mathematics and Computers in Simulation*, 82, 2012, 1221–1232.

M. D. Todorov and C. I. Christov. Collision Dynamics of Polarized Solitons in Linearly Coupled Nonlinear Schrцdinger Equations, in K.Sekigawa et al (Eds.), *The International Workshop on Complex Structures, Integrability, and Vector Fields, 2010, Sofia, Bulgaria,* American Institute of Physics CP1340, pages 144–153, 2011.

M. D. Todorov and C. I. Christov. Collision Dynamics of Circularly Polarized Solitons in Nonintegrable Coupled Nonlinear Schrödinger System. *Special Volume 2009 of Journal "Discrete and Continuous Dynamical Systems"*, 780—789.

V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media.*Soviet Physics JETP*, 34:62–9, 1972: 37:823, 1973.

C. I. Christov, M. D. Todorov, and M. A. Christou, Perturbation Solution for the 2D Shallow-water Waves. in *Application of Mathematics in Technical and Natural Sciences, 2011, Albena, Bulgaria,* American Institute of Physics CP1404, pages 49–56, 2012.

Thanks

Thank you for your kind attention !