Vector Schroedinger Equation: Coupling, Polarization, Phase Difference, Quasi-Particle Dynamics

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Outline

- Problem Formulation
- Choice and Generation of Initial Conditions
- Polarization and phase difference
- Linear and Nonlinear Coupling and Quasi-Particle Dynamics
- Conservation Laws
- Numerical Method. Conservation Properties
- Main Results and Discussion
- References

Problem Formulation: Equations

CNLSE is system of nonlinearly coupled Schrödinger equations (called the Gross-Pitaevskii or Manakov-type system):

$$
\begin{aligned}
\mathrm{i}\psi_t &= \beta\psi_{xx} + \left[\alpha_1|\psi|^2 + (\alpha_1 + 2\alpha_2)|\phi|^2\right]\psi(+\Gamma\phi),\\
\mathrm{i}\phi_t &= \beta\phi_{xx} + \left[\alpha_1|\phi|^2 + (\alpha_1 + 2\alpha_2)|\psi|^2\right]\phi(+\Gamma\psi),
\end{aligned}
\tag{1}
$$

where:
β is the dispersion coefficient;
α_1 describes the self-focusing of a signal for pulses in birefringent media;
$\Gamma = \Gamma_r + \mathrm{i}\Gamma_i$ is the magnitude of linear coupling. Γ_r governs the oscillations between states termed as breathing solitons, while Γ_i describes the gain behavior of soliton solutions.
α_2 (called cross-modulation parameter) governs the nonlinear coupling between the equations.

When $\alpha_2 = 0$, no nonlinear coupling is present despite the fact that "cross-terms" proportional to α_1 appear in the equations. For $\alpha_2 = 0$, the solutions of the two equations are identical, $\psi \equiv \phi$, and equal to the solution of single NLSE with nonlinearity coefficient $\alpha = 2\alpha_1$.

Problem Formulation: Choice of Initial Conditions

We concern ourselves with the soliton solutions whose modulation amplitude is of general form (non-sech) and which are localized envelops on a propagating carrier wave. This allows us to play various scenario of initial polarization. Unfortunately in sech-case the initial polarization can be only linear. Then we assume that for each of the functions ϕ, ψ the initial condition is of the form of a single propagating soliton, namely

$$\begin{Bmatrix} \psi(x,t) \\ \phi(x,t) \end{Bmatrix} = \begin{Bmatrix} A^{\psi} \\ A^{\phi} \end{Bmatrix} \operatorname{sech}\left[b(x-X-ct)\right] \exp\left\{ \mathrm{i}\left[\frac{c}{2\beta}(x-X)-nt\right]\right\}.$$

$$b^2 = \frac{1}{\beta}\left(n+\frac{c^2}{4\beta}\right), \quad A = b\sqrt{\frac{2\beta}{\alpha_1}}, \quad u_c = \frac{2n\beta}{c}, \qquad (2)$$

where X is the spatial position (center of soliton), c is the phase speed, n is the carrier frequency, and b^{-1} – a measure of the support of the localized wave.

Problem Formulation: Choice of Initial Conditions

We assume that for each of the functions ϕ, ψ the initial condition has the general type

$$\begin{aligned} \psi &= A_\psi(x + X - c_\psi t) \exp\left\{\mathrm{i}\left[n_\psi t - \tfrac{1}{2}c_\psi(x - X - c_\psi t) + \delta_\psi\right]\right\} \\ \phi &= A_\phi(x + X - c_\phi t) \exp\left\{\mathrm{i}\left[n_\phi t - \tfrac{1}{2}c_\phi(x - X - c_\phi t) + \delta_\phi\right]\right\}, \end{aligned} \tag{3}$$

where c_ψ, c_ϕ are the phase speeds and X's are the initial positions of the centers of the solitons; n_ψ, n_ϕ are the carrier frequencies for the two components; δ_ψ and δ_ϕ are the phases of the two components. Note that the phase speed must be the same for the two components ψ and ϕ. If they propagate with different phase speeds, after some time the two components will be in two different positions in space, and will no longer form a single structure. For the envelopes (A_ψ, A_ϕ),
$\theta \equiv \arctan(\max|\phi|/\max|\psi|)$ is a polarization angle.

Problem Formulation: Generation of Initial Conditions

Generally the carrier frequencies for the two components $n_\psi \neq n_\phi$ – elliptic polarization. When $n_\psi = n_\phi$ – circular polarization. If one of them vanishes – linear polarization (sech soliton, $\theta = 0; 90^\circ$).
In general case the initial condition is solution of the following system of nonlinear conjugated equations

$$\begin{aligned} A''_\psi + \left(n_\psi + \tfrac{1}{4}c_\psi^2\right) A_\psi + \left[\alpha_1 A_\psi^2 + (\alpha_1 + 2\alpha_2)A_\phi^2\right] A_\psi = 0 \\ A''_\phi + \left(n_\phi + \tfrac{1}{4}c_\phi^2\right) A_\phi + \left[\alpha_1 A_\phi^2 + (\alpha_1 + 2\alpha_2)A_\psi^2\right] A_\phi = 0. \end{aligned} \quad (4)$$

The system admits bifurcation solutions since the trivial solution obviously is always present.

Initial Conditions and Initial Polarization

We solve the auxiliary conjugated system (4) with asymptotic boundary conditions using Newton method and the initial approximation of sought nontrivial solution is sech-function. The final solution, however, is not obligatory sech-function. It is a two-component polarized soliton solution.

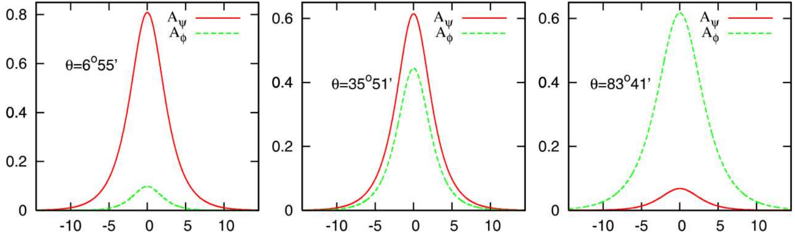

Figure: 1. Amplitudes A_ψ and A_ϕ for $c_l = -c_r = 1$, $\alpha_1 = 0.75$, $\alpha_2 = 0.2$. Left:$n_\psi = -0.68$; middle: $n_\psi = -0.55$; right: $n_\psi = -0.395$.

Initial Conditions and Initial Phase Difference

Another dimension of complexity is introduced by the phases of the different components. The initial difference in phases can have a profound influence on the polarizations of the solitons after the interaction and the magnitude of the full energy. The relative shift of real and imaginary parts is what matters in this case.

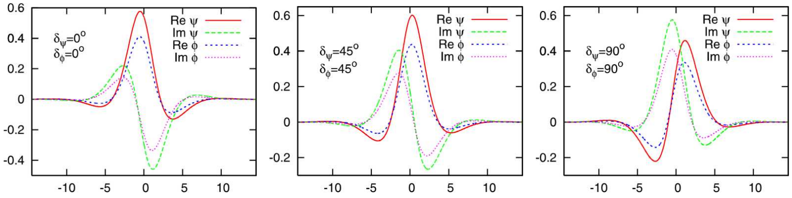

Figure: 2. Real and imaginary parts of the amplitudes from the case shown in the middle panel of Figure 1 and the dependence on phase angle.

Problem Formulation: Initial Conditions

After completing the initial conditions our aim is to understand better the influence of the initial polarization and initial phase difference on the particle-like behavior of the localized waves. We call a localized wave a quasi-particle (QP) if it survives the collision with other QPs (or some other kind of interactions) without losing its identity.

Linearly Coupled Problem Formulation: Equations and Initial Conditions

For the linearly coupled system of NLSE the magnitude of linear coupling Γ_r generates breathing the solitons although noninteracting The initial conditions must be

$$\Psi = \psi \cos(\Gamma t) + \mathrm{i}\phi \sin(\Gamma t), \quad \Phi = \phi \cos(\Gamma t) + \mathrm{i}\psi \sin(\Gamma t), \qquad (5)$$

where ϕ and ψ are assumed to be sech-solutions of (1) for $\alpha_2 = 0$. Hence (1) posses solutions, which are combinations of interacting solitons oscillating with frequency Γ_r and their motion gives rise to the so-called rotational polarization.

Problem Formulation: Conservation Laws

Define "mass", M, (pseudo)momentum, P, and energy, E:

$$M \stackrel{\text{def}}{=} \frac{1}{2\beta}\int_{-L_1}^{L_2}(|\psi|^2+|\phi|^2)\mathrm{d}x, \quad P \stackrel{\text{def}}{=} -\int_{-L_1}^{L_2}\mathcal{I}(\psi\bar{\psi}_x+\phi\bar{\phi}_x)\mathrm{d}x,$$

$$E \stackrel{\text{def}}{=} \int_{-L_1}^{L_2}\mathcal{H}\mathrm{d}x, \quad \text{where} \tag{6}$$

$$\mathcal{H} \stackrel{\text{def}}{=} \beta\left(|\psi_x|^2+|\phi_x|^2\right)-\tfrac{1}{2}\alpha_1(|\psi|^4+|\phi|^4) -(\alpha_1+2\alpha_2)\left(|\phi|^2|\psi|^2\right)-2\Gamma[\Re(\bar{\psi}\bar{\phi})]$$

is the Hamiltonian density of the system. Here $-L_1$ and L_2 are the left end and the right end of the interval under consideration. For the linear coupling case, $\alpha_2 = 0$ and $\Gamma \neq 0$ the functions ψ and ϕ correspond to notations in (5). The following conservation/balance laws hold, namely

$$\frac{dM}{dt}=0, \qquad \frac{dP}{dt}=\mathcal{H}\big|_{x=L_2}-\mathcal{H}\big|_{x=-L_1}, \qquad \frac{dE}{dt}=0, \tag{7}$$

Problem Formulation: Conservation Laws

In all considered cases we found that a conservation of the total polarization is present. Only for the linearly CNLSE ($\alpha_2 = 0$) the total polarizations breathe with an amplitude evidently depending on the initial phase difference but is conserved within one full period of the breathing.

$\delta_r - \delta_l$	θ_l^i	θ_r^i	$\theta_l^i + \theta_r^i$	θ_l^f	θ_r^f	$\theta_l^f + \theta_r^f$
45°	45°	45°	90°	$33^\circ 48'$	$56^\circ 12'$	90°
90°	45°	45°	90°	$24^\circ 06'$	$65^\circ 54'$	90°
0°	20°	20°	40°	$20^\circ 00'$	$20^\circ 00'$	40°
90°	20°	20°	40°	$28^\circ 48'$	$2^\circ 02'$	$30^\circ 50'$
0°	36°	36°	72°	$36^\circ 00'$	$36^\circ 00'$	72°
90°	36°	36°	72°	$53^\circ 00'$	$13^\circ 20'$	$66^\circ 20'$
0°	10°	80°	90°	$21^\circ 05'$	$68^\circ 54'$	$89^\circ 59'$
90°	10°	80°	90°	$9^\circ 27'$	$80^\circ 30'$	$89^\circ 57'$

To solve the main problem numerically, we use an implicit conservative scheme in complex arithmetic.

$$\mathrm{i}\frac{\psi_i^{n+1}-\psi_i^n}{\tau} = \frac{\beta}{2h^2}\left(\psi_{i-1}^{n+1}-2\psi_i^{n+1}+\psi_{i+1}^{n+1}+\psi_{i-1}^{n}-2\psi_i^{n}+\psi_{i+1}^{n}\right)$$
$$+\frac{\psi_i^{n+1}+\psi_i^n}{4}\left[\alpha_1\left(|\psi_i^{n+1}|^2+|\psi_i^n|^2\right)+(\alpha_1+2\alpha_2)\left(|\phi_i^{n+1}|^2+|\phi_i^n|^2\right)\right]$$
$$-\frac{1}{2}\Gamma\left(\phi_i^{n+1}+\phi_i^n\right),$$

$$\mathrm{i}\frac{\phi_i^{n+1}-\phi_i^n}{\tau} = \frac{\beta}{2h^2}\left(\phi_{i-1}^{n+1}-2\phi_i^{n+1}+\phi_{i+1}^{n+1}+\phi_{i-1}^{n}-2\phi_i^{n}+\phi_{i+1}^{n}\right)$$
$$+\frac{\phi_i^{n+1}+\phi_i^n}{4}\left[\alpha_1\left(|\phi_i^{n+1}|^2+|\phi_i^n|^2\right)+(\alpha_1+2\alpha_2)\left(|\psi_i^{n+1}|^2+|\psi_i^n|^2\right)\right]$$
$$-\frac{1}{2}\Gamma\left(\psi_i^{n+1}+\psi_i^n\right).$$

Numerical Method: Internal Iterations

$$
\begin{aligned}
\mathrm{i}\frac{\psi_i^{n+1,k+1}-\psi_i^n}{\tau} = & \frac{\beta}{2h^2}\Big(\psi_{i-1}^{n+1,k+1}-2\psi_i^{n+1,k+1}+\psi_{i+1}^{n+1,k+1} \\
& + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n\Big) \\
+ \frac{\psi_i^{n+1,k}+\psi_i^n}{4} & \Big[\alpha_1\big(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}|+|\psi_i^n|^2\big) \\
& +(\alpha_1+2\alpha_2)\big(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}|+|\phi_i^n|^2\big)\Big] \\
\mathrm{i}\frac{\phi_i^{n+1,k+1}-\phi_i^n}{\tau} = & \frac{\beta}{2h^2}\big(\phi_{i-1}^{n+1,k+1}-2\phi_i^{n+1,k+1}+\phi_{i+1}^{n+1,k+1} \\
& + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n\big) \\
+ \frac{\phi_i^{n+1,k}+\phi_i^n}{4} & \Big[\alpha_1\big(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}|+|\phi_i^n|^2\big) \\
& +(\alpha_1+2\alpha_2)\big(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}|+|\psi_i^n|^2\big)\Big].
\end{aligned}
$$

Numerical Method: Conservation Properties

It is not only convergent (consistent and stable), but also conserves mass and energy, i.e., there exist discrete analogs for (7), which arise from the scheme.

$$M^n = \sum_{i=2}^{N-1} \left(|\psi_i^n|^2 + |\phi_i^n|^2\right) = \text{const},$$

$$E^n = \sum_{i=2}^{N-1} \frac{-\beta}{2h^2} \left(|\psi_{i+1}^n - \psi_i^n|^2 + |\phi_{i+1}^n - \phi_i^n|^2\right) + \frac{\alpha_1}{4} \left(|\psi_i^n|^4 + |\phi_i^n|^4\right)$$
$$+ \tfrac{1}{2}(\alpha_1 + 2\alpha_2) \left(|\psi_i^n|^2 |\phi_i^n|^2\right) - \Gamma \Re[\bar{\phi}_i^n \psi_i^n] = \text{const},$$
$$\text{for} \quad \text{all} \quad n \geq 0.$$

These values are kept constant during the time stepping. The above scheme is of Crank-Nicolson type for the linear terms and we employ internal iterations to achieve implicit approximation of the nonlinear terms, i.e., we use its linearized implementation.

Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

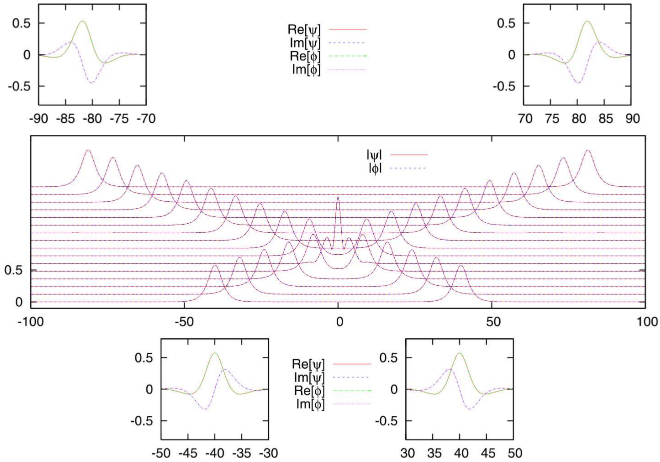

Figure: 3. $\delta_l = 0°$, $\delta_r = 0°$

Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

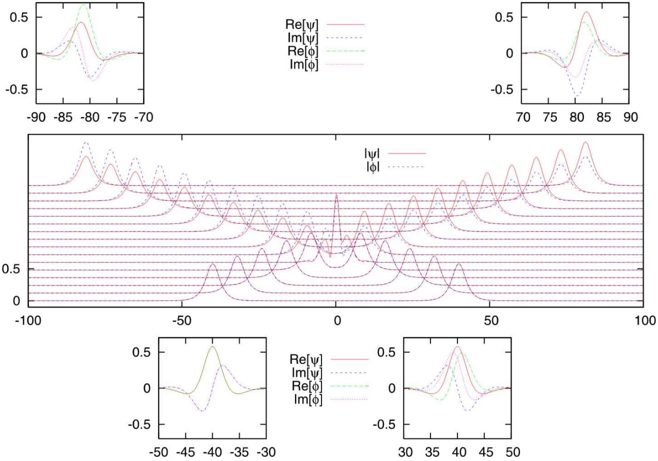

Figure: 4. $\delta_l = 0°$, $\delta_r = 45°$

Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

—When both of QPs have zero phases (Fig. 3), the interaction perfectly follows the analytical Manakov two-soliton solution.
—The surprise comes in Fig. 4 where is presented an interaction of two QPs, the right one of which has a nonzero phase $\delta_r = 45^\circ$. After the interaction, the two QPs become different Manakov solitons than the original two that entered the collision. The outgoing QPs have polarizations $33^\circ 48'$ and $56^\circ 12'$. Something that can be called a 'shock in polarization' takes place. All the solutions are perfectly smooth, but because the property called polarization cannot be defined in the cross-section of interaction and for this reason, it appears as undergoing a shock.

Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

Here is to be mentioned that when rescaled the moduli of ψ and ϕ from Fig. 4 perfectly match each other which means that the resulting solitons have circular polarization The Manakov solution is not unique. There exists a class of Manakov solution and in the place of interaction becomes a bifurcation between them.

Equal Elliptic Initial Polarizations of $50^\circ 08'$ for $\alpha_2 = 2$

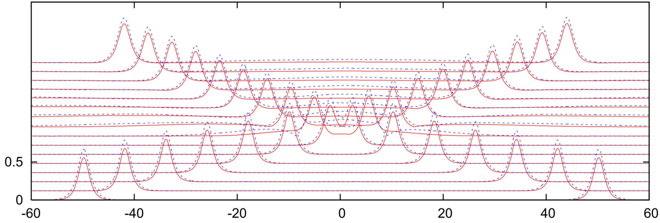

Figure: 5. $\delta_l = 0^\circ, \delta_r = 0^\circ$

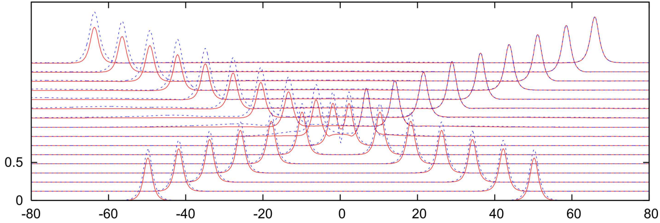

Figure: 6. $\delta_l = 0^\circ, \delta_r = 180^\circ$

Equal Elliptic Initial Polarizations of $50^\circ 08'$ for $\alpha_2 = 2$

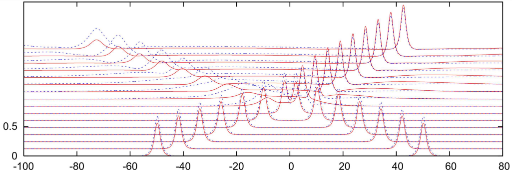

Figure: 7. $\delta_l = 0^\circ, \delta_r = 130^\circ$

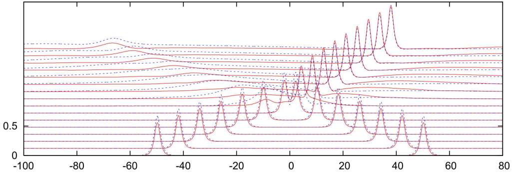

Figure: 8. $\delta_l = 0^\circ, \delta_r = 135^\circ$

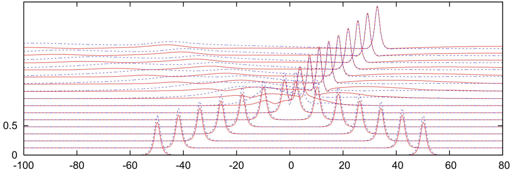

Figure: 9. $\delta_l = 0^\circ, \delta_r = 140^\circ$

Equal Elliptic Initial Polarizations of $50^\circ 08'$ for $\alpha_2 = 2$.

We choose $n_{l\psi} = n_{r\psi} = -1.5$, $n_{l\phi} = n_{r\phi} = -1.1$, $c_l = -c_r = 1$, $\alpha_1 = 0.75$, and focus on the effects of α_2 and $\vec{\delta}$.

One sees that the desynchronisations of the phases leads in the final stage to a superposition of two one-soliton solutions but with different polarizations from the initial polarization. Yet, for $\delta_r = 130^\circ \div 140^\circ$ one of the QPs loses its energy contributing it to the other QP during the collision and then virtually disappears: kind of energy trapping (Figs.7, 8, 9).

For $\delta_r = 180^\circ$ another interesting effect is seen, when the right outgoing QP is circularly polarized (Fig. 6).

All these interactions are accompanied by changes of phase speeds. The total polarization exhibits some kind of conservation.

Strong Nonlinear Interaction: $\alpha_2 = 10$

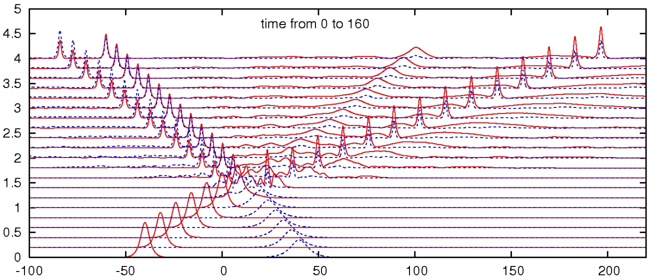

Figure: 10. $\alpha_2 = 10$, $c_l = 1$, $c_r = -0.5$.

Strong Nonlinear Interaction: $\alpha_2 = 10$

- Two new solitons are born after the collision.
- The kinetic energies of the newly created solitons correspond their phase speeds and masses, but the internal energy is very different for the different QP.
- the total energy of the QPs is radically different from the total energy of the initial wave profile. The differences are so drastic that the sum of QPs energies can even become negative. This means that the energy was carried away by the radiation.
- The predominant part of the energy is concentrated in the left and right forerunners because of the kinetic energies of the latter are very large. This is due to the fact that the forerunners propagate with very large phase speeds, and span large portions of the region.
- All four QPs have elliptic polarizations.
- Energy transformation is a specific trait of the coupled system considered here.

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23°44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

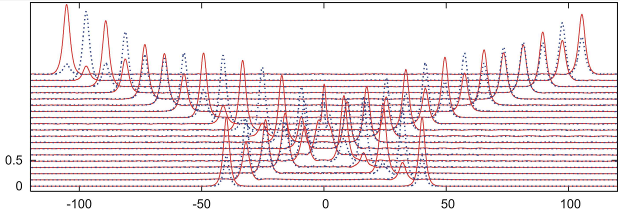

Figure: 11. $\delta_l = 0°$, $\delta_r = 0°$

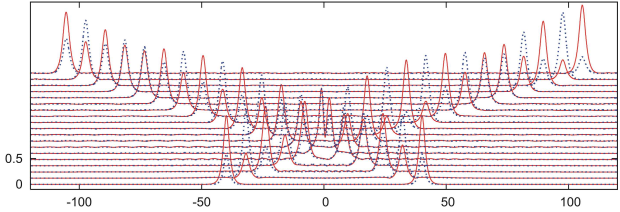

Figure: 12. $\delta_l = 0°$, $\delta_r = 90°$

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

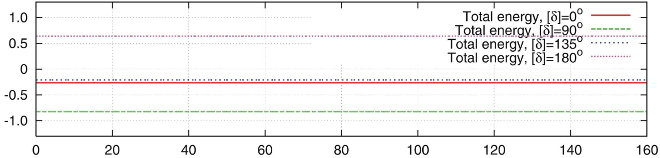

Figure: 13. Influence of the initial phase difference on the total energy:
$\delta_l = 0^\circ$, $\delta_r = 0^\circ$ – $E = -0.262$;
$\delta_l = 0^\circ$, $\delta_r = 90^\circ$ – $E = -0.821$;
$\delta_l = 0^\circ$, $\delta_r = 135^\circ$ – $E = -0.206$;
$\delta_l = 0^\circ$, $\delta_r = 180^\circ$ – $E = 0.640$

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

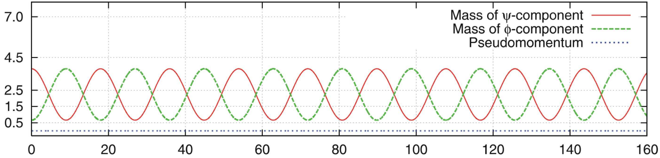

Figure: 14. $\delta_l = 0^\circ$, $\delta_r = 90^\circ$, $P = 10^{-3} \div 10^{-5}$

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

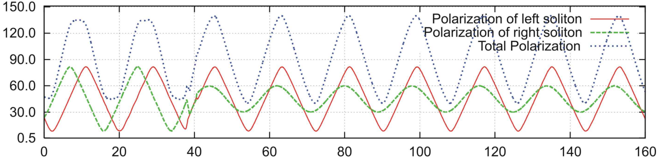

Figure: 15. $\delta_l = 0^\circ$, $\delta_r = 0^\circ$

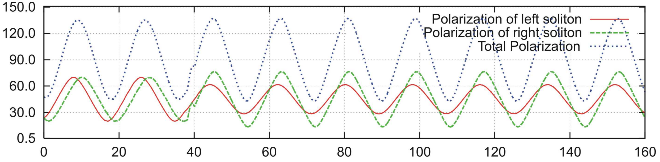

Figure: 16. $\delta_l = 0^\circ$, $\delta_r = 90^\circ$

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

- We have found that the phases of the components play an essential role on the full energy of QPs. The magnitude of the latter essentially depends on the choice of initial phase difference (Figure 13);
- The pseudomomentum is also conserved and it is trivial due to the symmetry (Figure 14);
- The individual masses, however, breathe together with the individual (rotational) polarizations. Their amplitude and period do not influenced from the initial phase difference (Figure 14);

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

- The total mass is constant while the total polarization oscillates and suffers a 'shock in polarization ' when QPs enter the collision. The polarization amplitude evidently depends on the initial phase difference (Figures 15,16);
- Due to the real linear coupling the polarization angle of QPs can change independently of the collision.

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Thanks

Thank you for your kind attention !