

# Paradigm and Long-Time Evolution of Localized Solutions of Wave Systems: Consistency versus Integrability

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- Problem Formulation
- Hamiltonian Representation
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- Results and discussion. Self-similar behaviour
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# Outline the problem

- Boussinesq's equation (BE) was the first model for the propagation of surface waves over shallow inviscid fluid layer. He found an analytical solution of his equation and thus proved that the balance between the steepening effect of the nonlinearity and the flattening effect of the dispersion maintains the shape of the wave. This discovery can be properly termed 'Boussinesq Paradigm.'
- Apart from the significance for the shallow water flows, this paradigm is very important for understanding the particle-like behavior of nonlinear localized waves. As it should have been expected, most of the physical systems are not fully integrable (even in one spatial dimension) and only a numerical approach can lead to unearthing the pertinent physical mechanisms of the interactions.

The equation derived by Boussinesq in 1871 r., called later “Original Boussinesq Equation”

$$\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2}{\partial x^2} \left[ (gH)h + \frac{3g}{2}h^2 + \frac{gH^3}{3} \frac{\partial^2 h}{\partial x^2} \right] \quad (1)$$

is fully integrable but incorrect in sense of Hadamard.  
Introducing the effect of the surface tension Kortevæg-de Vries  
derive the so-called Proper Boussinesq Equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2} \left[ u - \alpha u^2 + \beta \frac{\partial^2 u}{\partial x^2} \right] \quad (2)$$

that is correct in the sense of Hadamard only when  $\beta < 0$ . The  
above equation is “improved” many times – ‘good’ or ‘proper’  
Boussinesq Equation.

# Outline of the Problem

A different approach to removing the incorrectness is by changing the spatial fourth derivative to a mixed fourth derivative, which resulted into an equation known nowadays as the Regularized Long Wave Equation (RLWE) or Benjamin–Bona–Mahony equation (BBME):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2} \left[ u - \alpha u^2 + \beta \frac{\partial^2 u}{\partial t^2} \right]. \quad (3)$$

In fact, the mixed derivative occurs naturally in Boussinesq derivation (see Eq. (1)), and was changed by Boussinesq to a fourth spatial derivative under an assumption that  $\frac{\partial}{\partial t} \approx \frac{\partial}{\partial x}$ , which is currently known as the 'Linear Impedance Relation' (or LIA). The LIA has produced innumerable instances of unphysical results.

# Generalized Wave Equation

The overwhelming majority of the analytical and numerical results obtained so far are for one spatial dimension, while in multidimension, much less is possible to achieve analytically, and almost nothing is known about the unsteady solutions that involve interactions, especially when the full-fledged Boussinesq equations are involved. As shown in Christov(2001), the consistent implementation of the Boussinesq method yields the following Generalized Wave Equation (GWE) for  $f = \phi(x, y, z = 0; t)$ :

$$f_{tt} + 2\beta \nabla f \cdot \nabla f_t + \beta f_t \Delta f + \frac{3\beta^2}{2} (\nabla f)^2 \Delta f - \Delta f + \frac{\beta}{6} \Delta^2 f - \frac{\beta}{2} (\Delta f)_{tt} = 0$$

with Hamiltonian density

$$\mathcal{H} = \frac{1}{2} \left[ f_t^2 + (\nabla f)^2 - \frac{1}{4} \beta^2 (\nabla f)^4 + \frac{1}{6} \beta (\nabla f)^2 + \frac{1}{2} \beta (\nabla f_t)^2 \right].$$

# Generalized Wave Equation

This equation is the most rigorous amplitude equation that can be derived for the surface waves over an inviscid shallow layer, when the length of the wave is considered large in comparison with the depth of the layer. It was derived only in 2001. Besides it a plethora of different inconsistent Boussinesq equations are still vigorously investigated. The most popular are the versions that contain a quadratic nonlinearity which are useful from the paradigmatic point of view.

# Problem Formulation: Equations

Consider the equation when the velocity potential and surface elevation do not depend on the coordinate  $y$ . Having in mind that  $f$  is the velocity potential on the bottom of the layer we introduce a vertical velocity  $u = f_x$  and an auxiliary function  $q$ . Then we obtain so-called Dispersive Wave System is a progenitor of the different 1d Boussinesq equations:

$$u_t + \frac{\alpha}{2}(u^2)_x = q_{xx},$$
$$q_t + \alpha u q_x = u - \beta_2 u_{xx} + \beta_1 u_{tt},$$

where:  $\beta_1$  and  $\beta_2$  are two dispersion coefficients,  $\beta_1 = 3\beta_2 = \beta$ ;  $\alpha = \beta$  is an amplitude parameter.



# Problem Formulation: Boundary Conditions

The boundary conditions are

$$u|_{x=-L_1, L_2} = 0 \quad q_x|_{x=-L_1, L_2} = 0.$$

When the interval  $[-L_1, L_2]$  is finite they provide the conservation of the total energy.

# Problem Formulation: Choice of Initial Conditions

The initial condition is a superposition of two solitary waves traveling in opposite directions with phase velocities  $c_l$  and  $c_r$

$$u(x, t = 0) = \frac{a \operatorname{sgn}(c)}{\frac{|c|-1}{2} + \cosh^2[b(x - X - ct)]}$$

where  $a = \frac{c^2-1}{\alpha}$ ,  $b = \sqrt{\frac{c^2-1}{2(\beta_1 c^2 - \beta_2)}}$ . The sech-like solutions exist in two domains – subcritical (subsonic):  $0 < c < 1/\sqrt{3}$  and supercritical (supersonic):  $c > 1$ . The physically relevant are only supercritical modes because the subcritical do not admit long waves for small  $\beta$  and are out of our interest.

# Problem Formulation: Conservation Laws

Define “mass”,  $M$ , (pseudo)momentum,  $P$ , and (pseudo)energy,  $E$ :

$$M = \int_{-L_1}^{L_2} u dx, \quad P = \int_{-L_1}^{L_2} (uq_x - \frac{\beta}{2} u_t u_x) dx,$$
$$E = \frac{1}{2} \int_{-L_1}^{L_2} \left( u^2 + q_x^2 + \frac{\beta}{2} u^3 + \frac{\beta}{2} u_t^2 + \frac{\beta}{6} u_x^2 \right) dx,$$

Here  $-L_1$  and  $L_2$  are the left end and the right end of the interval under consideration. The following conservation/balance laws hold, namely

$$\frac{dM}{dt} = 0, \quad \frac{dP}{dt} = -\frac{\beta}{2} u_x^2 \Big|_{x=-L_1}^{x=L_2}, \quad \frac{dE}{dt} = 0.$$

We construct a staggered conservative scheme for the Galilean invariant case treated here. We introduce a regular mesh in the interval  $[-L_1, L_2]$ ,  $x_i = -L_1 + (i - 1)h$ ,  $h = (L_1 + L_2)/(N - 1)$ ,  $N$  is the total number of grid points. We use a simplest linearization combined with an internal iteration (referred to by the composite superscript  $k$ ). It appears to be robust enough and economical.

$$\begin{aligned}
 \frac{u_i^{n+1,k} - u_i^n}{\tau} &= \frac{q_{i+1}^{n+1/2,k} - 2q_i^{n+1/2,k} + q_{i-1}^{n+1/2,k}}{h^2} \\
 &\quad - \frac{\beta}{8h} \left[ (u_{i+1}^{n+1,k-1})^2 - (u_{i-1}^{n+1,k-1})^2 + (u_{i+1}^n)^2 - (u_{i-1}^n)^2 \right] \\
 \frac{q_i^{n+1/2,k} - q_i^{n-1/2}}{\tau} &= -\frac{\beta}{8h} \left( q_{i+1}^{n+1/2,k-1} - q_{i-1}^{n+1/2,k} + q_{i+1}^{n-1/2} - q_{i-1}^{n-1/2} \right) \\
 &\quad \times (u_i^{n+1,k} + u_i^{n-1}) \\
 &\quad - \frac{\beta}{12h^2} \left[ (u_{i+1}^{n+1,k} - 2u_i^{n+1,k} + u_{i-1}^{n+1,k}) + (u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}) \right] \\
 &\quad + \frac{\beta}{2} \frac{u_i^{n+1,k} - 2u_i^n + u_i^{n-1}}{\tau^2} + \frac{u_i^{n+1,k} + u_i^{n-1}}{2}.
 \end{aligned}$$

We prove that the above approximation secures the conservation of energy on difference level for arbitrary potential  $U(u)$ , namely the difference approximations of the mass and energy are conserved by the difference scheme in the sense that  $M^{n+1} = M^n$  and  $E^{n+1/2} = E^{n-1/2}$ .

The Airy function is defined by the improper integral

$$\text{Ai}(x) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{t^3}{3} + xt\right) dt.$$

It is oscillatory in the negative part of  $x$  and decays exponentially in the positive part of  $x$ . The asymptotic behavior in the negative direction is

$$\text{Ai}(-x) \sim \frac{\sin\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}x^{1/4}}.$$

Head-on collision.  $c_l = 0.22$ ,  $c_r = -1.2$ ,  $\beta_1 = 3$ ,  $\beta_2 = 1$ ,  
 $\alpha = -3$ ,  $\tau = 0.05$ ,  $h = 0.1$ ,  $M = -5.02002$ ,  $E = 3.584$

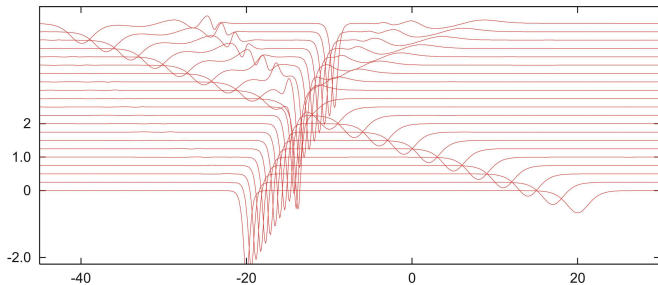


Рис.: 1. Short-times evolution. Forming of accompanying excitations



Head-on collision.  $c_l = 0.22$ ,  $c_r = -1.2$ ,  $\beta_1 = 3$ ,  $\beta_2 = 1$ ,  
 $\alpha = -3$ ,  $\tau = 0.05$ ,  $h = 0.1$ ,  $M = -5.02002$ ,  $E = 3.584$

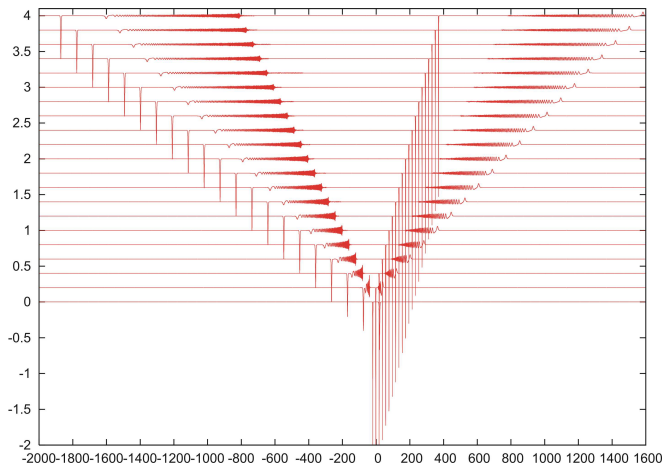


Рис.: 2. Long-times evolution.

Head-on collision.  $c_l = 0.22$ ,  $c_r = -1.2$ ,  $\beta_1 = 3$ ,  $\beta_2 = 1$ ,  
 $\alpha = -3$ ,  $\tau = 0.05$ ,  $h = 0.1$ ,  $M = -5.02002$ ,  $E = 3.584$

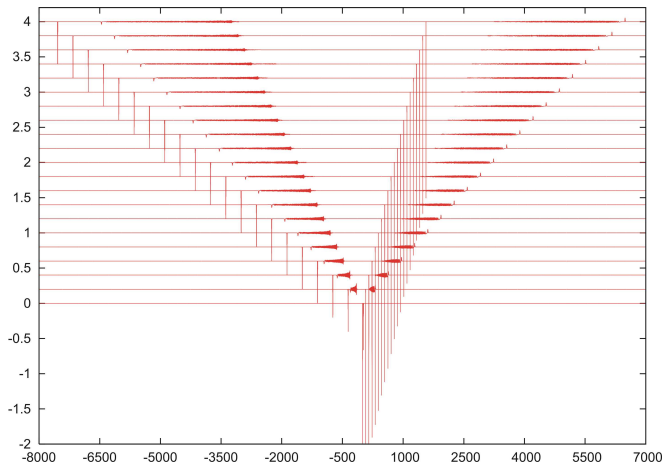


Рис.: 3. Very long-times evolution.

Head-on collision.  $c_l = 0.22$ ,  $c_r = -1.2$ ,  $\beta_1 = 3$ ,  $\beta_2 = 1$ ,  
 $\alpha = -3$ ,  $\tau = 0.05$ ,  $h = 0.1$ ,  $M = -5.02002$ ,  $E = 3.584$

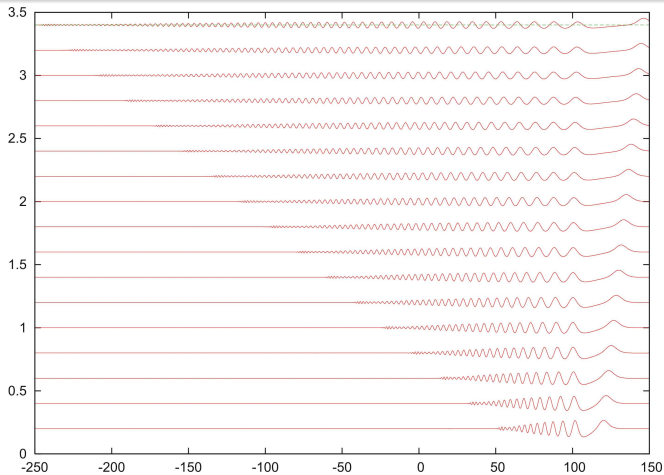


Рис.: 4. The evolution of the left excitation (shifted).

Head-on collision.  $c_l = 0.22$ ,  $c_r = -1.2$ ,  $\beta_1 = 3$ ,  $\beta_2 = 1$ ,  
 $\alpha = -3$ ,  $\tau = 0.05$ ,  $h = 0.1$ ,  $M = -5.02002$ ,  $E = 3.584$

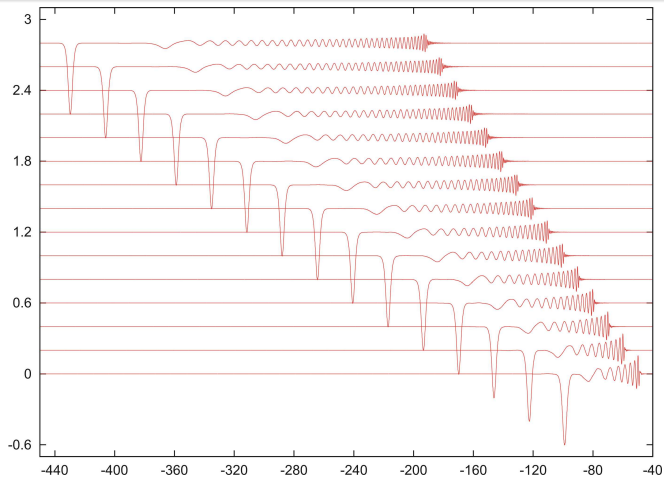


Рис.: 5. The right left-going soliton with its trail. Middle-times evolution.

Head-on collision.  $c_l = 0.22$ ,  $c_r = -1.2$ ,  $\beta_1 = 3$ ,  $\beta_2 = 1$ ,  
 $\alpha = -3$ ,  $\tau = 0.05$ ,  $h = 0.1$ ,  $M = -5.02002$ ,  $E = 3.584$

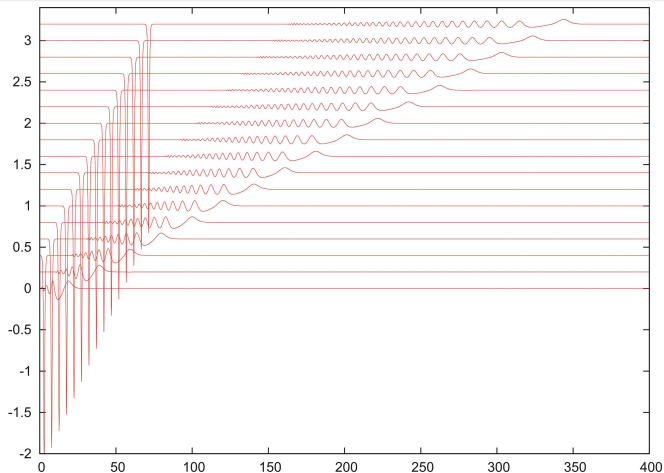


Рис.: 6. The left right-going soliton with its trail. Middle-times evolution.

# Problem Formulation: Equations

CNLSE is system of nonlinearly coupled Schrödinger equations (called the Gross-Pitaevskii or Manakov-type system):

$$\begin{aligned}i\psi_t &= \beta\psi_{xx} + [\alpha_1|\psi|^2 + (\alpha_1 + 2\alpha_2)|\phi|^2]\psi + \Gamma\phi, \\i\phi_t &= \beta\phi_{xx} + [\alpha_1|\phi|^2 + (\alpha_1 + 2\alpha_2)|\psi|^2]\phi + \Gamma\psi,\end{aligned}\tag{4}$$

where:

$\beta$  is the dispersion coefficient;

$\alpha_1$  describes the self-focusing of a signal for pulses in birefringent media;

$\Gamma = \Gamma_r + i\Gamma_i$  is the magnitude of linear coupling.  $\Gamma_r$  governs the oscillations between states termed as breathing solitons, while  $\Gamma_i$  describes the gain behavior of soliton solutions.

$\alpha_2$  (called cross-modulation parameter) governs the nonlinear coupling between the equations.

# Problem Formulation: Equations

When  $\alpha_2 = 0$ , no nonlinear coupling is present despite the fact that “cross-terms” proportional to  $\alpha_1$  appear in the equations. For  $\alpha_2 = 0$ , the solutions of the two equations are identical,  $\psi \equiv \phi$ , and equal to the solution of single NLSE with nonlinearity coefficient  $\alpha = 2\alpha_1$ .

For  $\Gamma = 0$ , CNSE is alternately called the Gross-Pitaevskii equation or an equation of Manakov type. It was derived independently by Gross and Pitaevskii to describe the behavior of Bose-Einstein condensates as well as optic pulse propagation. It was solved analytically for the case  $\alpha_2 = 0$ ,  $\beta = \frac{1}{2}$  by Manakov via Inverse Scattering Transform who generalized an earlier result by Zakharov & Shabat for the scalar cubic NLSE (i.e. Eq.(2- $\psi$ ) with  $\varphi(x, t) = 0$ ). Recently, Chow, Nakkeeran, and Malomed studied periodic waves in optic fibers using a version of CNSE with  $\Gamma \neq 0$ .

The single NLSE has the form

$$i\psi_t + \beta\psi_{xx} + \alpha|\psi|^2\psi = 0 \quad (5)$$

As far as the applications in nonlinear optics are concerned, the above equation describes the single-mode wave propagation in a fiber. Depending on the sign of  $\alpha$  (7) admits single and multiple sech-solutions (*bright solitons*), as well as tanh-profile, or *dark soliton* solutions. In this paper we concentrate on the case of bright solitons.

A dynamical system with infinite number of conservation laws (integrals) is called integrable. NSE is an integrable dynamical system. CNSE is a nonintegrable generalisation of NSE.



# Problem Formulation: Conservation Laws

Define “mass”,  $M$ , (pseudo)momentum,  $P$ , and energy,  $E$ :

$$M \stackrel{\text{def}}{=} \frac{1}{2\beta} \int_{-L_1}^{L_2} (|\psi|^2 + |\phi|^2) dx, \quad P \stackrel{\text{def}}{=} - \int_{-L_1}^{L_2} \mathcal{I}(\psi \bar{\psi}_x + \phi \bar{\phi}_x) dx,$$
$$E \stackrel{\text{def}}{=} \int_{-L_1}^{L_2} \mathcal{H} dx, \quad \text{where} \quad (6)$$
$$\mathcal{H} \stackrel{\text{def}}{=} \beta (|\psi_x|^2 + |\phi_x|^2) - \frac{1}{2} \alpha_1 (|\psi|^4 + |\phi|^4) - (\alpha_1 + 2\alpha_2) (|\phi|^2 |\psi|^2) - 2\Gamma [\Re(\bar{\psi}\phi)]$$

is the Hamiltonian density of the system. Here  $-L_1$  and  $L_2$  are the left end and the right end of the interval under consideration.

The following conservation/balance laws hold, namely

$$\frac{dM}{dt} = 0, \quad \frac{dP}{dt} = \mathcal{H}|_{x=L_2} - \mathcal{H}|_{x=-L_1}, \quad \frac{dE}{dt} = 0, \quad (7)$$

# Problem Formulation: Choice of Initial Conditions

We concern ourselves with the soliton solutions whose modulation amplitude is of general form (non-sech) and which are localized envelopes on a propagating carrier wave. This allows us to play various scenario of initial polarization. Unfortunately in sech-case the initial polarization can be only linear. Then we assume that for each of the functions  $\phi, \psi$  the initial condition is of the form of a single propagating soliton, namely

$$\begin{aligned} \begin{cases} \psi(x, t) \\ \phi(x, t) \end{cases} &= \begin{cases} A^\psi \\ A^\phi \end{cases} \operatorname{sech} [b(x - X - ct)] \exp \left\{ i \left[ \frac{c}{2\beta} (x - X) - nt \right] \right\}. \\ b^2 &= \frac{1}{\beta} \left( n + \frac{c^2}{4\beta} \right), \quad A = b \sqrt{\frac{2\beta}{\alpha_1}}, \quad u_c = \frac{2n\beta}{c}, \end{aligned} \quad (8)$$

where  $X$  is the spatial position (center of soliton),  $c$  is the phase speed,  $n$  is the carrier frequency, and  $b^{-1}$  – a measure of the support of the localized wave.

# Problem Formulation: Choice of Initial Conditions

We assume that for each of the functions  $\phi, \psi$  the initial condition has the general type

$$\begin{aligned}\psi &= A_\psi(x + X - c_\psi t) \exp \left\{ i \left[ n_\psi t - \frac{1}{2} c_\psi (x - X - c_\psi t) + \delta_\psi \right] \right\} \\ \phi &= A_\phi(x + X - c_\phi t) \exp \left\{ i \left[ n_\phi t - \frac{1}{2} c_\phi (x - X - c_\phi t) + \delta_\phi \right] \right\},\end{aligned}\tag{9}$$

where  $c_\psi, c_\phi$  are the phase speeds and  $X$ 's are the initial positions of the centers of the solitons;  $n_\psi, n_\phi$  are the carrier frequencies for the two components;  $\delta_\psi$  and  $\delta_\phi$  are the phases of the two components. Note that the phase speed must be the same for the two components  $\psi$  and  $\phi$ . If they propagate with different phase speeds, after some time the two components will be in two different positions in space, and will no longer form a single structure. For the envelopes  $(A_\psi, A_\phi)$ ,  $\theta \equiv \arctan(\max |\phi| / \max |\psi|)$  is a polarization angle.

# Problem Formulation: Generation of Initial Conditions

Generally the carrier frequencies for the two components  $n_\psi \neq n_\phi$  – elliptic polarization. When  $n_\psi = n_\phi$  – circular polarization. If one of them vanishes – linear polarization (sech soliton,  $\theta = 0; 90^\circ$ ). In general case the initial condition is solution of the following system of nonlinear conjugated equations

$$\begin{aligned} A''_\psi + \left(n_\psi + \frac{1}{4}c_\psi^2\right) A_\psi + [\alpha_1 A_\psi^2 + (\alpha_1 + 2\alpha_2)A_\phi^2] A_\psi &= 0 \\ A''_\phi + \left(n_\phi + \frac{1}{4}c_\phi^2\right) A_\phi + [\alpha_1 A_\phi^2 + (\alpha_1 + 2\alpha_2)A_\psi^2] A_\phi &= 0. \end{aligned} \quad (10)$$

The system admits bifurcation solutions since the trivial solution obviously is always present.

# Initial Conditions and Initial Polarization

We solve the auxiliary conjugated system (12) with asymptotic boundary conditions using Newton method and the initial approximation of sought nontrivial solution is sech-function. The final solution, however, is not obligatory sech-function. It is a two-component polarized soliton solution.

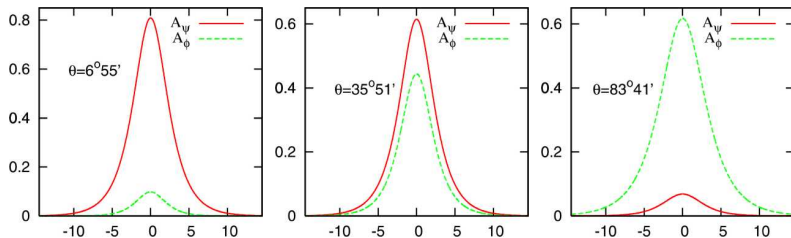


Рис.: 7. Amplitudes  $A_\psi$  and  $A_\phi$  for  $c_l = -c_r = 1$ ,  $\alpha_1 = 0.75$ ,  $\alpha_2 = 0.2$ .  
Left:  $n_\psi = -0.68$ ; middle:  $n_\psi = -0.55$ ; right:  $n_\psi = -0.395$ .

# Initial Conditions and Initial Phase Difference

Another dimension of complexity is introduced by the phases of the different components. The initial difference in phases can have a profound influence on the polarizations of the solitons after the interaction and the magnitude of the full energy. The relative shift of real and imaginary parts is what matters in this case.

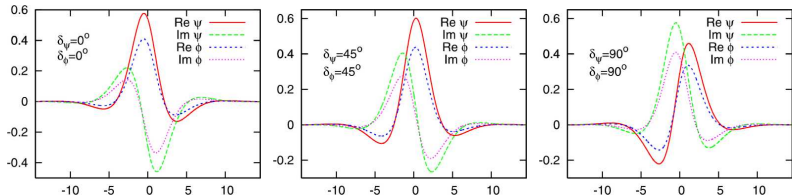


Рис.: 8. Real and imaginary parts of the amplitudes from the case shown in the middle panel of Figure 7 and the dependence on phase angle.

# Problem Formulation: Initial Conditions

After completing the initial conditions our aim is to understand better the influence of the initial polarization and initial phase difference on the particle-like behavior of the localized waves. We call a localized wave a quasi-particle (QP) if it survives the collision with other QPs (or some other kind of interactions) without losing its identity.

# Linearly Coupled Problem Formulation: Equations and Initial Conditions

For the linearly coupled system of NLSE the magnitude of linear coupling  $\Gamma_r$  generates breathing the solitons although noninteracting The initial conditions must be

$$\Psi = \psi \cos(\Gamma t) + i\phi \sin(\Gamma t), \quad \Phi = \phi \cos(\Gamma t) + i\psi \sin(\Gamma t), \quad (11)$$

where  $\phi$  and  $\psi$  are assumed to be sech-solutions of (6) for  $\alpha_2 = 0$ . Hence (6) posses solutions, which are combinations of interacting solitons oscillating with frequency  $\Gamma_r$  and their motion gives rise to the so-called rotational polarization.



# Problem Formulation: Conservation Laws

In all considered cases we found that a conservation of the total polarization is present. Only for the linearly CNLSE ( $\alpha_2 = 0$ ) the total polarizations breathe with an amplitude evidently depending on the initial phase difference but is conserved within one full period of the breathing.

$\delta_r - \delta_l$	$\theta_l^i$	$\theta_r^i$	$\theta_l^i + \theta_r^i$	$\theta_l^f$	$\theta_r^f$	$\theta_l^f + \theta_r^f$
45°	45°	45°	90°	33°48'	56°12'	90°
90°	45°	45°	90°	24°06'	65°54'	90°
0°	20°	20°	40°	20°00'	20°00'	40°
90°	20°	20°	40°	28°48'	2°02'	30°50'
0°	36°	36°	72°	36°00'	36°00'	72°
90°	36°	36°	72°	53°00'	13°20'	66°20'
0°	10°	80°	90°	21°05'	68°54'	89°59'
90°	10°	80°	90°	9°27'	80°30'	89°57'

To solve the main problem numerically, we use an implicit conservative scheme in complex arithmetic.

$$\begin{aligned} i \frac{\psi_i^{n+1} - \psi_i^n}{\tau} &= \frac{\beta}{2h^2} (\psi_{i-1}^{n+1} - 2\psi_i^{n+1} + \psi_{i+1}^{n+1} + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n) \\ &+ \frac{\psi_i^{n+1} + \psi_i^n}{4} \left[ \alpha_1 (|\psi_i^{n+1}|^2 + |\psi_i^n|^2) + (\alpha_1 + 2\alpha_2) (|\phi_i^{n+1}|^2 + |\phi_i^n|^2) \right] \\ &- \frac{1}{2} \Gamma (\phi_i^{n+1} + \phi_i^n), \end{aligned}$$

$$\begin{aligned} i \frac{\phi_i^{n+1} - \phi_i^n}{\tau} &= \frac{\beta}{2h^2} (\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1} + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n) \\ &+ \frac{\phi_i^{n+1} + \phi_i^n}{4} \left[ \alpha_1 (|\phi_i^{n+1}|^2 + |\phi_i^n|^2) + (\alpha_1 + 2\alpha_2) (|\psi_i^{n+1}|^2 + |\psi_i^n|^2) \right] \\ &- \frac{1}{2} \Gamma (\psi_i^{n+1} + \psi_i^n). \end{aligned}$$

$$\begin{aligned}
 i \frac{\psi_i^{n+1,k+1} - \psi_i^n}{\tau} &= \frac{\beta}{2h^2} \left( \psi_{i-1}^{n+1,k+1} - 2\psi_i^{n+1,k+1} + \psi_{i+1}^{n+1,k+1} \right. \\
 &\quad \left. + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n \right) \\
 &+ \frac{\psi_i^{n+1,k} + \psi_i^n}{4} \left[ \alpha_1 (|\psi_i^{n+1,k+1}| |\psi_i^{n+1,k}| + |\psi_i^n|^2) \right. \\
 &\quad \left. + (\alpha_1 + 2\alpha_2) (|\phi_i^{n+1,k+1}| |\phi_i^{n+1,k}| + |\phi_i^n|^2) \right] \\
 i \frac{\phi_i^{n+1,k+1} - \phi_i^n}{\tau} &= \frac{\beta}{2h^2} \left( \phi_{i-1}^{n+1,k+1} - 2\phi_i^{n+1,k+1} + \phi_{i+1}^{n+1,k+1} \right. \\
 &\quad \left. + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n \right) \\
 &+ \frac{\phi_i^{n+1,k} + \phi_i^n}{4} \left[ \alpha_1 (|\phi_i^{n+1,k+1}| |\phi_i^{n+1,k}| + |\phi_i^n|^2) \right. \\
 &\quad \left. + (\alpha_1 + 2\alpha_2) (|\psi_i^{n+1,k+1}| |\psi_i^{n+1,k}| + |\psi_i^n|^2) \right].
 \end{aligned}$$

# Numerical Method: Conservation Properties

It is not only convergent (consistent and stable), but also conserves mass and energy, i.e., there exist discrete analogs for (9), which arise from the scheme.

$$M^n = \sum_{i=2}^{N-1} (|\psi_i^n|^2 + |\phi_i^n|^2) = \text{const},$$

$$E^n = \sum_{i=2}^{N-1} \frac{-\beta}{2h^2} (|\psi_{i+1}^n - \psi_i^n|^2 + |\phi_{i+1}^n - \phi_i^n|^2) + \frac{\alpha_1}{4} (|\psi_i^n|^4 + |\phi_i^n|^4) \\ + \frac{1}{2}(\alpha_1 + 2\alpha_2) (|\psi_i^n|^2 |\phi_i^n|^2) - \Gamma \Re[\bar{\phi}_i^n \psi_i^n] = \text{const}, \\ \text{for all } n \geq 0.$$

These values are kept constant during the time stepping. The above scheme is of Crank-Nicolson type for the linear terms and we employ internal iterations to achieve implicit approximation of the nonlinear terms, i.e., we use its linearized implementation.

# Results and Discussion: Initial Circular Polarizations of $45^\circ$ , $\alpha_2 = 0$

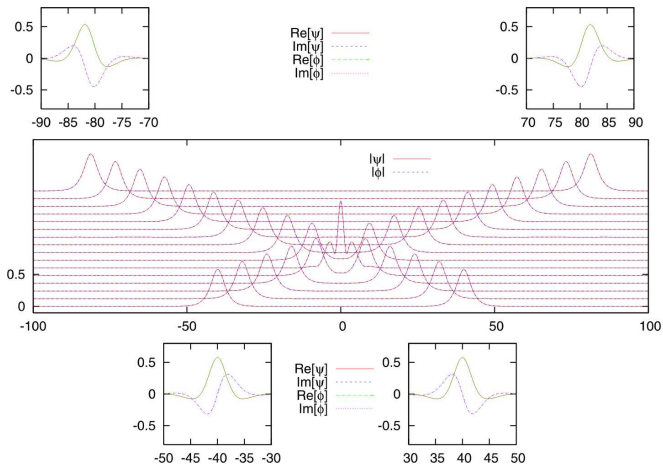


Рис.: 9.  $\delta_l = 0^\circ$ ,  $\delta_r = 0^\circ$

# Results and Discussion: Initial Circular Polarizations of $45^\circ$ , $\alpha_2 = 0$

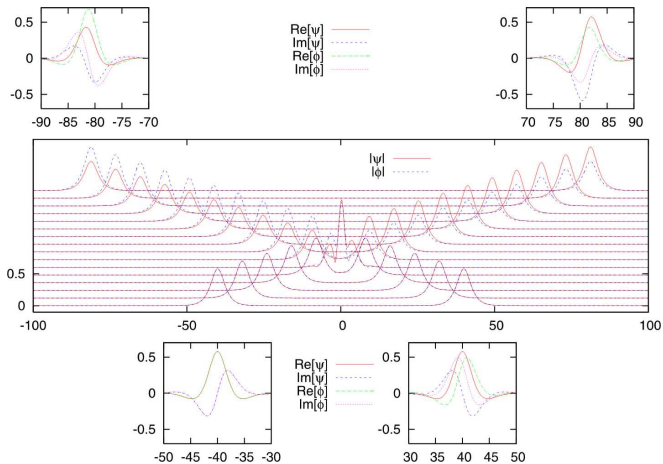


Рис.: 10.  $\delta_l = 0^\circ$ ,  $\delta_r = 45^\circ$

# Results and Discussion: Initial Circular Polarizations of $45^\circ$ , $\alpha_2 = 0$

- When both of QPs have zero phases (Fig. 9), the interaction perfectly follows the analytical Manakov two-soliton solution.
- The surprise comes in Fig. 10 where is presented an interaction of two QPs, the right one of which has a nonzero phase  $\delta_r = 45^\circ$ . After the interaction, the two QPs become different Manakov solitons than the original two that entered the collision. The outgoing QPs have polarizations  $33^\circ 48'$  and  $56^\circ 12'$ . Something that can be called a 'shock in polarization' takes place. All the solutions are perfectly smooth, but because the property called polarization cannot be defined in the cross-section of interaction and for this reason, it appears as undergoing a shock.

# Results and Discussion: Initial Circular Polarizations of $45^\circ$ , $\alpha_2 = 0$

Here is to be mentioned that when rescaled the moduli of  $\psi$  and  $\phi$  from Fig. 10 perfectly match each other which means that the resulting solitons have circular polarization (see left panel of Fig. 11 below). The Manakov solution is not unique. There exists a class of Manakov solution and in the place of interaction becomes a bifurcation between them.



# Results and Discussion: Initial Circular Polarization and Nonuniqueness of the Manakov Solution

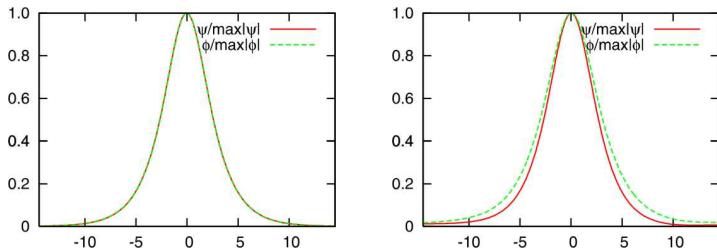


Рис.: 11. Circular polarization (left); Elliptical Polarization (right).

# Equal Elliptic Initial Polarizations of $50^{\circ}08'$ for $\alpha_2 = 2$

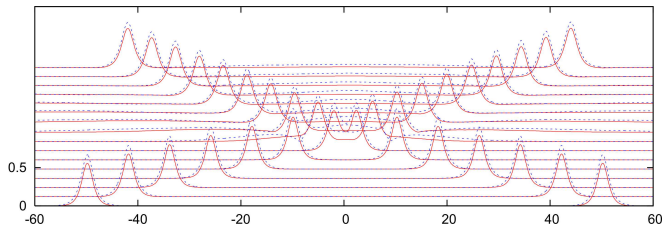


Рис.: 12.  $\delta_l = 0^{\circ}, \delta_r = 0^{\circ}$

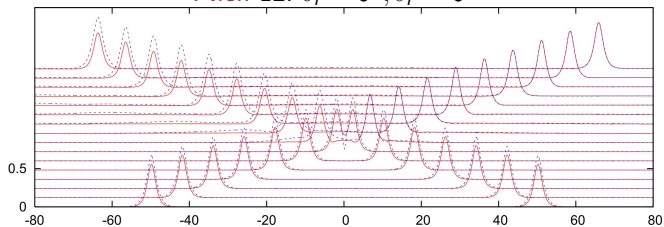


Рис.: 13.  $\delta_l = 0^{\circ}, \delta_r = 180^{\circ}$

# Equal Elliptic Initial Polarizations of $50^{\circ}08'$ for $\alpha_2 = 2$

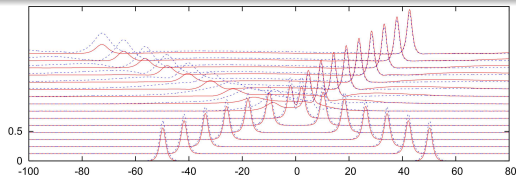


Рис.: 14.  $\delta_l = 0^\circ, \delta_r = 130^\circ$

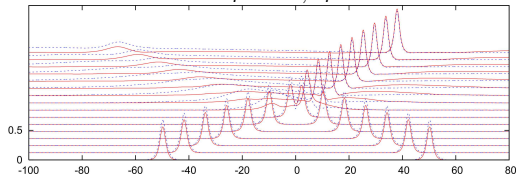


Рис.: 15.  $\delta_l = 0^\circ, \delta_r = 135^\circ$

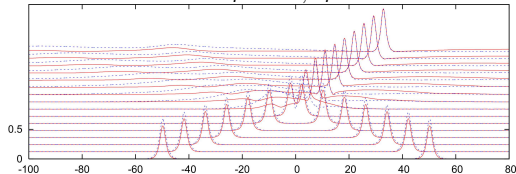


Рис.: 16.  $\delta_l = 0^\circ, \delta_r = 140^\circ$

## Equal Elliptic Initial Polarizations of $50^{\circ}08'$ for $\alpha_2 = 2$ .

We choose  $n_{l\psi} = n_{r\psi} = -1.5$ ,  $n_{l\phi} = n_{r\phi} = -1.1$ ,  $c_l = -c_r = 1$ ,  $\alpha_1 = 0.75$ , and focus on the effects of  $\alpha_2$  and  $\vec{\delta}$ .

One sees that the desynchronisations of the phases leads in the final stage to a superposition of two one-soliton solutions but with different polarizations from the initial polarization. Yet, for  $\delta_r = 130^{\circ} \div 140^{\circ}$  one of the QPs loses its energy contributing it to the other QP during the collision and then virtually disappears: kind of energy trapping (Figs.14, 15, 16).

For  $\delta_r = 180^{\circ}$  another interesting effect is seen, when the right outgoing QP is circularly polarized (Fig. 13).

All these interactions are accompanied by changes of phase speeds. The total polarization exhibits some kind of conservation.

# Strong Nonlinear Interaction: $\alpha_2 = 10$

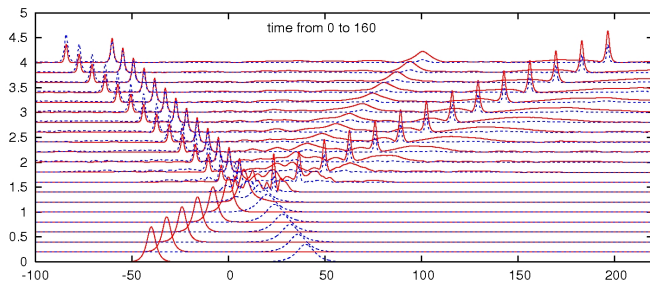


Рис.: 17.  $\alpha_2 = 10$ ,  $c_l = 1$ ,  $c_r = -0.5$ .

# Strong Nonlinear Interaction: $\alpha_2 = 10$

- Two new solitons are born after the collision.
- The kinetic energies of the newly created solitons correspond their phase speeds and masses, but the internal energy is very different for the different QP.
- the total energy of the QPs is radically different from the total energy of the initial wave profile. The differences are so drastic that the sum of QPs energies can even become negative. This means that the energy was carried away by the radiation.
- The predominant part of the energy is concentrated in the left and right forerunners because of the kinetic energies of the latter are very large. This is due to the fact that the forerunners propagate with very large phase speeds, and span large portions of the region.
- All four QPs have elliptic polarizations.
- Energy transformation is a specific trait of the coupled system considered here.

Linear Coupling: Initial Elliptic Polarizations:  $\theta = 23^\circ 44'$ ,  
 $\alpha_2 = 0$ ,  $c_l = -c_r = 1$ ,  $n_{l\psi} = n_{r\psi} = -1.1$ ,  
 $n_{l\phi} = n_{r\phi} = -1.5$ ,  $\Gamma = 0.175$

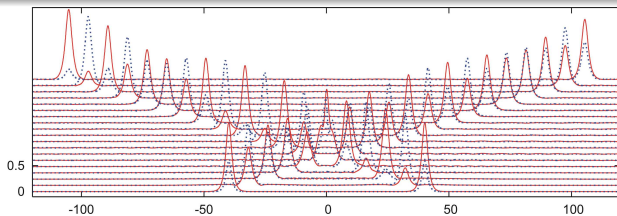


Рис.: 18.  $\delta_l = 0^\circ$ ,  $\delta_r = 0^\circ$

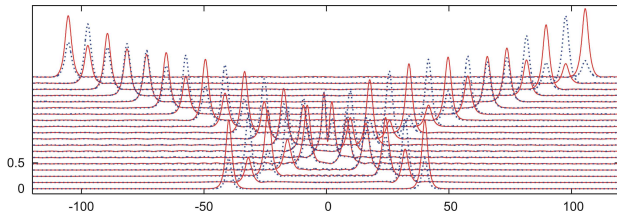


Рис.: 19.  $\delta_l = 0^\circ$ ,  $\delta_r = 90^\circ$

Linear Coupling: Initial Elliptic Polarizations:  $\theta = 23^{\circ}44'$ ,

$\alpha_2 = 0$ ,  $c_l = -c_r = 1$ ,  $n_{l\psi} = n_{r\psi} = -1.1$ ,

$n_{l\phi} = n_{r\phi} = -1.5$ ,  $\Gamma = 0.175$

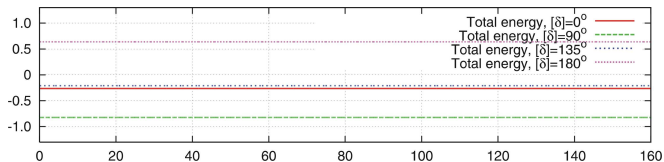


Рис.: 20. Influence of the initial phase difference on the total energy:

$$\delta_l = 0^{\circ}, \delta_r = 0^{\circ} - E = -0.262;$$

$$\delta_l = 0^{\circ}, \delta_r = 90^{\circ} - E = -0.821;$$

$$\delta_l = 0^{\circ}, \delta_r = 135^{\circ} - E = -0.206;$$

$$\delta_l = 0^{\circ}, \delta_r = 180^{\circ} - E = 0.640$$



Linear Coupling: Initial Elliptic Polarizations:  $\theta = 23^\circ 44'$ ,

$\alpha_2 = 0$ ,  $c_l = -c_r = 1$ ,  $n_{l\psi} = n_{r\psi} = -1.1$ ,

$n_{l\phi} = n_{r\phi} = -1.5$ ,  $\Gamma = 0.175$

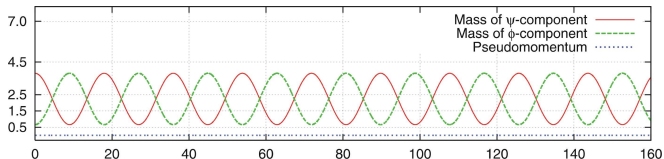


Рис.: 21.  $\delta_l = 0^\circ$ ,  $\delta_r = 90^\circ$ ,  $P = 10^{-3} \div 10^{-5}$

Linear Coupling: Initial Elliptic Polarizations:  $\theta = 23^\circ 44'$ ,

$\alpha_2 = 0$ ,  $c_l = -c_r = 1$ ,  $n_{l\psi} = n_{r\psi} = -1.1$ ,

$n_{l\phi} = n_{r\phi} = -1.5$ ,  $\Gamma = 0.175$

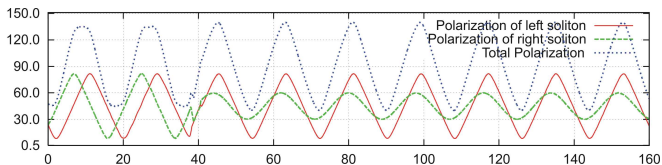


Рис.: 22.  $\delta_l = 0^\circ$ ,  $\delta_r = 0^\circ$

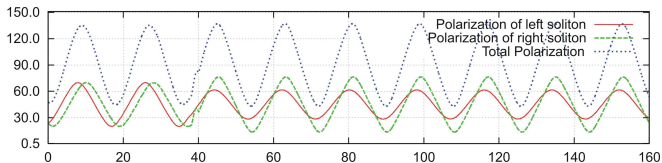
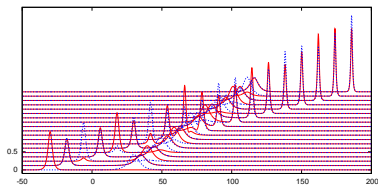
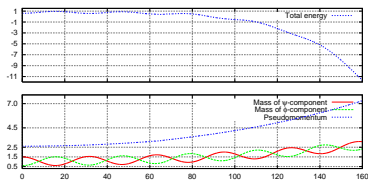


Рис.: 23.  $\delta_l = 0^\circ$ ,  $\delta_r = 90^\circ$

Linear Coupling: Initial Linear Polarizations:  $\theta_l = 0^\circ$ ,  
 $\theta_r = 90^\circ$ ,  $\alpha_2 = 0$ ,  $c_l = 1.5$ ,  $c_r = 0.6$ ,  $\Gamma = 0.175 + 0.005i$



a) Profiles of the components



b) Masses, pseudomomentum, and total energy as functions of time.

Рис.: 24.  $\delta_l = 0^\circ$ ,  $\delta_r = 90^\circ$

Linear Coupling: Initial Elliptic Polarizations:  $\theta = 23^\circ 44'$ ,

$\alpha_2 = 0$ ,  $c_l = -c_r = 1$ ,  $n_{l\psi} = n_{r\psi} = -1.1$ ,

$n_{l\phi} = n_{r\phi} = -1.5$ ,  $\Gamma = 0.175$

- We have found that the phases of the components play an essential role on the full energy of QPs. The magnitude of the latter essentially depends on the choice of initial phase difference (Figure 20);
- The pseudomomentum is also conserved and it is trivial due to the symmetry (Figure 21);
- The individual masses, however, breathe together with the individual (rotational) polarizations. Their amplitude and period do not influenced from the initial phase difference (Figure 21);

Linear Coupling: Initial Elliptic Polarizations:  $\theta = 23^\circ 44'$ ,

$\alpha_2 = 0$ ,  $c_l = -c_r = 1$ ,  $n_{l\psi} = n_{r\psi} = -1.1$ ,

$n_{l\phi} = n_{r\phi} = -1.5$ ,  $\Gamma = 0.175$

- The total mass is constant while the total polarization oscillates and suffers a 'shock in polarization' when QPs enter the collision. The polarization amplitude evidently depends on the initial phase difference (Figures 22,23);
- Due to the real linear coupling the polarization angle of QPs can change independently of the collision.
- Complex parameter of linear coupling: Along with the oscillations of the energy and masses the (negative) energy decreases very fast, while the masses  $M_\psi$  and  $M_\phi$  increase all of them oscillating. The pseudomomentum  $P$  increases without appreciable oscillation (Figure 24).

Thank you for your kind attention !