# Wave Equations and Quasi-Particle Concept of Dynamics of Solitary Waves: Integrability vs. Consistency, Nonlinearity vs. Dispersion, Adiabatic Approximation

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# Outline

- Problem Formulation. Integrability and Nonintegrability
- Conservation Laws and Boussinesq Paradigm
- Choice or Generation of Initial Conditions
- Quasi-Particle Dynamics and Polarization
- Variational Approach. Adiabatic Approximation. Particle Dynamics
- Numerics
- Some Results and Discussion

# Problem Formulation: Equations

CNLSE is system of nonlinearly coupled Schrödinger equations (called the Gross-Pitaevskii or Manakov-type system):

$$
\begin{aligned}
\mathrm{i}\psi_t &= \beta\psi_{xx} + \left[\alpha_1|\psi|^2 + (\alpha_1 + 2\alpha_2)|\phi|^2\right]\psi(+\Gamma\phi),\\
\mathrm{i}\phi_t &= \beta\phi_{xx} + \left[\alpha_1|\phi|^2 + (\alpha_1 + 2\alpha_2)|\psi|^2\right]\phi(+\Gamma\psi),
\end{aligned}
\tag{1}
$$

where:
$\beta$ is the dispersion coefficient;
$\alpha_1$ describes the self-focusing of a signal for pulses in birefringent media;
$\Gamma = \Gamma_r + \mathrm{i}\Gamma_i$ is the magnitude of linear coupling. $\Gamma_r$ governs the oscillations between states termed as breathing solitons, while $\Gamma_i$ describes the gain behavior of soliton solutions.
$\alpha_2$ (called cross-modulation parameter) governs the nonlinear coupling between the equations.

# Problem Formulation: Equations

When $\alpha_2 = 0$, no nonlinear coupling is present despite the fact that "cross-terms" proportional to $\alpha_1$ appear in the equations. For $\alpha_2 = 0$, the solutions of the two equations are identical, $\psi \equiv \phi$, and equal to the solution of single NLSE with nonlinearity coefficient $\alpha = 2\alpha_1$.

For $\Gamma = 0$, CNSE is alternately called the Gross-Pitaevskii equation or an equation of Manakov type. It was derived independently by Gross and Pitaevskii to describe the behavior of Bose-Einstein condensates as well as optic pulse propagation. It was solved analytically for the case $\alpha_2 = 0$, $\beta = \frac{1}{2}$ by Manakov via Inverse Scattering Transform who generalized an earlier result by Zakharov & Shabat for the scalar cubic NLSE (i.e. Eq.(2-$\psi$) with $\varphi(x,t) = 0$). Recently, Chow, Nakkeeran, and Malomed studied periodic waves in optic fibers using a version of CNSE with $\Gamma \neq 0$.

The single NLSE has the form

$$i\psi_t + \beta\psi_{xx} + \alpha|\psi|^2\psi = 0 \tag{2}$$

As far as the applications in nonlinear optics are concerned, the above equation describes the single-mode wave propagation in a fiber. Depending on the sign of (2) admits single and multiple sech-solutions (*bright solitons*), as well as tanh-profile, or *dark soliton* solutions. In this paper we concentrate on the case of bright solitons.

A dynamical system with infinite number of conservation laws (integrals) is called integrable. NSE is an integrable dynamical system. CNSE is a nonintegrable generalisation of NSE.

# Problem Formulation: Conservation Laws

Define "mass", $M$, (pseudo)momentum, $P$, and energy, $E$:

$$M \stackrel{\mathrm{def}}{=} \frac{1}{2\beta}\int_{-L_1}^{L_2}(|\psi|^2+|\phi|^2)\mathrm{d}x, \quad P \stackrel{\mathrm{def}}{=} -\int_{-L_1}^{L_2}\mathcal{I}(\psi\bar{\psi}_x+\phi\bar{\phi}_x)\mathrm{d}x,$$

$$E \stackrel{\mathrm{def}}{=} \int_{-L_1}^{L_2}\mathcal{H}\mathrm{d}x, \quad \text{where} \tag{3}$$

$$\mathcal{H} \stackrel{\mathrm{def}}{=} \beta\left(|\psi_x|^2+|\phi_x|^2\right)-\tfrac{1}{2}\alpha_1(|\psi|^4+|\phi|^4) \\ -(\alpha_1+2\alpha_2)\left(|\phi|^2|\psi|^2\right)-2\Gamma[\Re(\bar{\psi}\bar{\phi})]$$

is the Hamiltonian density of the system. Here $-L_1$ and $L_2$ are the left end and the right end of the interval under consideration. The following conservation/balance laws hold, namely

$$\frac{dM}{dt}=0, \qquad \frac{dP}{dt}=\mathcal{H}\big|_{x=L_2}-\mathcal{H}\big|_{x=-L_1}, \qquad \frac{dE}{dt}=0, \tag{4}$$

# Boussinesq Paradigm. Quasi-particles Concept

Boussinesq 's equation (BE) was the first model for the propagation of surface waves over shallow inviscid layer which contains dispersion. Boussinesq found an analytical solution of his equation and thus proved that the balance between the steepening effect of the nonlinearity and the flattening effect of the dispersion maintains the shape of the wave. This discovery can be properly termed 'Boussinesq Paradigm'. Apart from the significance for the shallow water flows, this paradigm is very important for understanding the particle-like behavior of localized waves which behavior was discovered in the 1960ies (the so-called 'collision property'), and the localized waves were called *solitons* (Zabusky & Kruskal). The localized waves which can retain their identity during interaction appear to be a rather pertinent model for particles, especially if some mechanical properties are conserved by the governing equations (such as mass, energy, momentum). For this reason they are called quasi-particles.

The equation derived by Boussinesq (1871) referred as "Original Boussinesq Equation"

$$\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[(gH)h + \frac{3g}{2}h^2 + \frac{gH^3}{3}\frac{\partial^2 h}{\partial x^2}\right] \tag{5}$$

is fully integrable but incorrect in the sence of Hadamard due to the positive sign of the foutrh spatial derivative.
Including the effect of surface tension as was done by Korteweg & de Vries one arrives so called Proper Boussinesq Equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - \alpha u^2 + \beta\frac{\partial^2 u}{\partial x^2}\right] \tag{6}$$

which is correct in the sense of Hadamard (well-posed as IVP) only when $\beta < 0$.

The ill-posedness of OBE can be removed if the Boussinesq assumption that $\frac{\partial}{\partial t} \approx \frac{\partial}{\partial x}$ is used in "reverse":

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - \alpha u^2 + \beta \frac{\partial^2 u}{\partial t^2}\right]. \tag{7}$$

This equation is called "improper" or RLWE. All versions of the Boussinesq equations considered here possess solitary-wave solutions of *sech* type. For example, for the BE the solution reads

$$u = -\frac{3}{2}\frac{c^2-1}{\alpha}\operatorname{sech}^2\left(\frac{x-ct}{2}\sqrt{\frac{c^2-1}{\beta}}\right)$$

The overwhelming majority of the analytical and numerical results so far are for one spatial dimension, while in multidimension much less is possible to achieve analytically, and almost nothing is known about the unsteady solutions, especially when the Boussinesq equations contain different dispersions and nonlinearities are involved. An exception is the case of so called KPE, which has fourth derivative only in one spatial directions, while in the other direction, the highest order is second.

Boussinesq Paradigm Equation (BPE)

$$u_{tt} = \Delta[u - \alpha u^2 + \beta_1 u_{tt} - \beta_2 \Delta u] \tag{8}$$

where $u$ is the surface elevation, $\beta_1, \beta_1 > 0$ are two dispersion coefficients, $\alpha$ is an amplitude parameter.

Cubic-Quintic Boussinesq Paradigm Equation (CQBPE)

$$u_{tt} = \Delta[u - \alpha(u^3 - \sigma u^5) + \beta_1 u_{tt} - \beta_2 \Delta u] \tag{9}$$

where $u$ again is the surface elevation, $\beta_1, \beta_1 > 0$ are two dispersion coefficients. The parameter $\sigma$ accounts for the relative importance of the quintic nonlinearity term. Then energy law reads

$$\frac{dE}{dt} = 0 \quad \text{with}$$

$$E = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(\nabla u_t)^2 + u^2 - \frac{1}{2}u^4 + \frac{\sigma}{3}u^6 + \beta_1 u^2 + \beta_2(\nabla u)^2]dxdy.$$

Unlike the BPE with quadratic nonlinearity, when the amplitude increases, the quintic term in CQBPE for reasonably large $\sigma$ will dominate and will make the energy functional positive, which limits the increase of the amplitude. All this means that no blow-up can be expected for CQBPE when $\sigma > \frac{3}{16}$.

# Boussinesq Paradigm. (Energy)-Consistent Boussinesq System

This a further generalization of the Boussinesq Paradigm equation. The following system was derived rigorously from the main Boussinesq assumption

$$\frac{\partial\chi}{\partial t} = -\beta\nabla\cdot\chi\nabla f - \Delta f + \frac{\beta}{6}\Delta^2 f - \frac{\beta}{2}\frac{\partial^2\Delta f}{\partial t^2}\,, \tag{10a}$$

$$\frac{\partial f}{\partial t} = -\frac{\beta}{2}(\nabla f)^2 - \chi, \tag{10b}$$

# Boussinesq Paradigm. (Energy)-Consistent Boussinesq System

Upon multiplying the left-hand side of (10a) by the right-hand side of Eq. (10b) and the right-hand side of Eq. (10a) by the left-hand side of Eq. (10b), adding the results and integrating over the surface region $D$ under consideration one gets the following energy balance law

$$E = \frac{1}{2}\int\limits_{D}\left[\chi^2 + (1+\beta\chi)(\nabla f)^2 + \frac{\beta}{6}(\Delta f)^2 + \frac{\beta}{2}(\nabla f_t)^2\right] dx$$

$$\frac{dE}{dt} = \oint\limits_{\partial D}\left[(1+\beta\chi) f_t \frac{\partial f}{\partial n} + \frac{\beta}{2} f_t \frac{\partial f_{tt}}{\partial n} + \frac{\beta}{2} f_t \frac{\partial \Delta f}{\partial n} - \frac{\beta}{2}\Delta f \frac{\partial f_t}{\partial n}\right] ds, \tag{11}$$

which allows us to call the Eqs. (10) ‘Energy Consistent Boussinesq Paradigm.’

# Problem Formulation: Choice of Initial Conditions

We concern ourselves with the soliton solutions whose modulation amplitude is of general form (non-sech) and which are localized envelops on a propagating carrier wave. This allows us to play various scenario of initial polarization. Unfortunately in sech-case the initial polarization can be only linear. Then we assume that for each of the functions $\phi, \psi$ the initial condition is of the form of a single propagating soliton, namely

$$\begin{Bmatrix} \psi(x,t) \\ \phi(x,t) \end{Bmatrix} = \begin{Bmatrix} A^{\psi} \\ A^{\phi} \end{Bmatrix} \operatorname{sech}\left[b(x-X-ct)\right] \exp\left\{ \mathrm{i}\left[\frac{c}{2\beta}(x-X)-nt\right]\right\}.$$

$$b^2 = \frac{1}{\beta}\left(n+\frac{c^2}{4\beta}\right), \quad A = b\sqrt{\frac{2\beta}{\alpha_1}}, \quad u_c = \frac{2n\beta}{c}, \tag{12}$$

where $X$ is the spatial position (center of soliton), $c$ is the phase speed, $n$ is the carrier frequency, and $b^{-1}$ – a measure of the support of the localized wave.

We assume that for each of the functions $\phi, \psi$ the initial condition has the general type

$$\begin{aligned} \psi &= A_\psi(x + X - c_\psi t) \exp\left\{\mathrm{i}\left[n_\psi t - \tfrac{1}{2} c_\psi (x - X - c_\psi t) + \delta_\psi\right]\right\} \\ \phi &= A_\phi(x + X - c_\phi t) \exp\left\{\mathrm{i}\left[n_\phi t - \tfrac{1}{2} c_\phi (x - X - c_\phi t) + \delta_\phi\right]\right\}, \end{aligned} \tag{13}$$

where $c_\psi$, $c_\phi$ are the phase speeds and $X$'s are the initial positions of the centers of the solitons; $n_\psi$, $n_\phi$ are the carrier frequencies for the two components; $\delta_\psi$ and $\delta_\phi$ are the phases of the two components. Note that the phase speed must be the same for the two components $\psi$ and $\phi$. If they propagate with different phase speeds, after some time the two components will be in two different positions in space, and will no longer form a single structure. For the envelopes $(A_\psi, A_\phi)$, $\theta \equiv \arctan(\max|\phi|/\max|\psi|)$ is a polarization angle.

# Problem Formulation: Generation of Initial Conditions

Generally the carrier frequencies for the two components $n_\psi \neq n_\phi$ – elliptic polarization. When $n_\psi = n_\phi$ – circular polarization. If one of them vanishes – linear polarization (sech soliton, $\theta = 0; 90^\circ$).
In general case the initial condition is solution of the following system of nonlinear conjugated equations

$$\begin{aligned} A_\psi'' + \left(n_\psi + \tfrac{1}{4}c_\psi^2\right) A_\psi + \left[\alpha_1 A_\psi^2 + (\alpha_1 + 2\alpha_2)A_\phi^2\right] A_\psi &= 0 \\ A_\phi'' + \left(n_\phi + \tfrac{1}{4}c_\phi^2\right) A_\phi + \left[\alpha_1 A_\phi^2 + (\alpha_1 + 2\alpha_2)A_\psi^2\right] A_\phi &= 0. \end{aligned} \tag{14}$$

The system admits bifurcation solutions since the trivial solution obviously is always present.

We solve the auxiliary conjugated system (14) with asymptotic boundary conditions using Newton method and the initial approximation of sought nontrivial solution is sech-function. The final solution, however, is not obligatory sech-function. It is a two-component polarized soliton solution.

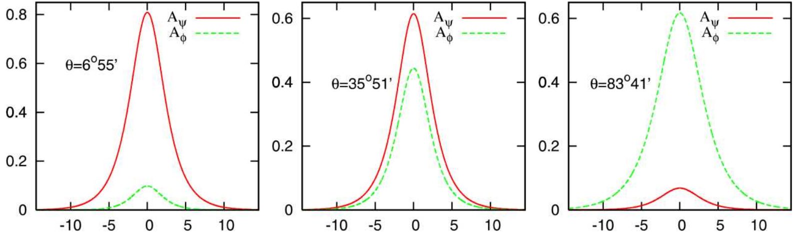



Рис.: 1. Amplitudes $A_\psi$ and $A_\phi$ for $c_l = -c_r = 1$, $\alpha_1 = 0.75$, $\alpha_2 = 0.2$. Left:$n_\psi = -0.68$; middle: $n_\psi = -0.55$; right: $n_\psi = -0.395$.

# Initial Conditions and Initial Phase Difference

Another dimension of complexity is introduced by the phases of the different components. The initial difference in phases can have a profound influence on the polarizations of the solitons after the interaction and the magnitude of the full energy. The relative shift of real and imaginary parts is what matters in this case.

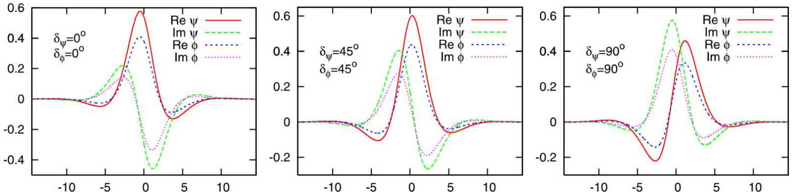



Рис.: 2. Real and imaginary parts of the amplitudes from the case shown in the middle panel of Figure 1 and the dependence on phase angle.

# Problem Formulation: Initial Conditions

After completing the initial conditions our aim is to understand better the influence of the initial polarization and initial phase difference on the particle-like behavior of the localized waves. We call a localized wave a quasi-particle (QP) if it survives the collision with other QPs (or some other kind of interactions) without losing its identity.

The idea of the adiabatic approximation to the soliton interactions (Karpman&Solov'ev (1981)) led to effective modeling of the $N$-soliton trains of the perturbed both scalar NLS eq. and vector NLSE eq.:

$$i\vec{u}_t + \frac{1}{2}\vec{u}_{xx} + (\vec{u}^\dagger, \vec{u})\vec{u}(x,t) = iR[\vec{u}]. \qquad (15)$$

The corresponding vector $N$-soliton train is determined by the initial condition

$$\vec{u}(x,t=0) = \sum_{k=1}^{N} \vec{u}_k(x,t=0), \quad \vec{u}_k(x,t) = 2\nu_k e^{i\phi_k}\operatorname{sech} z_k \vec{n}_k, \qquad (16)$$

and the amplitudes, the velocities, the phase shifts, and the centers of solitons are

$$u_k(x,t) = 2\nu_k \mathrm{e}^{i\phi_k}\operatorname{sech} z_k, \quad z_k = 2\nu_k(x - \xi_k(t)), \quad \xi_k(t) = 2\mu_k t + \xi_{k,0},$$

$$\phi_k = \frac{\mu_k}{\nu_k} z_k + \delta_k(t), \quad \delta_k(t) = 2(\mu_k^2 + \nu_k^2)t + \delta_{k,0}.$$

Here $\mu_k$ are the amplitudes, $\nu_k$ – the velocities, $\delta_k$ – the phase shifts, $\xi_k$ - the centers of solitons, $\beta = \frac{1}{2}$, $\alpha_1 = 1$, $\alpha_2 = 0$. So, this approach is applicable only for the (perturbed) Manakov system The phenomenology, however, is enriched by introducing a constant polarization vectors $\vec{n}_k$ that are normalized by the conditions

$$(\vec{n}_k^\dagger, \vec{n}_k) = 1, \qquad \sum_{s=1}^{n} \arg \vec{n}_{k;s} = 0.$$

# Idea of Adiabatic Approximation

The adiabatic approximation holds if the soliton parameters satisfy the restrictions

$$|\nu_k - \nu_0| \ll \nu_0, \; |\mu_k - \mu_0| \ll \mu_0, \; |\nu_k - \nu_0||\xi_{k+1,0} - \xi_{k,0}| \gg 1, \quad (17)$$

where $\nu_0$ and $\mu_0$ are the average amplitude and velocity respectively. In fact we have two different scales:

$$|\nu_k - \nu_0| \simeq \varepsilon_0^{1/2}, \quad |\mu_k - \mu_0| \simeq \varepsilon_0^{1/2}, \quad |\xi_{k+1,0} - \xi_{k,0}| \simeq \varepsilon_0^{-1}.$$

In this approximation the dynamics of the $N$-soliton train is described by a dynamical system for the $4N$ soliton parameters. We are interested in what follow perturbation(s) by external sech-potentials:

$$iR[u] \equiv V(x)u(x,t), \qquad V(x) = \sum_s c_s \mathrm{sech}^2(2\nu_0 x - y_s). \quad (18)$$

The latter allows us to realize the idea about localized potential wells(depressions) and humps.

# Potential Perturbations

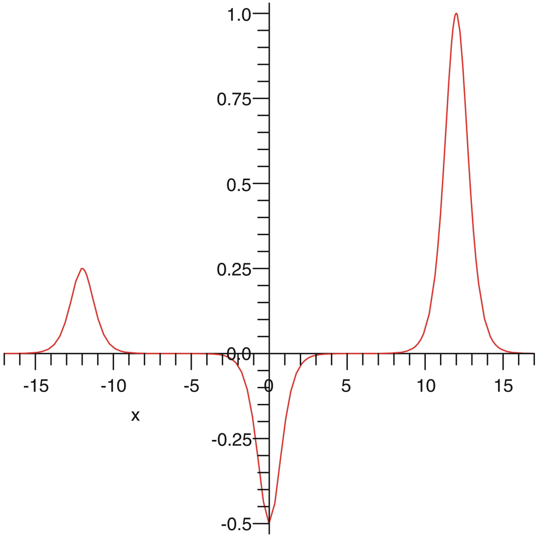



Рис.: 3. Sketch of potential sech-like wells and humps used.

# Perturbed Vector Complex Toda Chain Model. Initial Conditions

We use the variational approach (Anderson and Lisak (1986)) and derive the GCTC model. GCTC is a finite dimensional completely integrable model allowing Lax representation. The resulting equations can be written down in the form

$$\begin{aligned} \frac{d\lambda_k}{dt} &= -4\nu_0 \left( e^{Q_{k+1}-Q_k} - e^{Q_k-Q_{k-1}} \right) + M_k + iN_k, \\ \frac{dQ_k}{dt} &= -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \end{aligned} \qquad (19)$$

where $\lambda_k = \mu_k + i\nu_k$ and $X_k = 2\mu_k\Xi_k + D_k$ and

$$\begin{gathered} Q_k = -2\nu_0\xi_k + k\ln 4\nu_0^2 - i(\delta_k + \delta_0 + k\pi - 2\mu_0\xi_k), \\ \nu_0 = \frac{1}{N}\sum_{s=1}^{N}\nu_s, \qquad \mu_0 = \frac{1}{N}\sum_{s=1}^{N}\mu_s, \qquad \delta_0 = \frac{1}{N}\sum_{s=1}^{N}\delta_s. \end{gathered} \qquad (20)$$

# Effects of the polarization vectors on the soliton interaction

The GCTC is also a completely integrable model because it allows Lax representation $\tilde{L}_t = [\tilde{A}.\tilde{L}]$, where:

$$\begin{aligned} \tilde{L} &= \sum_{s=1}^{N} \left( \tilde{b}_s E_{ss} + \tilde{a}_s (E_{s,s+1} + E_{s+1,s}) \right), \\ \tilde{A} &= \sum_{s=1}^{N} \left( \tilde{a}_s (E_{s,s+1} - E_{s+1,s}) \right), \end{aligned} \tag{21}$$

where $\tilde{a}_s = m_{0k}^2 e^{2i\phi_{0k}} a_s$, $b_s = \mu_{s,t} + i\nu_{s,t}$. Like for the scalar case, the eigenvalues of $\tilde{L}$ are integrals of motion. If we denote by $\zeta_s = \kappa_s + i\eta_s$ (resp. $\tilde{\zeta}_s = \tilde{\kappa} + i\tilde{\eta}_s$) the set of eigenvalues of $L$ (resp. $\tilde{L}$) then their real parts $\kappa_s$ (resp. $\tilde{\kappa}_s$) determine the asymptotic velocities for the soliton train described by CTC (resp. GCTC).

# RTC and CTC. Asymptotic regimes

While for the RTC the set of eigenvalues $\zeta_s$ of the Lax matrix are all real, for the CTC they generically take complex values, e.g., $\zeta_s = \kappa_s + \mathrm{i}\eta_s$.

Hence, the only possible asymptotic behavior in the RTC is an asymptotically separating, free motion of the solitons. In opposite, for the CTC the real parts $\kappa_s \equiv \mathfrak{Re}\zeta_s$ of eigenvalues of the Lax matrix $\zeta_s$ determines the asymptotic velocity of the $s$th soliton.

Thus, starting from the set of initial soliton parameters we can calculate $L|_{t=0}$ (resp. $\tilde{L}|_{t=0}$), evaluate the real parts of their eigenvalues and thus determine the asymptotic regime of the soliton train.

- Regime (i) $\kappa_k \neq \kappa_j$ ($\tilde{\kappa}_k \neq \tilde{\kappa}_j$) for $k \neq j$ – asymptotically separating, free solitons;
- Regime (ii) $\kappa_1 = \kappa_2 = \cdots = \kappa_N = 0$ ($\tilde{\kappa}_1 = \tilde{\kappa}_2 = \cdots = \tilde{\kappa}_N = 0$) – a "bound state;"
- Regime (iii) group of particles move with the same mean asymptotic velocity and the rest of the particles will have free asymptotic motion.

Varying only the polarization vectors one can change the asymptotic regime of the soliton train.

# Effects of the external potentials on the GCTC. Numeric checks vs Variational approach

The predictions and validity of the CTC and GCTC are compared and verified with the numerical solutions of the corresponding CNSE using fully explicit difference scheme of Crank-Nicolson type, which conserves the energy, the mass, and the pseudomomentum. Such comparison is conducted for all dynamical regimes considered.

- First we study the soliton interaction of the pure Manakov model (without perturbations, $V(x) \equiv 0$) and with vanishing cross-modulation $\alpha_2 = 0$;
- 2-soliton configurations and transitions between different asymptotic regimes;
- 3-soliton configurations and transitions between different asymptotic regimes;
- 2- and 3-soliton configurations and transitions under the effect of well- and hump-like external potential.

# Two-soliton configurations and transitions between different asymptotic regimes: free asymptotic regime

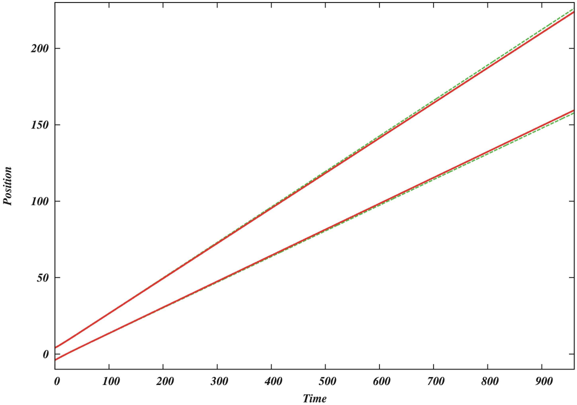



Рис.: 4. $\Delta\nu = |\nu_2 - \nu_1| < \nu_{\rm cr} = 0.01786$.
$\nu_{\rm cr} = 2\sqrt{2\cos(\theta_1 - \theta_2)}\nu_0 \exp\left(-\nu_0 r_0\right)$, $\mu_{k0} = 0.1$, $\nu_{10} = 0.49$, $\nu_{20} = 0.51$,
$\xi_{10,20} = \pm 4$, $\delta_{10} = 0$, $\delta_{20} = \pi + 2\mu_0 r_0$, $\theta_{10} = 2\pi/10$, $\theta_{20} = \theta_{10} - \pi/10$.

# Three-soliton configurations in mixed asymptotic regimes: two-bound state + free soliton

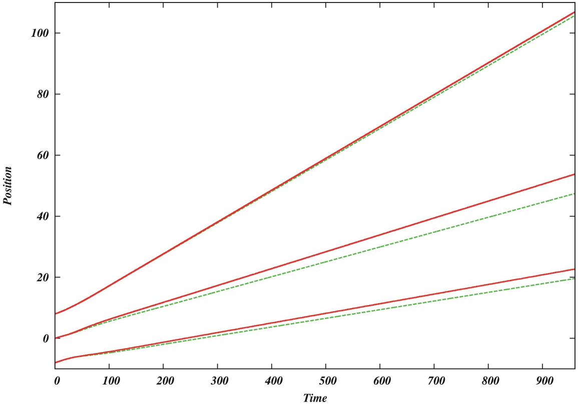



Рис.: 5. $\Delta\nu = 0.01 < \nu_{\rm cr}$. $\nu_{\rm cr} = 2\sqrt{2\cos(\theta_1 - \theta_2)}\nu_0 \exp\left(-\nu_0 r_0\right)$, $\mu_{k0} = 0.03$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$.

# Effects of the external potentials on the GCTC – 3-soliton configurations

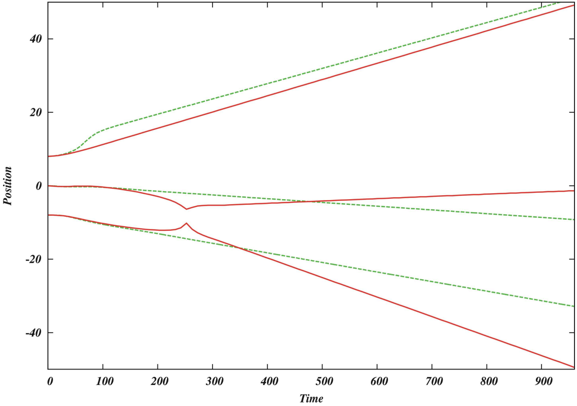



Рис.: 6. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, one potential hump at $y = 12$, $c_s = 0.01$.

# Effects of the external potentials on the GCTC – 3-soliton configurations

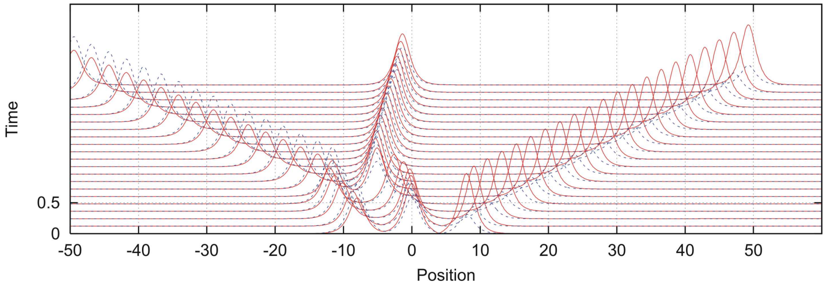



Рис.: 7. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, one potential hump at $y = 12$, $c_s = 0.01$.

# Effects of the external potentials on the GCTC – 3-soliton configurations

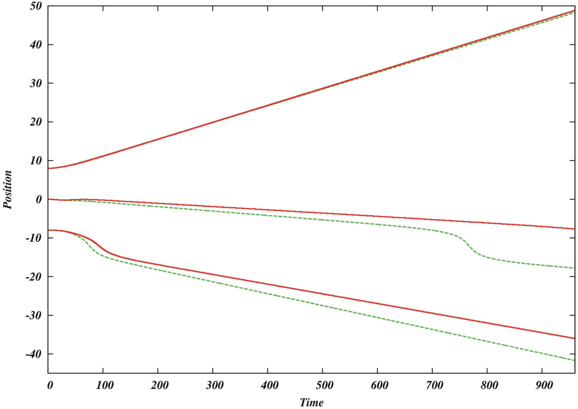



Рис.: 8. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, one potential well at $y = -12$, $c_s = 0.01$.

# Effects of the external potentials on the GCTC – 3-soliton configurations

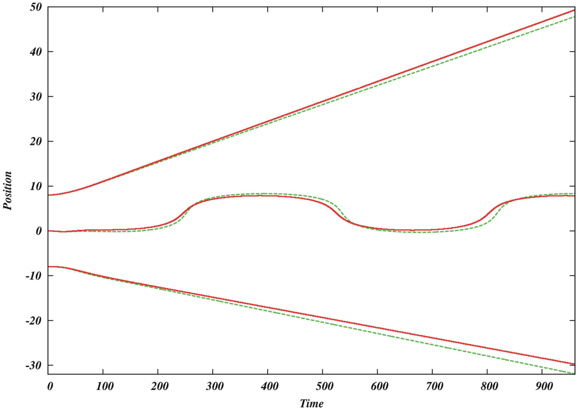



Рис.: 9. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, one potential well at $y = 4$, $c_s = 0.01$.

# Effects of the external potentials on the GCTC – 3-soliton configurations

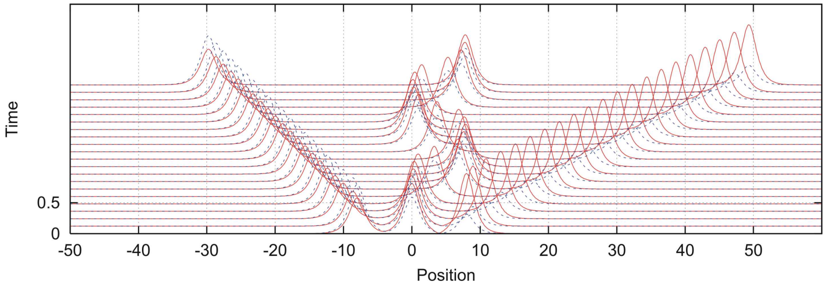



Рис.: 10. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, one potential well at $y = 4$, $c_s = 0.01$.

# Effects of the external potentials on the GCTC. Numeric checks vs Variational approach

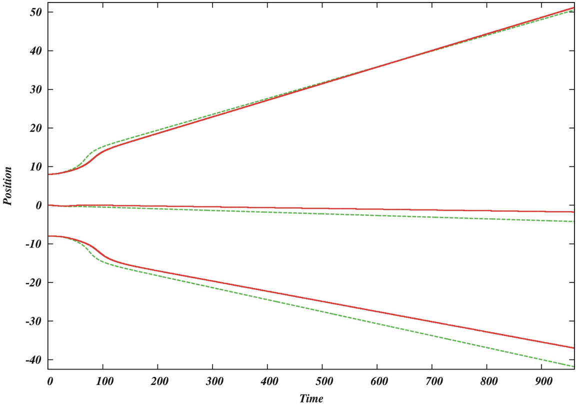



Рис.: 11. $\Delta\nu = 0.01$, $\mu_{k0} = 0$, $\nu_{20} = 0.5$, $\nu_{10,30} = \nu_{20} \pm \Delta\nu$, $\xi_{20} = 0$, $\xi_{10,30} = \pm 8$, $\delta_{10} = 0$, $\delta_{20,30} = \pm\pi/2 + 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{k0} = \theta_{k-1,0} - \pi/10$, $k = 2, 3$, two potential wells at $y = \pm 12$, $c_s = 0.01$.

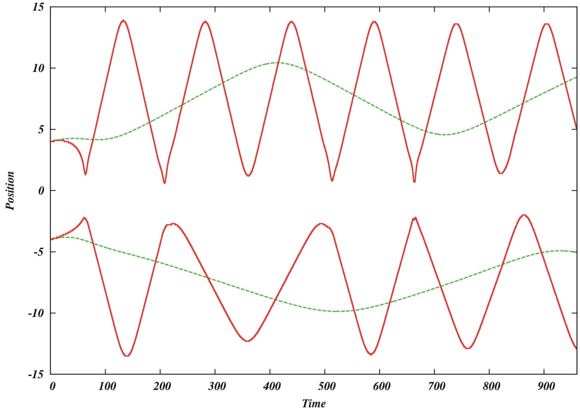



Рис.: 12. $\Delta\nu = 0$, $\mu_{k0} = 0.005$, $\nu_{k0} = 0.5$, $\xi_{10,20} = \pm 4$, $\xi_{10,20} = \pm 4$, $\delta_{10} = 0$, $\Delta_{20} = 2\mu_0 r_0$, $\theta_{10} = 3\pi/10$, $\theta_{20} = \theta_{10} - \pi/10$, three potential humps at $y = \pm 15$, $y = 0$, $c_s = 0.0485$.

# Linearly Coupled Problem Formulation: Equations and Initial Conditions

For the linearly coupled system of NLSE the magnitude of linear coupling $\Gamma_r$ generates breathing the solitons although noninteracting The initial conditions must be

$$\Psi = \psi \cos(\Gamma t) + \mathrm{i}\phi \sin(\Gamma t), \quad \Phi = \phi \cos(\Gamma t) + \mathrm{i}\psi \sin(\Gamma t), \qquad (22)$$

where $\phi$ and $\psi$ are assumed to be sech-solutions of (1) for $\alpha_2 = 0$. Hence (1) posses solutions, which are combinations of interacting solitons oscillating with frequency $\Gamma_r$ and their motion gives rise to the so-called rotational polarization.

In all considered cases we found that a conservation of the total polarization is present. Only for the linearly CNLSE ($\alpha_2 = 0$) the total polarizations breathe with an amplitude evidently depending on the initial phase difference but is conserved within one full period of the breathing.

| $\delta_r - \delta_l$ | $\theta_l^i$ | $\theta_r^i$ | $\theta_l^i + \theta_r^i$ | $\theta_l^f$ | $\theta_r^f$ | $\theta_l^f + \theta_r^f$ |
|---|---|---|---|---|---|---|
| 45° | 45° | 45° | 90° | 33°48′ | 56°12′ | 90° |
| 90° | 45° | 45° | 90° | 24°06′ | 65°54′ | 90° |
| 0° | 20° | 20° | 40° | 20°00′ | 20°00′ | 40° |
| 90° | 20° | 20° | 40° | 28°48′ | 2°02′ | 30°50′ |
| 0° | 36° | 36° | 72° | 36°00′ | 36°00′ | 72° |
| 90° | 36° | 36° | 72° | 53°00′ | 13°20′ | 66°20′ |
| 0° | 10° | 80° | 90° | 21°05′ | 68°54′ | 89°59′ |
| 90° | 10° | 80° | 90° | 9°27′ | 80°30′ | 89°57′ |

# Numerical Method

To solve the main problem numerically, we use an implicit conservative scheme in complex arithmetic.

$$
\begin{aligned}
\mathrm{i}\frac{\psi_i^{n+1}-\psi_i^n}{\tau} &= \frac{\beta}{2h^2}\left(\psi_{i-1}^{n+1}-2\psi_i^{n+1}+\psi_{i+1}^{n+1}+\psi_{i-1}^{n}-2\psi_i^{n}+\psi_{i+1}^{n}\right)\\
&+\frac{\psi_i^{n+1}+\psi_i^n}{4}\left[\alpha_1\left(|\psi_i^{n+1}|^2+|\psi_i^n|^2\right)+(\alpha_1+2\alpha_2)\left(|\phi_i^{n+1}|^2+|\phi_i^n|^2\right)\right]\\
&-\tfrac{1}{2}\Gamma\left(\phi_i^{n+1}+\phi_i^n\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{i}\frac{\phi_i^{n+1}-\phi_i^n}{\tau} &= \frac{\beta}{2h^2}\left(\phi_{i-1}^{n+1}-2\phi_i^{n+1}+\phi_{i+1}^{n+1}+\phi_{i-1}^{n}-2\phi_i^{n}+\phi_{i+1}^{n}\right)\\
&+\frac{\phi_i^{n+1}+\phi_i^n}{4}\left[\alpha_1\left(|\phi_i^{n+1}|^2+|\phi_i^n|^2\right)+(\alpha_1+2\alpha_2)\left(|\psi_i^{n+1}|^2+|\psi_i^n|^2\right)\right]\\
&-\tfrac{1}{2}\Gamma\left(\psi_i^{n+1}+\psi_i^n\right).
\end{aligned}
$$

# Numerical Method: Internal Iterations

$$
\begin{aligned}
\mathrm{i}\frac{\psi_i^{n+1,k+1}-\psi_i^n}{\tau} =& \frac{\beta}{2h^2}\Big(\psi_{i-1}^{n+1,k+1}-2\psi_i^{n+1,k+1}+\psi_{i+1}^{n+1,k+1} \\
&+\psi_{i-1}^n-2\psi_i^n+\psi_{i+1}^n\Big) \\
+\frac{\psi_i^{n+1,k}+\psi_i^n}{4}&\Big[\alpha_1\big(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}|+|\psi_i^n|^2\big) \\
&+(\alpha_1+2\alpha_2)\big(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}|+|\phi_i^n|^2\big)\Big] \\
\mathrm{i}\frac{\phi_i^{n+1,k+1}-\phi_i^n}{\tau} =& \frac{\beta}{2h^2}\big(\phi_{i-1}^{n+1,k+1}-2\phi_i^{n+1,k+1}+\phi_{i+1}^{n+1,k+1} \\
&+\phi_{i-1}^n-2\phi_i^n+\phi_{i+1}^n\big) \\
+\frac{\phi_i^{n+1,k}+\phi_i^n}{4}&\Big[\alpha_1\big(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}|+|\phi_i^n|^2\big) \\
&+(\alpha_1+2\alpha_2)\big(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}|+|\psi_i^n|^2\big)\Big].
\end{aligned}
$$

It is not only convergent (consistent and stable), but also conserves mass and energy, i.e., there exist discrete analogs for (4), which arise from the scheme.

$$M^n = \sum_{i=2}^{N-1} \left(|\psi_i^n|^2 + |\phi_i^n|^2\right) = \mathrm{const},$$

$$E^n = \sum_{i=2}^{N-1} \frac{-\beta}{2h^2} \left(|\psi_{i+1}^{n} - \psi_i^{n}|^2 + |\phi_{i+1}^{n} - \phi_i^{n}|^2\right) + \frac{\alpha_1}{4}\left(|\psi_i^n|^4 + |\phi_i^n|^4\right)$$
$$+ \tfrac{1}{2}(\alpha_1 + 2\alpha_2)\left(|\psi_i^n|^2 |\phi_i^n|^2\right) - \Gamma \Re[\bar{\phi}_i^n \psi_i^n] = \mathrm{const},$$
$$\text{for} \quad \text{all} \quad n \geq 0.$$

These values are kept constant during the time stepping. The above scheme is of Crank-Nicolson type for the linear terms and we employ internal iterations to achieve implicit approximation of the nonlinear terms, i.e., we use its linearized implementation.

# Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

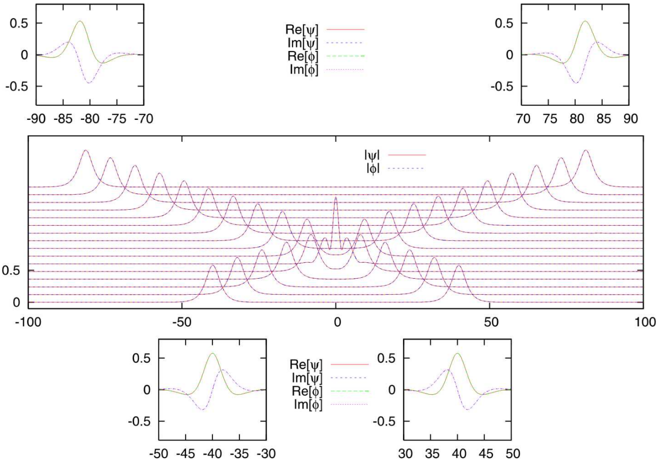



Рис.: 13. $\delta_l = 0°$, $\delta_r = 0°$

# Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

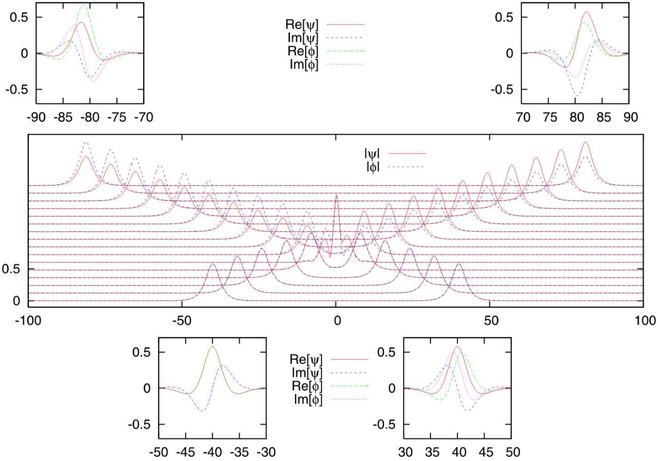


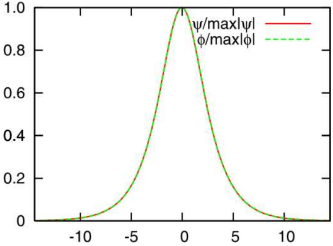

Рис.: 14. $\delta_l = 0^\circ$, $\delta_r = 45^\circ$

# Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

—When both of QPs have zero phases (Fig. 13), the interaction perfectly follows the analytical Manakov two-soliton solution.
—The surprise comes in Fig. 14 where is presented an interaction of two QPs, the right one of which has a nonzero phase $\delta_r = 45^\circ$. After the interaction, the two QPs become different Manakov solitons than the original two that entered the collision. The outgoing QPs have polarizations $33^\circ 48'$ and $56^\circ 12'$. Something that can be called a 'shock in polarization' takes place. All the solutions are perfectly smooth, but because the property called polarization cannot be defined in the cross-section of interaction and for this reason, it appears as undergoing a shock.

Here is to be mentioned that when rescaled the moduli of $\psi$ and $\phi$ from Fig. 14 perfectly match each other which means that the resulting solitons have circular polarization (see left panel of Fig. 15 below). The Manakov solution is not unique. There exists a class of Manakov solution and in the place of interaction becomes a bifurcation between them.

# Results and Discussion: Initial Circular Polarization and Nonuniqueness of the Manakov Solution



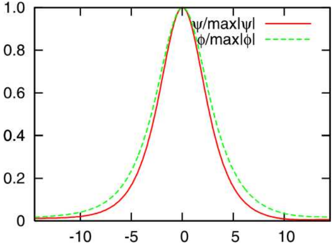




Рис.: 15. Circular polarization (left); Elliptic Polarization (right).

# Equal Elliptic Initial Polarizations of $50°08'$ for $\alpha_2 = 2$

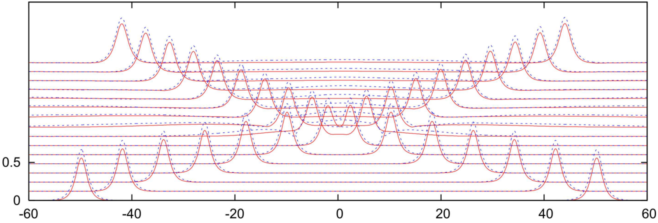



Рис.: 16. $\delta_l = 0°, \delta_r = 0°$

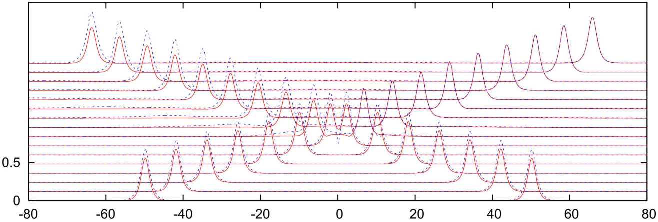



Рис.: 17. $\delta_l = 0°, \delta_r = 180°$

# Equal Elliptic Initial Polarizations of $50^\circ 08'$ for $\alpha_2 = 2$

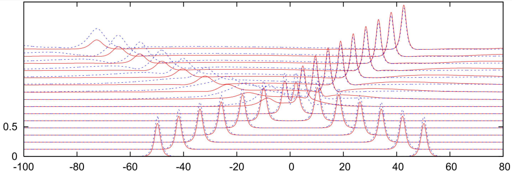



Рис.: 18. $\delta_l = 0^\circ, \delta_r = 130^\circ$

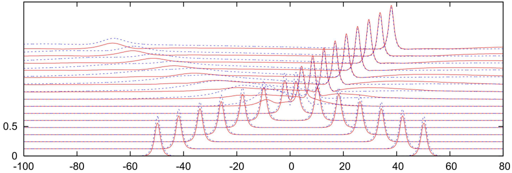



Рис.: 19. $\delta_l = 0^\circ, \delta_r = 135^\circ$

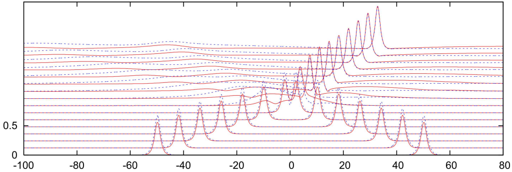



Рис.: 20. $\delta_l = 0^\circ, \delta_r = 140^\circ$

# Equal Elliptic Initial Polarizations of $50^\circ 08'$ for $\alpha_2 = 2$.

We choose $n_{l\psi} = n_{r\psi} = -1.5$, $n_{l\phi} = n_{r\phi} = -1.1$, $c_l = -c_r = 1$, $\alpha_1 = 0.75$, and focus on the effects of $\alpha_2$ and $\vec{\delta}$.

One sees that the desynchronisations of the phases leads in the final stage to a superposition of two one-soliton solutions but with different polarizations from the initial polarization. Yet, for $\delta_r = 130^\circ \div 140^\circ$ one of the QPs loses its energy contributing it to the other QP during the collision and then virtually disappears: kind of energy trapping (Figs.18, 19, 20).

For $\delta_r = 180^\circ$ another interesting effect is seen, when the right outgoing QP is circularly polarized (Fig. 17).

All these interactions are accompanied by changes of phase speeds. The total polarization exhibits some kind of conservation.

# Strong Nonlinear Interaction: $\alpha_2 = 10$

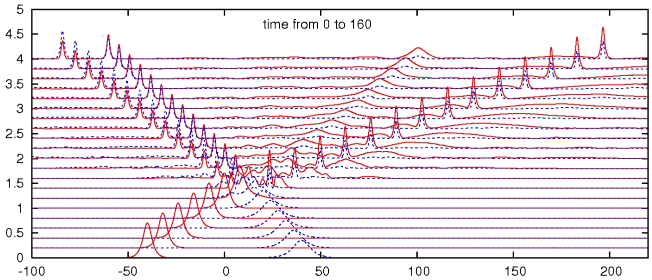



Рис.: 21. $\alpha_2 = 10$, $c_l = 1$, $c_r = -0.5$.

# Strong Nonlinear Interaction: $\alpha_2 = 10$

- Two new solitons are born after the collision.
- The kinetic energies of the newly created solitons correspond their phase speeds and masses, but the internal energy is very different for the different QP.
- the total energy of the QPs is radically different from the total energy of the initial wave profile. The differences are so drastic that the sum of QPs energies can even become negative. This means that the energy was carried away by the radiation.
- The predominant part of the energy is concentrated in the left and right forerunners because of the kinetic energies of the latter are very large. This is due to the fact that the forerunners propagate with very large phase speeds, and span large portions of the region.
- All four QPs have elliptic polarizations.
- Energy transformation is a specific trait of the coupled system considered here.

# Linear Coupling: Initial Elliptic Polarizations: $\theta = 23°44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

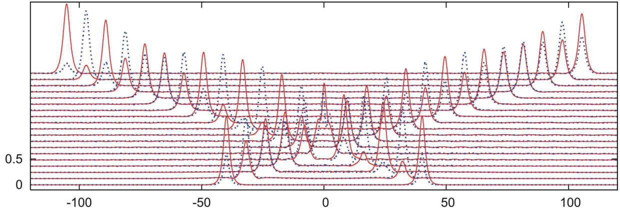



Рис.: 22. $\delta_l = 0°$, $\delta_r = 0°$

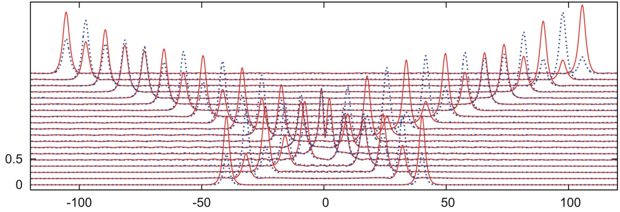



Рис.: 23. $\delta_l = 0°$, $\delta_r = 90°$

# Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

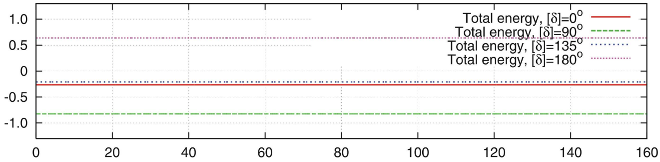



Рис.: 24. Influence of the initial phase difference on the total energy:

$\delta_l = 0^\circ,\ \delta_r = 0^\circ$ – $E = -0.262$;

$\delta_l = 0^\circ,\ \delta_r = 90^\circ$ – $E = -0.821$;

$\delta_l = 0^\circ,\ \delta_r = 135^\circ$ – $E = -0.206$;

$\delta_l = 0^\circ,\ \delta_r = 180^\circ$ – $E = 0.640$

# Linear Coupling: Initial Elliptic Polarizations: $\theta = 23°44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

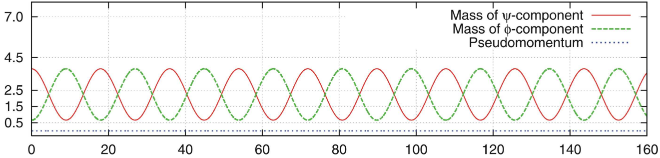



Рис.: 25. $\delta_l = 0°$, $\delta_r = 90°$, $P = 10^{-3} \div 10^{-5}$

# Linear Coupling: Initial Elliptic Polarizations: $\theta = 23°44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

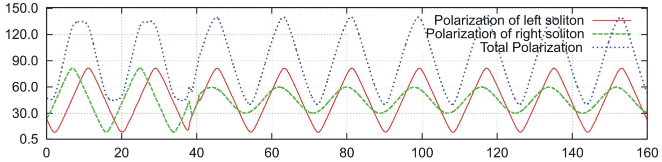



Рис.: 26. $\delta_l = 0°$, $\delta_r = 0°$

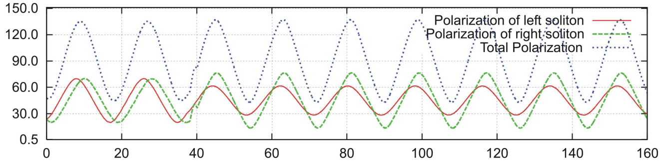
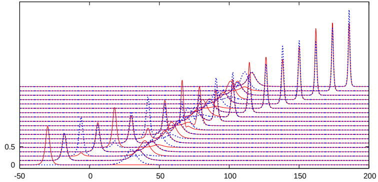
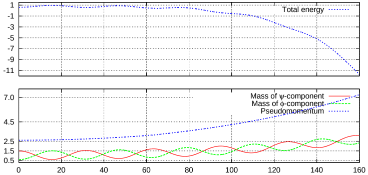



Рис.: 27. $\delta_l = 0°$, $\delta_r = 90°$

# Linear Coupling: Initial Linear Polarizations: $\theta_l = 0^\circ$, $\theta_r = 90^\circ$, $\alpha_2 = 0$, $c_l = 1.5$, $c_r = 0.6$, $\Gamma = 0.175 + 0.005i$




a) Profiles of the components




b) Masses, psedomomentum, and total energy as functions of time.

Рис.: 28. $\delta_l = 0^\circ$, $\delta_r = 90^\circ$

# Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

- We have found that the phases of the components play an essential role on the full energy of QPs. The magnitude of the latter essentially depends on the choice of initial phase difference (Figure 24);
- The pseudomomentum is also conserved and it is trivial due to the symmetry (Figure 25);
- The individual masses, however, breathe together with the individual (rotational) polarizations. Their amplitude and period do not influenced from the initial phase difference (Figure 25);

# Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

- The total mass is constant while the total polarization oscillates and suffers a 'shock in polarization ' when QPs enter the collision. The polarization amplitude evidently depends on the initial phase difference (Figures 26,27);
- Due to the real linear coupling the polarization angle of QPs can change independently of the collision.
- Complex parameter of linear coupling: Along with the oscillations of the energy and masses the (negative) energy decreases very fast, while the masses $M_\psi$ and $M_\phi$ increase all of them oscillating. The pseudomomentum $P$ increases without appreciable oscillation (Figure 28).

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# Thanks

Thank you for your kind attention !