Investigation of the long-Time Evolution of Localized Solutions of a Dispersive Wave System

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(Work done in collaboration with late Prof. C. I. Christov from University of Louisiana at Lafayette, USA and dedicated to his memory)

Outline

- Problem Formulation

- Hamiltonian Representation

- Numerical Method

- Results and discussion. Self-similar behaviour

- References

Outline of the Problem

- Boussinesq's equation (BE) was the first model for the propagation of surface waves over shallow inviscid fluid layer. He found an analytical solution of his equation and thus proved that the balance between the steepening effect of the nonlinearity and the flattening effect of the dispersion maintains the shape of the wave. This discovery can be properly termed 'Boussinesq Paradigm.'

- Apart from the significance for the shallow water flows, this paradigm is very important for understanding the particle-like behavior of nonlinear localized waves. As it should have been expected, most of the physical systems are not fully integrable (even in one spatial dimension) and only a numerical approach can lead to unearthing the pertinent physical mechanisms of the interactions.

Outline of the Problem

- The overwhelming majority of the analytical and numerical results obtained so far are for one spatial dimension, while in multidimension, much less is possible to achieve analytically, and almost nothing is known about the unsteady solutions that involve interactions, especially when the full-fledged Boussinesq equations are involved.

  As shown in Christov(2001), the consistent implementation of the Boussinesq method yields the following Generalized Wave Equation (GWE) for $f = \phi(x, y, z = 0; t)$:

  $$f_{tt} + 2\beta \nabla f \cdot \nabla f_t + \beta f_t \Delta f$$
  $$+ \tfrac{3\beta^2}{2}(\nabla f)^2 \Delta f - \Delta f + \tfrac{\beta}{6}\Delta^2 f - \tfrac{\beta}{2}(\Delta f)_{tt} = 0. \quad (1)$$

  with Hamiltonian density
  $$\mathcal{H} = \tfrac{1}{2}\left[f_t^2 + (\nabla f)^2 - \tfrac{1}{4}\beta^2(\nabla f)^4 + \tfrac{1}{6}\beta(\Delta f)^2 + \tfrac{1}{2}\beta(\nabla f_t)^2 \right].$$

Outline of the Problem

- Eq. (1) is the most rigorous amplitude equation that can be derived for the surface waves over an inviscid shallow layer, when the length of the wave is considered large in comparison with the depth of the layer. It was derived only in 2001. Besides it a plethora of different inconsistent Boussinesq equations are still vigorously investigated. The most popular are the versions that contain a quadratic nonlinearity which are useful from the paradigmatic point of view.

Outline of the Problem

- Unfortunately, Boussinesq did some additional unnecessary assumptions, which rendered his equation incorrect in the sense of Hadamard, when the dispersion is positive. We term the original model the 'Boussinesq's Boussinesq Equation' (BBE):

$$u_{tt} = (u - \alpha u^2 + \beta u_{xx})_{xx}, \qquad \beta < 0. \qquad (2)$$

During the years, it was 'improved' in a number of works. The mere change of the incorrect sign of the fourth derivative in BBE yields the so-called 'good' or 'proper' Boussinesq equation (BE).

Outline of the Problem

A different approach to removing the incorrectness is by changing the spatial fourth derivative to a mixed fourth derivative, which resulted into an equation know nowadays as the Regularized Long Wave Equation (RLWE) or Benjamin–Bona–Mahony equation (BBME):

$$u_{tt} = (u - \alpha u^2 + \beta u_{tt})_{xx}. \qquad (3)$$

In fact, the mixed derivative occurs naturally in Boussinesq derivation (see Eq. (1)), and was changed by Boussinesq to a fourth spatial derivative under an assumption that $\partial_t \approx c\partial_x$, which is currently known as the 'Linear Impedance Relation' (or LIA). The LIA has produced innumerable instances of unphysical results.

Problem Formulation: Equations

Consider the equation when the velocity potential and surface elevation do not depend on the coordinate y. Having in mind that f is the velocity potential on the bottom of the layer we introduce a vertical velocity $u = f_x$ and an auxiliary function q. Then we obtain so-called Dispersive Wave System is a progenitor of the different 1d Boussinesq equations:

$$u_t + \frac{\alpha}{2}(u^2)_x = q_{xx},$$

$$q_t + \alpha u q_x = u - \beta_2 u_{xx} + \beta_1 u_{tt}, \tag{4}$$

where:

β_1 and β_2 are two dispersion coefficients, $\beta_1 = 3\beta_2 = \beta$;

$\alpha = \beta$ is an amplitude parameter;

Problem Formulation: Boundary Conditions

The boundary conditions are

$$u|_{x=-L_1, L_2} = 0 \qquad q_x|_{x=-L_1, L_2} = 0.$$

When the interval $[-L_1, L_2]$ is finite they provide the conservation of the total energy.

Problem Formulation: Choice of Initial Conditions

The initial condition is a superposition of two solitary waves traveling in opposite directions with a phase velocities c_l and c_r

$$u(x, t = 0) = \frac{a \, \text{sgn}(c)}{\frac{|c|-1}{2} + \cosh^2[b(x - X - ct)]}$$

where $a = \frac{c^2 - 1}{\alpha}$, $b = \sqrt{\frac{c^2 - 1}{2(\beta_1 c^2 - \beta_2)}}$.

The sech-like solutions exit in two domains – subcritical (subsonic): $0 < c < 1/\sqrt{3}$ and supercritical (supersonic): $c > 1$. The physicaly relevant are only supercritical modes because the subcritical do not admit long waves for small β and are out of our interest.

Boussinesq Paradigm Equation

If f_t is replaced by f_x in the quadratic nonlinear term one arrives at

$$u_{tt} = (u + \frac{3\beta}{2}u^2 + \frac{\beta}{2}u_{tt} - \frac{\beta}{6}u_{xx})_{xx} \qquad (5)$$

which was called "Boussinesq Paradigm Equation" (BPE). Note that it is not the equation derived by Boussinesq himself. The above simplification, however, destroys the Galilean invariance of the system.

Problem Formulation: Conservation Laws

Define "mass", M, (pseudo)momentum, P, and (pseudo)energy, E:

$$M \stackrel{\text{def}}{=} \int_{-L_1}^{L_2} u \mathrm{d}x, \quad P \stackrel{\text{def}}{=} \int_{-L_1}^{L_2} (uq_x - \tfrac{\beta}{2}u_t u_x)\mathrm{d}x,$$

$$E \stackrel{\text{def}}{=} \tfrac{1}{2} \int_{-L_1}^{L_2} \left(u^2 + q_x^2 + \tfrac{\beta}{2}u^3 + \tfrac{\beta}{2}u_t^2 + \tfrac{\beta}{6}u_x^2\right) \mathrm{d}x,$$

Here $-L_1$ and L_2 are the left end and the right end of the interval under consideration.

The following conservation/balance laws hold, namely

$$\frac{dM}{dt} = 0, \qquad \frac{dP}{dt} = -\frac{\beta}{2}u_x^2\Big|_{x=-L_1}^{x=L_2}, \qquad \frac{dE}{dt} = 0, \qquad (6)$$

Numerical Method

To solve the main problem numerically, we use a strictly conservative implicit staggered scheme, which is inevitably nonlinear

$$\frac{q_i^{n+1/2} - q_i^{n-1/2}}{\tau} = \frac{u_i^{n+1} + u_i^{n-1}}{2} - \frac{\alpha}{2h}(q_{i+1}^{n+1/2} - q_{i-1}^{n+1/2})(u_i^{n+1} + u_i^{n-1})$$
$$+ \frac{\beta_1}{\tau^2}(u_i^{n+1} - 2u_i^n + u_i^{n-1}) - \frac{\beta_2}{h^2}(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}),$$

$$\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{q_{i+1}^{n+1/2} - 2q_i^{n+1/2} + q_{i-1}^{n+1/2}}{h^2}$$
$$- \frac{\alpha}{2h^2}\left[(u_{i+1}^{n+1})^2 - 2(u_i^{n+1})^2 + (u_{i-1}^{n+1})^2\right]$$

Numerical Method. Quasilinearization and Inner Iterations

We construct a conservative scheme for the Galilean invariant case treated here. We introduce a regular mesh in the interval $[-L_1, L_2]$, $x_i = -L_1 + (i-1)h$, $h = (L_1 + L_2)/(N-1)$, where N is the total number of grid points. We use a simplest linearization combined with an internal iteration (referred to by the composite

superscript k). It appears to be robust enough and economical.

$$\frac{u_i^{n+1,k} - u_i^n}{\tau} = \frac{q_{i+1}^{n+1/2,k} - 2q_i^{n+1/2,k} + q_{i-1}^{n+1/2,k}}{h^2}$$
$$- \frac{\beta}{8h} \left[(u_{i+1}^{n+1,k-1})^2 - (u_{i-1}^{n+1,k-1})^2 + (u_{i+1}^n)^2 - (u_{i-1}^n)^2 \right]$$

$$\frac{q_i^{n+1/2,k} - q_i^{n-1/2}}{\tau} = -\frac{\beta}{8h} \left(q_{i+1}^{n+1/2,k-1} - q_{i-1}^{n+1/2,k-1} + q_{i+1}^{n-1/2} - q_{i-1}^{n-1/2} \right)$$
$$\times (u_i^{n+1,k} + u_i^{n-1})$$
$$- \frac{\beta}{12h^2} \left[(u_{i+1}^{n+1,k} - 2u_i^{n+1,k} + u_{i-1}^{n+1,k}) + (u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}) \right]$$
$$+ \frac{\beta}{2} \frac{u_i^{n+1,k} - 2u_i^n + u_i^{n-1}}{\tau^2} + \frac{u_i^{n+1,k} + u_i^{n-1}}{2}.$$

Numerical Method. Quasilinearization and Inner Iterations

The inner iterations start from the functions obtained at the previous time stage $u_i^{n+1,0} = u_i^0$ and $q_i^{n+1/2,0} = q_i^n$ and are terminated at certain $k = K$ when

$$\max |u_i^{n+1,K} i - u_i^{n,K-1}| \leq 10^{-13} \max |u_i^{n+1,K}|$$

The value 10^{-13} is selected to be large enough in comparison with the round-off error. In general, the number of iterations K (in our calculations we keep them around six to eight) depends on the size of time increment.

The linearized scheme has inextricably coupled five-diagonal banded matrix.

Numerical Method

We prove that the above approximation secures the conservation of energy on difference level for arbitrary potential $U(u)$, namely the difference approximations of the mass and energy are conserved by the difference scheme in the sense that $M^{n+1} = M^n$ and $E^{n+1/2} = E^{n-1/2}$.

Some Results

 Using a strongly implicit difference scheme with internal iterations allows us to follow the evolution of the solution at very long times. The system is stable even for 2.500.000 points of spatial resolution. We focus on the dynamical behavior of traveling localized solutions developing from critical initial data. The main solitary waves appear virtually non-deformed from the interaction, but additional oscillations are excited at the trailing edge of each one of them. We extract the perturbations and track their evolution for very long times when they tend to adopt a self-similar shape: their amplitudes decrease with the time while the length scales increase. We test a hypothesis about the dependence on time of the amplitude and the support of Airy-function shaped coherent structures which gives a very good quantitative agreement with the numerically obtained solutions.

   We consider the nonlinear case $c_l = 0.22$, $c_r = -1.2$. The interaction is virtually ellastic and nosignificant phase shift occurs. In the place of collision residual signals like wriggles appear and they trail the two main humps.

Airy function

The Airy function is defined by the improper integral

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos(\tfrac{t^3}{3} + xt)\mathrm{d}t.$$

It is oscillatory in the negative part of x and decays exponentially in the positive part of x. The asymptotic behavior in the negative direction is

$$Ai(-x) \sim \frac{\sin(\tfrac{2}{3}x^{3/2} + \tfrac{\pi}{4})}{\sqrt{\pi}x^{1/4}}.$$

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$
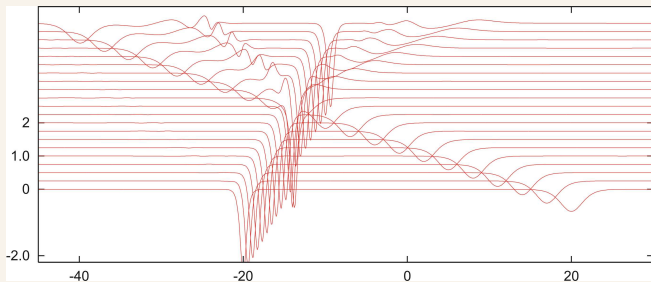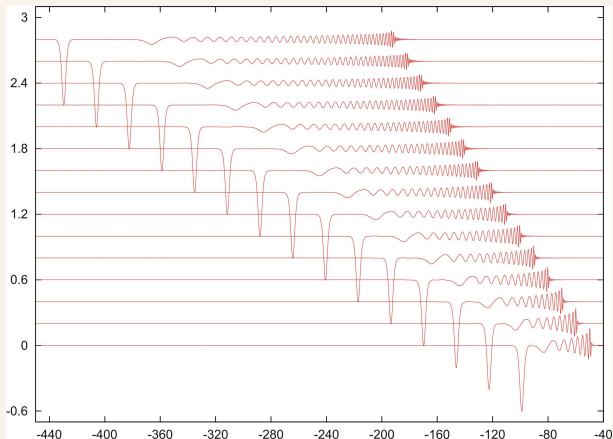

Figure 1: Short-times evolution. Forming of accompanying excitations.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$
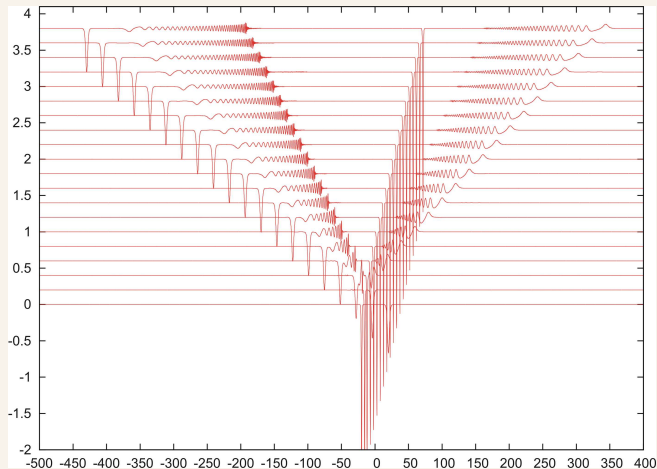
Figure 2: Middle-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$
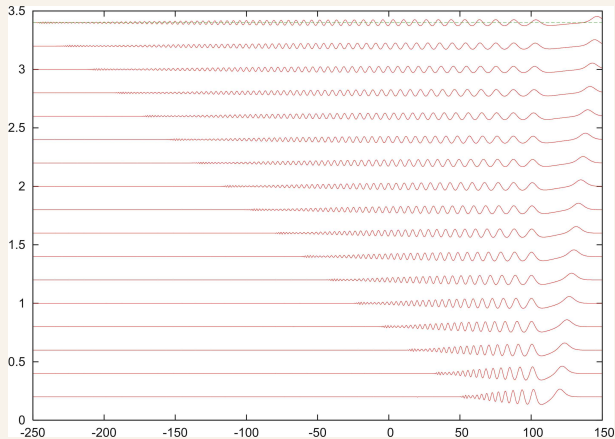

Figure 3: The evolution of the left excitation (shifted and scaled).

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

Figure 4: The right left-going soliton with its trail. Middle-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$
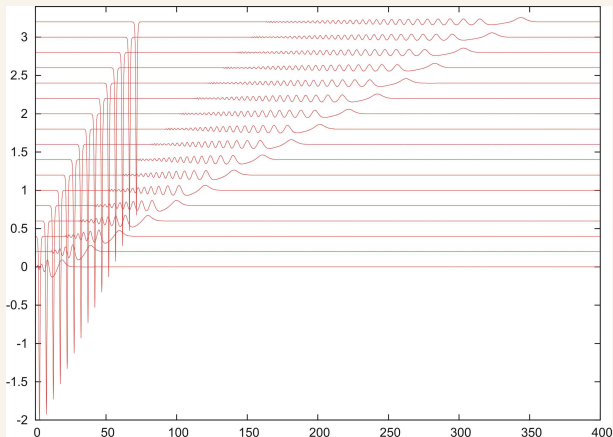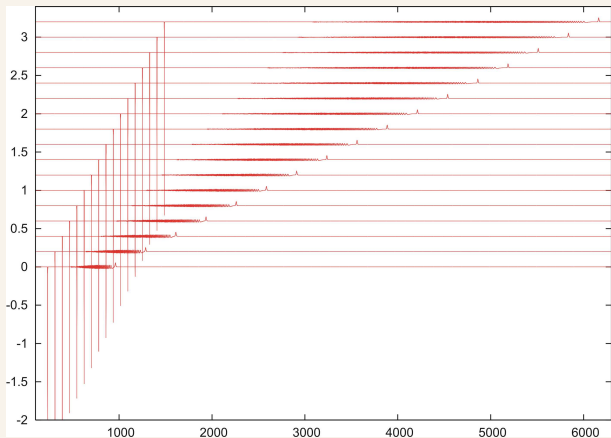
Figure 5: The left right-going soliton with its trail. Middle-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$
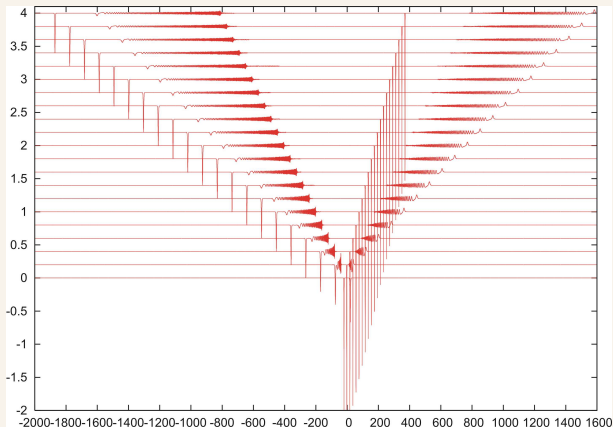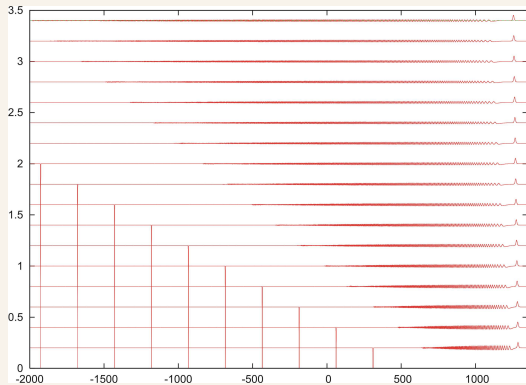

Figure 6: Long-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

Figure 7: The left right-going soliton with its trail. Very long-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$
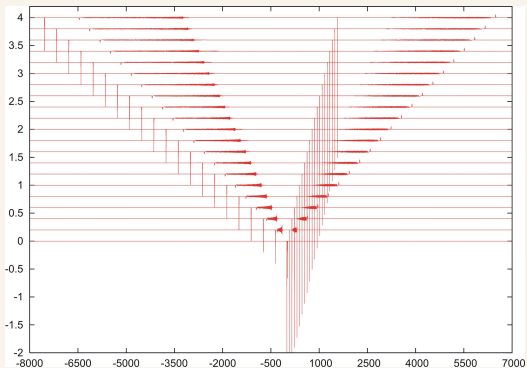

Figure 8: The right left-going soliton with its trail. Long-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$
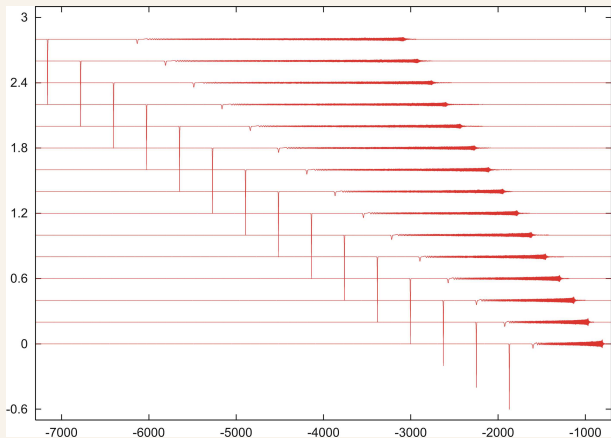

Figure 9: Very long-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

Figure 10: The right left-going soliton with its trail. Very long-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$
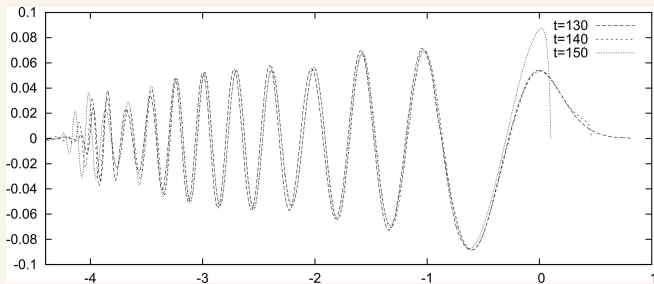

Figure 11: Graphic comparison of the trail for different times

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$
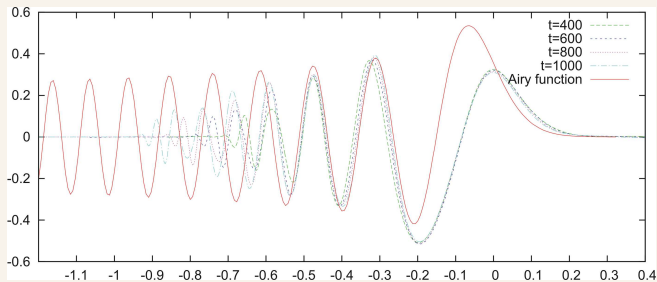

Figure 12: Graphic comparison of the trails with the Airy function.

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Thanks

Thank you for your kind attention !