NUMERICAL IMPLEMENTATION OF FOURIER-TRANSFORM METHOD FOR GENERALIZED WAVE EQUATIONS

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1. Problem Formulation

Consider the Boussinesq equation in two spatial dimensions (so called Boussinesq Paradigm Equation)

$$u_{tt} = \Delta(u - \alpha u^2 + \beta_1 u_{tt} - \beta_2 \Delta u) \tag{1}$$

where $u = u(x, y, t)$ is the surface elevation, t is the time, $\beta_1, \beta_2 > 0$ are two dispersion coefficients and α is an amplitude parameter. The initial conditions can be prepared by a single soliton (computed numerically and semi-analytically) or as a superposition of two solitons (see, for example [2], [1], [3] and [4]). The possible ways to solve numerically the above problem can be summarized in three groups: (i) by using a semi-implicit difference scheme; (ii) by using a fully implicit difference scheme; (iii) by using pseudospectral methods. In this paper we focus our attentions to the last ones.

2. Fourier Integral-Transform Method

Instead of using a multigrid solver (see, for example [5]) we can use a 2D Fourier transform. Applying it to the original equation (1) we get a second order Ordinary Differential Equation (ODE) with respect the time in the configurational space

$$\begin{aligned}[1 + 4\pi\beta_1(\xi^2 + \eta^2)]\hat{u}_{tt} \\ = -4\pi^2(\xi^2 + \eta^2)\left[1 + 4\beta_2\pi^2(\xi^2 + \eta^2)\right]\hat{u} + 4\pi^2\alpha(\xi^2 + \eta^2)\hat{N}\end{aligned} \tag{2}$$

where $\hat{u}(\xi, \eta, t) := \mathcal{F}[u]$ and $\hat{N}(\xi, \eta, t) := \mathcal{F}[u^2]$. Solving the last ODE is straightforward and requires very few operations per time step for given $\hat{N}$ but the lion's share of the computational resources are consumed by the computation of the contribution of the nonlinear term. An implicit scheme would require inverting the matrix that results from the discrete approximation of the convolution integral representing the Fourier transform of the nonlinear term u^2. The concept of the pseudospectral method is to use inverse Fourier transform to represent the sought function in the configurational space and to compute the square there, and then to "return" to the spectral space via the Fourier transform. The straightforward application of the pseudo-spectral method leads to an inherently explicit scheme, and in many case the latter us fully enough. Yet, for computations at very large times, one needs a fully conservative energy-conserving scheme. The latter is the object of the present note. We use the concept of "internal iterations" as introduced in [6].

3. Numerical Implementation of the Pseudo-Spectral Method

We introduce a uniform grid (ξ_m, η_n) in the Fourier space and discretize the Fourier integral. Suppose that we know $\hat{u}^k$, $\hat{u}^{k-1}$,..., $\hat{u}^0$. Then the next $(n+1)$-st time stage is computed from the following three-stage difference scheme

$$[1 + 4\pi\beta_1(\xi_m^2 + \eta_n^2)]\frac{\hat{u}_{mn}^{k,l+1} - 2\hat{u}_{mn}^k + \hat{u}_{mn}^{k-1}}{\tau^2}$$
$$= -2\pi^2(\xi_m^2 + \eta_n^2)[1 + 4\beta_2\pi^2(\xi_m^2 + \eta_n^2)](\hat{u}_{mn}^{k,l+1} + \hat{u}_{mn}^{k-1})$$
$$+ \frac{4}{3}\pi^2\alpha(\xi_m^2 + \eta_n^2)\mathcal{D}_F\left[(\mathcal{D}_F^{-1}[\hat{u}_{mn}^{k,l}])^2 + \mathcal{D}_F^{-1}[\hat{u}_{mn}^{k,l}]\mathcal{D}_F^{-1}[\hat{u}_{mn}^{k-1}] + (\mathcal{D}_F^{-1}[\hat{u}_{mn}^{k-1}])^2\right], \quad (3)$$

where τ is the time step, and $\mathcal{D}_F[\cdot]$ denotes the discrete Fourier transform, and $\mathcal{D}_F^{-1}[\cdot]$ is the inverse, respectively. The concept of internal iterations requires that at each time stage the linear scheme Eq. (3) starts with $u_{mn}^{k,l} = u_{mn}^k$, $l = 0$ and is repeated with increasing the number l until convergence is reached for some $l + 1 = L$. Then it is set up that $u_{mn}^{k+1} := u_{mn}^{k,L}$. Then, following [6], we show that the scheme is fully nonlinear and fully implicit and conserves the energy within the tolerance level set for the convergence of the internal iterations (can be chosen close the the round-off error of the computer). Note that the inverse Fourier transform gives a discrete function $u_{ij}^k := \mathcal{D}_F^{-1}[\hat{u}_{mn}^k]$, where i and j are the indices of a specific grid point in the configurational space.

4. Numerical Tests and Validation

We treat two 1D wave equations. In order to approximate the Fourier integrals we use specialized Filon's quadrature [8] on a uniform mesh

$$\int_{-x_\infty}^{x_\infty} u(x)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{d}x \approx \left(\frac{1}{\mathrm{i}\xi} + \frac{1 - \mathrm{e}^{-\mathrm{i}\xi h}}{\xi^2 h}\right)v_M - \left(\frac{1}{\mathrm{i}\xi} + \frac{\mathrm{e}^{-\mathrm{i}\xi h} - 1}{\xi^2 h}\right)v_0 + \frac{4}{\xi^2 h}\sin^2\frac{\xi h}{2}\sum_{m=1}^{M-1} v_m,$$

with $v \equiv u(x)\mathrm{e}^{\mathrm{i}\xi x}$, spatial step h and "actual" infinities $[-x_\infty, x_\infty]$.
The advantage of above quadrature consists in both – for $\xi h \leq 1$ it becomes a generalized trapezoidal formula with $O(h^2)$ error and when $\xi h > 1$ the order of error is as $O(M\xi^{-3}u_{xx})$ [7], [9]. Having in mind the localized nature of the sought solutions it is obvious that $\lim_{x\to\pm\infty} u_{xx} = 0$ and the decay of the quadrature error for $\xi \gg 1$ and given x_∞ in the problems in question is obeyed.

4.1. Cauchy problem for 1D string equation. Let us consider the well-known Cauchy problem

$$u_{tt} = c^2 u_{xx}, \qquad c = \text{const} > 0, \quad -\infty < x < \infty, \quad t > 0 \quad (4)$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \tag{5}$$

with exact solution given by D'Alembert's formula

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g d\theta.$$

The image of the problem (4)-(5) in the configurational space (again Cauchy problem with respect an ODE with algebraic right hand side) reads

$$\hat{u}_{tt} = -c^2\xi^2\hat{u}, \quad \hat{u}(\xi,0) = \hat{f}(\xi) \qquad \hat{u}_t(\xi,0) = \hat{g}(\xi) \tag{6}$$

and exact solution $\hat{u}(\xi,t) = \hat{f}(\xi)\cos c\xi t + \frac{\hat{g}(\xi)}{c\xi}\sin c\xi t$ where $\mathcal{F}^{-1}[\hat{u}] = u(x,t)$. Following the idea in (3), we build a standard three-stage explicit difference scheme for (6)

$$\frac{\hat{u}_m^{k+1} - 2\hat{u}_m^k + \hat{u}_m^{k-1}}{\tau^2} = -\frac{c^2\xi^2}{2}(\hat{u}_m^{k+1} + \hat{u}_m^{k-1}) \tag{7}$$

setting the phase velocity $c = 1$, and (i) $f(x) = \mathrm{e}^{-(x-X)^2}$, $g(x) = 2(x-X)\mathrm{e}^{-(x-X)^2}$, X stands for the initial position of the center of the solitary wave; (ii) the functions $f(x)$ and $g(x)$ in the initial conditions are sech-like (see the next subsection). Let us note that the scheme is stable, when $ch/\tau \leq 1$.

4.2. Regularized Long Wave Equation. If $\beta_1 = 0$ the Boussinesq equation reduces to the so-called Regularized Long Wave Equation (RLWE)

$$u_{tt} = (u - \alpha u^2 + \beta u_{tt})_{xx} \tag{8}$$

and possesses the following exact solitary-wave solution (see [6]):

$$w = -\frac{3}{2}\frac{c^2-1}{\alpha}\mathrm{sech}^2\left(\frac{x-ct}{2c}\sqrt{\frac{c^2-1}{\beta}}\right). \tag{9}$$

Here c is the phase velocity, α is the parameter of the nonlinearity, and β is the dispersion parameter. For the mechanical meaning of Eq. (8) we refer the reader to [6]. To begin the time stepping, we set

$$u(x,0) = w(x,0) \quad \text{and} \quad u(x,\tau) = w_t(x,0)\tau + w(x,0) \tag{10}$$

and transform the latter to spectral space, thus providing the two initial conditions for the 1D version of the scheme (3).

5. Results and Discussion

We show and discuss two groups of results concerning the 1D linear string equation and the 1D nonlinear RLWE. Figure 1 demonstrates the excellent comparison between the D'Alembert solution (dashed lines) and the numerical solution by the pseudospectral method (solid lines). Two running waves with Gaussian shape start

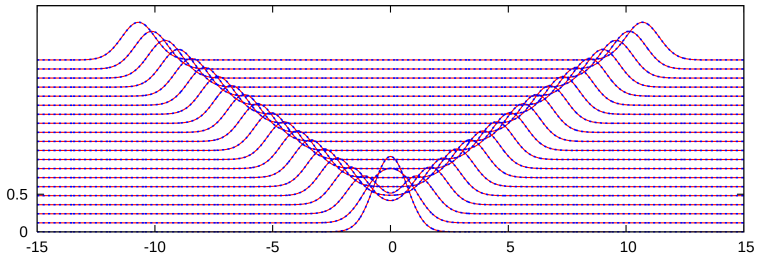

Figure 1: Comparison of the numerical solution with the D'Alembert formula.

from the coordinate origin $X = 0$ and go unchanged to the left and to the right with phase velocities $c_l = -c_r = 1$. The conclusion is that the linear wave equations can be discretized and solved numerically in the spectral space and only after the solution is obtained at each time stage, the inverse Fourier transformation can be used to restore the solution in the configuration space. As rule, the mapped differential equations are simpler compared to the original ones.

In Figure 2 the wave shapes are the same but the initial condition is a superposition of two running waves starting from different positions $-X_l = X_r = 3.5$ again with phase velocities $c_l = -c_r = 1$ which collide between them elastically.

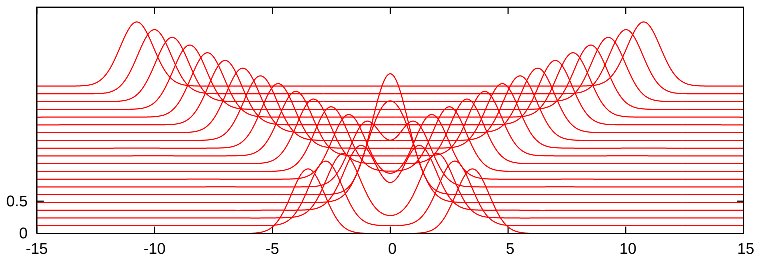

Figure 2: Superposition and elastic interaction of two Gaussian pulses.

The second part of investigation concerns 1D nonlinear dispersive generalized wave equations using RLWE as a featuring example. In the following figures the obtained numerical solutions with the described here algorithm are presented. To test the reliability of the method we compare the obtained results with these obtained by a finite difference method in [6].

In Figures 3 and 4 the head-on collisions for supercritical phase speeds that are still below the threshold of the blow-up are presented. The first figure presents a case where the nonlinearity is weaker, while in the second of these figures, the nonlinearity is considerable. In both cases, the solitons retain their individualities after the collision and no significant radiation is observed despite the fact that RLWE is not a fully integrable case. The only sign of inelasticity is the phase shift experienced by the colliding waves. For the sake of saving space it is not presented here.

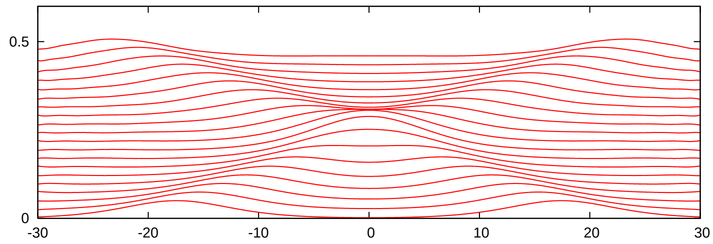

Figure 3: The inelastic interaction in RLWE for slightly supercritical phase velocities, $c_l = -c_r = 1.05$, $\alpha = -3$, $\beta = 1$.

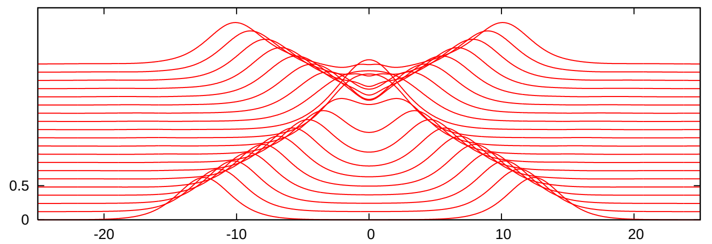

Figure 4: The interaction in RLWE near to the threshold of nonlinear blow-up, $c_l = -c_r = 1.5$, $\alpha = -3$, $\beta = 1$.

In the end, we present in Figure 5 a case known to lead to a blow-up of the solution.

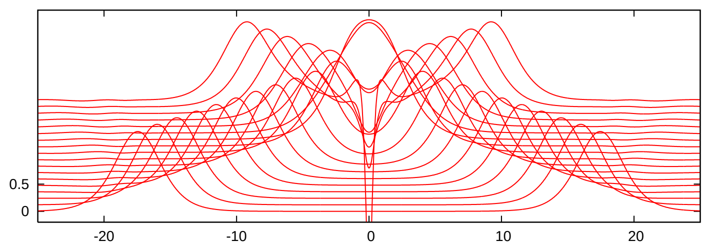

Figure 5: The blow-up in RLWE for large supercritical phase velocities, $c_l = -c_r = 2$, $\alpha = -3$, $\beta = 1$.

In all considered cases an excellent comparison with [6] is observed.

6. Conclusion

We have demonstrated that the pseudospectral methods and in particular Fourier transform can be efficient both for numerical treatment of linear and nonlinear wave equations. For the 2D and 3D equations one needs to apply 2D and 3D Fourier transforms and to follow the procedures described above.

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