Разностные схемы для уравнения Буссинеска Finite Difference Scheme for Boussinesq Paradigm Equation

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Introduction

In the present work we study the Cauchy problem for the Boussinesq Paradigm Equation (BPE)

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \alpha \Delta f(u), \quad x \in \mathbb{R}^n, \; t > 0,$$
$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

on the unbounded region $\mathbb{R}^n$ with asymptotic boundary conditions
$u(x,t) \to 0$, $\Delta u(x,t) \to 0$ as $|x| \to \infty$,
where Δ is the Laplace operator, α, β_1 and β_2 are positive constants.

This is a 4-th order equation in x and t on unbounded region with non-linearity contained in the term $f(u) = u^2$.

Referencies

BPE appears in the modeling of surface waves in shallow waters.

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For $\beta_2 > 0$ the problem is *well-posed in the sense of Hadamar*

- the derivation of BPE- Christov C.I., Wave motion, 34, 2001
- Xu&Liu (2009) – existence of a global weak solution; sufficient conditions for both the existence and the lack of a global solution.
- Polat&Ertas (2009) – local and global solution, blow-up of solutions – under different conditions for the nonlinear function $f(u)$.

We assume that the functions u_0, u_1 and $f(u)$ satisfy some regularity conditions so that a unique solution for BPE exists and is smooth enough.

theoretical study of numerical methods for 'good'BE (BPE with $\beta_1 = 0$)

- finite difference method- Ortega, Sanz-Serna, Numerische Math., 1990, 58
- finite element method, optimal error estimates- A. Pani, Saranga, Nonlinear Analysis, 29, 1997;
- pseudospectral method- Ortega, Sanz-Serna , Math. Comp., 1991, 57; for the damped BE- S. Choo, Comm. Korean Math. Soc., 13 , 1998;

numerical simulations and physical interpretations - 1D, 2D:

- Bogolubsky, I.L., Comput. Phys. Commun.,1977;
- Christov, Velarde, Intern. J Bifurcation Chaos, 4, 1994;
- Christou, M., Christov, C., AIP, 1186, 2009;
- Chertock, A., Christov, C., Kurganov, A., 2011;
- Christov, C., Kolkovska, N., Vasileva, D., LNCS, 6046, 2011;

Properties to the BPE

Let $\|\cdot\|$ denote the standard norm in $L_2(R^n)$.
Define the energy functional

$$E\left(u(t)\right) = \left\|(-\Delta)^{-1/2}\frac{\partial u}{\partial t}\right\|^2 + \beta_1\left\|\frac{\partial u}{\partial t}\right\|^2 + \|u\|^2 + \beta_2\|\Delta u\|^2 + 2\int_{R^n} F(u)dx$$

with

$$F(u) = \alpha \int_0^u f(s)ds$$

Theorem (Conservation law)

The solution u to Boussinesq problem satisfies the following energy identity

$$E\left(u(t)\right) = E\left(u(0)\right).$$

We obtain similar energy identities for the solutions of the FDS employed in the discretization of BPE.

Notations

- Domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$, L_1, L_2 – sufficiently large;
- a uniform mesh with steps h_1, h_2 in Ω: $x_i = ih_1, i = -M_1, M_1$; $y_j = jh_2, j = -M_2, M_2$;
- τ - the time step, $t_k = k\tau, k = 0, 1, 2, \ldots$;
- mesh points (x_i, y_j, t_k);
- $v^k_{(i,j)}$ denotes the discrete approximation $u(x_i, y_j, t_k)$;
- notations for some discrete derivatives of mesh functions:
  - $v^k_{x,(i,j)} = (v^k_{(i+1,j)} - v^k_{(i,j)})/h_1$; $\quad v^k_{\bar{x},(i,j)} = (v^k_{(i,j)} - v^k_{(i-1,j)})/h_1$;
  - $v^k_{\bar{x}x,(i,j)} = \left(v^k_{(i+1,j)} - 2v^k_{(i,j)} + v^k_{(i-1,j)}\right)/h_1^2$;
  - $v^k_{\bar{t}t,(i,j)} = \left(v^{k+1}_{(i,j)} - 2v^k_{(i,j)} + v^{k-1}_{(i,j)}\right)/\tau^2$;
  - $\Delta_h v = v_{\bar{x}x} + v_{\bar{y}y}$ – the 5-point discrete Laplacian.
  - $(\Delta_h)^2 v = v_{\bar{x}x\bar{x}x} + v_{\bar{y}y\bar{y}y} + 2v_{\bar{x}x\bar{y}y}$ – the discrete biLaplacian

Whenever possible the arguments of the mesh functions ${}^k_{(i,j)}$ are omitted.

Finite Difference Schemes

In approximation of $\Delta_h v$ and $(\Delta_h)^2 v$ we use v^θ – the symmetric θ-weighted approximation to $v^k_{(i,j)}$:

$v^{\theta,k}_{(i,j)} = \theta v^{k+1}_{(i,j)} + (1 - 2\theta) v^k_{(i,j)} + \theta v^{k-1}_{(i,j)}, \ \theta \in R.$

for approximation of non-linear term $f(u(x_i, y_j, t_k))$ we use

- $f(v^k)$;
- $f_1(v^k) = \dfrac{F(v^{k+1}) - F(v^{k-1})}{v^{k+1} - v^{k-1}}, \quad F(u) = \alpha \int_0^u f(s)ds;$
- $f_2(v^k) = 2\dfrac{F(0.5(v^{k+1} + v^k)) - F(0.5(v^k + v^{k-1}))}{v^{k+1} - v^{k-1}}.$

$f(v)$ is a polynomial of v, thus the integral $F(v)$ used in f_1, f_2 is explicitly evaluated!

Explicit (with respect to the nonlinearity) Method (EM)

$$v^k_{\bar{t}t} - \beta_1 \Delta_h v^k_{\bar{t}t} - \Delta_h v^{\theta,k} + \beta_2 (\Delta_h)^2 v^{\theta,k} = \Delta_h f(v^k).$$

Implicit (with respect to the nonlinearity) Methods (IM 1,2)

$$v^k_{\bar{t}t} - \beta_1 \Delta_h v^k_{\bar{t}t} - \Delta_h v^{\theta,k} + \beta_2 (\Delta_h)^2 v^{\theta,k} = \Delta_h f_1(v^k); \;\; f_2(v^k)$$

Initial conditions

$$\begin{aligned} v^0_{(i,j)} &= u_0(x_i, y_j), \\ v^1_{(i,j)} &= u_0(x_i, y_j) + \tau u_1(x_i, y_j) \\ &+ 0.5\tau^2 (I - \beta_1 \Delta_h)^{-1} \left(\Delta_h u_0 - \beta_2 (\Delta_h)^2 u_0 + \alpha \Delta_h f(u_0) \right) (x_i, y_j). \end{aligned}$$

The equations, boundary and initial conditions form three families of finite difference schemes.

Algorithm

$$\left(v^{k+1} - 2v^k + v^{k-1}\right)/\tau^2 - \beta_1 \Delta_h \left(v^{k+1} - 2v^k + v^{k-1}\right)/\tau^2$$
$$- \theta\Delta_h v^{k+1} - (1-2\theta)\Delta_h v^k - \theta\Delta_h v^{k-1}$$
$$+ \beta_2\theta(\Delta_h)^2 v^{k+1} + \beta_2(1-\theta)(\Delta_h)^2 v^k + \beta_2\theta(\Delta_h)^2 v^{k-1}$$
$$= 2\Delta_h \frac{F(0.5(v^{k+1}+v^k)) - F(0.5(v^k+v^{k-1}))}{v^{k+1}-v^{k-1}}$$

The inner iterations for evaluation of v^{k+1} start from v^k. They stop when the relative error between two successive iterations is less than a given threshold $\epsilon = 10^{-13}$.

- 1D case – 5-diagonal linear system of equations
- 2D case – splitting procedure and 5-diagonal linear systems in each direction

Analysis of the nonlinear schemes

Preliminaries:

the space of mesh functions which vanish on ω;

the scalar product at time t^k with respect to the spatial variables

$\langle v, w \rangle = \sum_{i,j} h_1 h_2 v_{(i,j)}^{(k)} w_{(i,j)}^{(k)}$;

operators $A = -\Delta_h$

$$B = I - \beta_1 \Delta_h + \tau^2 \theta(-\Delta_h + \beta_2 (\Delta_h)^2);$$

A, B are self-adjoint positive definite operators.

The energy functional E_h^L (obtained from the linear part of the equation) at the k-th time level is

$$E_h^L(v^{(k)}) =$$
$$\left\langle A^{-1} v_t^{(k)}, v_t^{(k)} \right\rangle + \beta_1 \left\langle v_t^{(k)}, v_t^{(k)} \right\rangle + \tau^2(\theta - 1/4) \left\langle (I + \beta_2 A) v_t^{(k)}, v_t^{(k)} \right\rangle$$
$$+ 1/4 \left\langle v^{(k)} + v^{(k+1)} + \beta_2 A(v^{(k)} + v^{(k+1)}), v^{(k)} + v^{(k+1)} \right\rangle$$

The *full discrete energy functional* for IM1 is (including the non-linearity)

$$E_h(v^{(k)}) = E_h^L(v^{(k)}) + \left\langle F(v^{(k+1)}), 1 \right\rangle + \left\langle F(v^{(k)}), 1 \right\rangle$$

The *full discrete energy functional* for IM2 is

$$E_h(v^{(k)}) = E_h^L(v^{(k)}) + 2 \left\langle F(0.5(v^{(k+1)} + v^{(k)})), 1 \right\rangle$$

Theorem (Discrete conservation laws)

The solutions to the implicit methods IM1 and IM2 satisfy the energy equalities

$$E_h(v^{(k)}) = E_h(v^{(0)}), \qquad k = 1, 2, \ldots .$$

i.e. the discrete energy is conserved in time.

Theorem (Discrete identities for EM)

The solution to the explicit scheme EM satisfies the equalities (k=1,2,...)

$$E_h^L(v^{(k)}) + (f(v^k), v^{k+1}) = E_h^L(v^{(k-1)}) + (f(v^{(k)}), v^{(k-1)}).$$

$$\theta > \frac{1}{4} - \frac{\beta_1}{\tau^2 ||I + \beta_2 A||}. \quad (1)$$

Note that if parameter θ satisfies (**??**), then functional $E_h^L(v^k)$ is nonnegative and can be viewed as a norm. Such combined norms depending on the values of solution on several layers are typical for three-layer schemes.

The local truncation error of the schemes is $O(|h|^2 + \tau^2)$.

Theorem (Convergence of the EM)

Let the parameter θ satisfies (??). Assume that the solution u to BPE obeys $u \in C^{4,4}\left(\mathbb{R}^2 \times (0,T)\right)$ Then the solution v to EM converges to u as $|h|, \tau \to 0$ and the following estimate holds for the error $z = y - u$ of the scheme:

$$\left(z^{(k)}, z^{(k)}\right) + \left(Az^{(k)}, z^{(k)}\right) \leq Ce^{Mt_k}\left(|h|^2 + \tau^2\right)^2.$$

with a constant M chosen so that $\max\limits_{i,j,s \leq k}(|u(x_i, y_j, t_s)|, |v_{i,j}^{(s)}|) \leq M$.

Theorem (Convergence of the Implicit Methods IM1, IM2)

Let $f(u) = u^2$ and the parameter θ satisfies (??). Assume that the solution u to BPE obeys $u \in C^{4,4}\left(\mathbb{R}^2 \times (0,T)\right)$ and the solution v to IM1 is bounded in the maximal norm. Let M be a constant such that

$$M \geq \max_{i,j,s \leq k} \left(|u(x_i, y_j, t_s)|, \left| \frac{\partial^2 u}{\partial t^2}(x_i, y_j, t_s) \right|, |v_{i,j}^{(s)}| \right)$$

and τ be sufficiently small, $\tau < (2C_2M)^{-1}$. Then v converges to the exact solution u as $|h|, \tau \to 0$ and the following estimate holds for the error $z = y - u$:

$$\left(z^{(k)}, z^{(k)}\right) + \left(Az^{(k)}, z^{(k)}\right) \leq Ce^{Mt_k}\left(|h|^2 + \tau^2\right)^2.$$

The main feature of theorems is the established second order of convergence in discrete W_2^1 norm, which is compatible with the rate of convergence of the similar linear problem.

Corollary

(i) The convergence of the solution to FDS with $\theta \geq 0.25$ to the exact solution is of second order when $|h|$ and τ go independently to zero.
(ii) For the scheme with $\theta = 0$ the convergence of the numerical solution to the exact solution is of second order when $|h|$ and τ go to 0 provided $\tau^2 < \frac{4}{9}\frac{\beta_1}{\beta_2}h^2$.

Corollary

Under the assumptions of the main theorems the FDS admits the following error estimates in the uniform norm $(z = y - u)$:

$$\max_i |z_i^{(k)}| < Ce^{Mt_k}\left(|h|^2 + \tau^2\right), \qquad d = 1;$$

$$\max_{i,j} |z_{i,j}^{(k)}| < Ce^{Mt_k}\sqrt{\ln N}\left(|h|^2 + \tau^2\right), \qquad d = 2.$$

The above estimates are optimal for the 1D case and *almost* optimal (up to a logarithmic factor) for the 2D case.

- The boundedness of the exact solution u to the BPE on the time interval $[0, T]$ is a main assumption in the convergence theorems.
- BPE may have both bounded on the time interval $[0, \infty)$ solutions or blowing up solutions
- the L_∞ norm of the exact solution is included in the exponent in the right-hand sides of the error estimates
- if u blows up at a moment T_0, $T_0 > T$, then: $\|u\|_{L_\infty[0,T]}$ will be big ; the term e^{MT} will be big ; the convergence will slow up!
- additional restriction on the time step in the convergence theorem is

$$\tau < (2C_2M)^{-1}, \quad M \geq \|u\|_{L_\infty[0,T]}.$$

In any case the FDS should be applied with very small τ's if one would like to evaluate the solution in a neighborhood of the blow up moment.

Preliminaries

- An analytical solution of the 1D equation (one solitary wave):

$$u(x,t;x_0,c) = \frac{3}{2}\frac{c^2-1}{\alpha}\operatorname{sech}^2\left(\frac{x-x_0-ct}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}\right),$$

  where x_0 is the initial position of the peak of the solitary wave,
- Parameters: $\alpha = 3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, c is the wave speed.
- Initial conditions for one solitary wave or two solitary waves:

$$u(x,0) = u(x,0;-40,2) + u(x,0;50,-1.5)$$

$$\frac{du}{dt}(x,0) = u(x,0;-40,2)_t + u(x,0;50,-1.5)_t$$

- Two conservative implicit schemes with $\theta = 0.5$; inner iterations until relative error $< \epsilon$, $\epsilon = 10^{-13}$.
  - IM1 (2010) , f_1
  - IM2 (2011) , f_2

One solitary wave, EM and IM1

Rate of convergence and errors
for $x \in [-100, 100]$, $t \in [0, 20]$, $c = 2$

$h = \tau$	Rate EM	Rate IM1	Error EM	Error IM1	IM1/ EM
0.1	–	–	0.02559	0.32271	12.60931
0.05	2.02762	1.87037	0.00628	0.08826	14.06140
0.025	2.00675	1.96892	0.00156	0.02255	14.43498
0.0125	2.00142	1.99221	0.00039	0.00567	14.52742

- The error is the difference between the calculated and the exact solution in uniform norm for $t = 20$.
- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.
- For one solitary wave the explicit scheme (EM) is about 14 times more precise than the implicit scheme (IM1).

Interaction of two solitary waves with different speeds

Rate of convergence and errors
for $x \in [-150, 150]$, $t \in [0, 40]$, $c_1 = 2$, $c_2 = -1.5$

$h = \tau$	Rate EM	Rate IM1	Error EM	Error IM1	IM1/ EM
0.04	2.09561	1.97465	0.017375	0.102754	5.913796
0.02	1.94485	1.99369	0.017375	0.026027	6.187079
0.01	1.97704	1.99838	0.001084	0.006528	6.021106

- For every h the error is calculated by Runge method as $E_1^2/(E_1 - E_2)$ with $E_1 = \|u_{[h]} - u_{[h/2]}\|$, $E_2 = \|u_{[h/2]} - u_{[h/4]}\|$, where $u_{[h]}$ is the calculated solution with step h for $t = 40$.
- The numerical rate of convergence is $(\log E_1 - \log E_2)/\log 2$.
- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.
- For two solitary waves the scheme EM is about 6 times more precise than the scheme IM1.

Rate of convergence and errors, implicit schemes, case of one solitary wave

$$\beta_1 = 1.5,\ \beta_2 = 0.5,\ \alpha = 3,\ c = 2,\ x \in [-40, 120],\ T = 40.$$

$h = \tau$	Rate IM1	Rate IM2	Er. IM1	Er. IM2	IM1/IM2
0.2	–	–	0.265115	0.144106	1.83
0.1	1.8836	1.9411	0.071849	0.037527	1.91
0.05	1.9720	1.9852	0.018315	0.009478	1.93
0.025	1.9929	1.9961	0.004601	0.002376	1.94
0.0125	1.9966	1.9961	0.001153	0.000596	1.93

$$E_1 = ||\tilde{u} - u_{[h]}||, \quad E_2 = ||\tilde{u} - u_{[h/2]}|| \quad \text{Rate} = \log_2(E_1/E_2)$$

$$\text{Error} = \max_{0\le i\le N} |\tilde{u}_i - u_{[h],i}|$$

Rate of convergence and errors, case of two solitary waves

$$\beta_1 = 1.5,\ \beta_2 = 0.5,\ \alpha = 3,\ c_1 = 2,\ c_2 = -1.5,\ x \in [-160, 170],\ T = 80.$$

$h = \tau$	Rate IM1	Rate IM2	Er. IM1	Er. IM2	IM1/IM2
0.1	–	–	–	–	–
0.05	1.9634	1.9819	0.126497	0.066214	1.91
0.025	1.9931	2.0000	0.032210	0.016692	1.93
0.0125	2.1730	2.1789	0.007785	0.004034	1.93

$$\text{Error} = E_1^2/(E_1 - E_2), \quad E_1 = ||u_{[h]} - u_{[h/2]}||, \quad E_2 = ||u_{[h/2]} - u_{[h/4]}||$$

- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.
- For one soliton and two solitary waves the scheme IM2 is about 2 times more precise than the implicit scheme IM1.

With respect to the error magnitude the scheme with RHS f_2 (IM2) performs twice better than the scheme with RHS f_1 (IM1)!

Justification: Consider the right-hand side of the FDS. We expand f_1, f_2 in Taylor series about the point (x_i, t^k) and get

$$f_1(u(x_i,t^k)) = f(u(x_i,t^k)) + \tau^2 R_1 + O(\tau^3),$$
$$f_2(u(x_i,t^k)) = f(u(x_i,t^k)) + \tau^2 R_2 + O(\tau^3),$$
$$R_1 = \frac{1}{2}\alpha\frac{\partial f}{\partial u}(x_i,t^k)\frac{\partial^2 u}{\partial t^2}(x_i,t^k),$$
$$R_2 = \frac{1}{4}\alpha\frac{\partial f}{\partial u}(x_i,t^k)\frac{\partial^2 u}{\partial t^2}(x_i,t^k).$$

Thus, $R_1 = 2R_2$. This has essential impact on the error, when the solution has large derivatives $(f(u) = u^3)$!

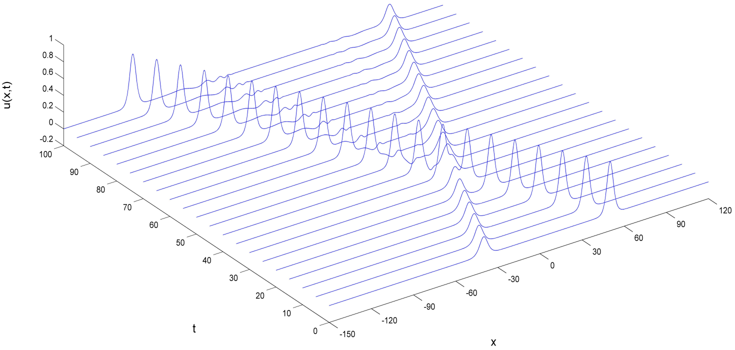
Discrete identities errors

The error is maximum for every $t \in [0, 40]$
of the numerical integral for $x \in [-150, 150]$,
either for one solitary wave with $c_1 = 2$
or for two solitary waves with $c_1 = 2$, $c_2 = -1.5$.

$\tau = h$	1 soliton EM	1 soliton IM1	2 solitons EM	2 soliton IM1
0.1	3.1264e-13	2.3152e-13	9.3245e-11	5.6192e-10
0.05	3.9790e-13	4.1866e-13	1.3416e-11	7.3909e-11
0.025	6.2528e-13	5.3321e-13	2.1630e-12	9.3973e-12
0.0125	1.0232e-13	8.9952e-13	1.2921e-12	1.2091e-12

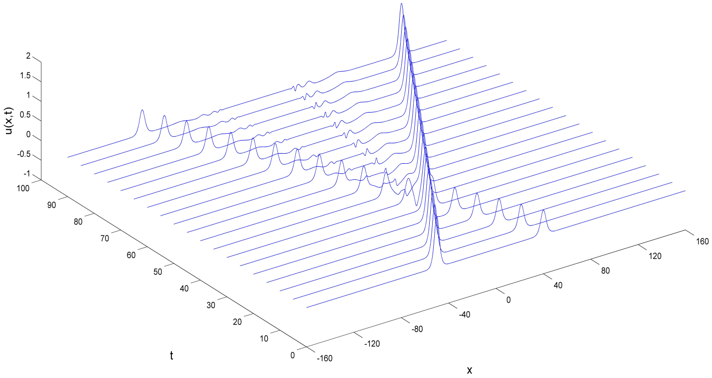
- The discrete identities are different for the iterative and for the non-iterative schemes (conservation laws for IM1, IM2 and discrete identities for EM)
- The table shows the numerical solution satisfies the respective discrete identities.

parameteres: $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$



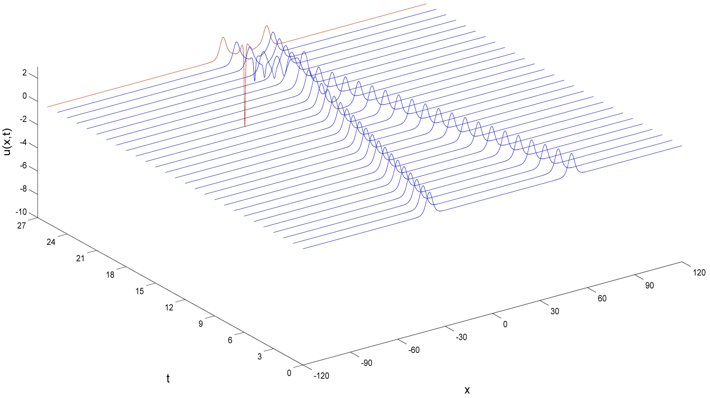
Фигура: Iteraction of two solitons: $c_1 = 1.2$, $c_2 = -1.5$.

parameters: $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$



Фигура: Iteraction of two solitons: $c_1 = 1.9$, $c_2 = -1.5$.

$\beta_1 = 1.5,\ \beta_2 = 0.5,\ \alpha = 3$



Фигура: Iteraction of two solitons: $c_1 = -c_2 = -2.2$,

$t^* \approx 27$, t^* - blow up time

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Thank you
for your attention!