A Multicomponent Alternating Direction Method for Numerical Solving of Boussinesq Paradigm Equation

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In the present work we study the Cauchy problem for the Boussinesq Paradigm Equation (BPE)

$$\frac{\partial^2 u}{\partial t^2} - \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} = \Delta u - \beta_2 \Delta^2 u + \alpha \Delta f(u), \quad x \in \mathbb{R}^n, \; t > 0,$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

on the unbounded region $\mathbb{R}^n$ with asymptotic boundary conditions

$u(x,t) \to 0$, $\Delta u(x,t) \to 0$ as $|x| \to \infty$,

where Δ is the Laplace operator, α, β_1 and β_2 are positive constants.

This is a 4-th order equation in x on unbounded region with non-linearity contained in the term $f(u) = u^p, p \geq 2$. The BPE is unsolved relative to the time derivative $\frac{\partial^2 u}{\partial t^2}$. (Sobolev type equation)

References

BPE appears in the modeling of surface waves in shallow waters.

For $\beta_2 > 0$ the problem is *well-posed in the sense of Hadamar*

- the derivation of BPE- Christov C.I., Wave motion, 34, 2001
- Xu&Liu (2009) – existence of a global weak solution; sufficient conditions for both the existence and the lack of a global solution.
- Polat&Ertas (2009) – local and global solution, blow-up of solutions – under different conditions for the nonlinear function $f(u)$.

We assume that the functions u_0, u_1 and $f(u)$ satisfy some regularity conditions so that a unique solution for BPE exists and is smooth enough.

theoretical study of numerical methods for 'good'BE (BPE with $\beta_1 = 0$)

- finite difference method (Ortega, Sanz-Serna, Numerische Math., 1990)
- finite element method, optimal error estimates (A. Pani, Saranga, Nonlinear Analysis, 1997);
- pseudo spectral method (Frutos, Ortega, Sanz-Serna , Math. Comp., 1991); for the damped BE (Choo, Comm. Korean Math. Soc., 1998);

numerical simulations and physical interpretations - 1D, 2D:

- Christov, C.I., Wave motion, 2001; Christov, Velarde, Intern. J Bifurcation Chaos, 1994;
- Chertock, A., Christov, C., Kurganov, A. 2011;
- Christov, C., Kolkovska, N., Vasileva, D., LNCS, 2011;
- Kolkovska, N., Dimova, LNCS (2011); CEJM (2012)

Splitting methods for multidimensional problems

- splitting with respect to physical processes
- coordinate (spatial) splitting
- locally one dimensional methods (Yanenko, Samarskii, Marchuk...) the alternating triangular method (Samarskii (1964),...)
- alternating direction implicit methods (ADI): Peaceman and Rachford (1955), Douglas (1955), 2D parabolic problems in 3D the scheme is not absolutely stable (Yanenko, 1965)
- multicomponent ADI schemes or vector additive schemes (Abrashin (1990), Zhadaeva, Samarskii, Vabishchevich, ...) At each time step we obtain n discrete solutions satisfying n discrete schemes, approximating the differential equation.

The aim of the lecture: To construct and analyze a multicomponent ADI method for numerical solving of BPE.

Notations

- We consider 2D case, $n = 2$.
- Domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$, L_1, L_2 – sufficiently large;
- a uniform mesh with steps h_1, h_2 in Ω:
  $x_i = ih_1, i = -M_1, M_1;$ $y_j = jh_2, j = -M_2, M_2;$
- τ - the time step, $t_k = k\tau, k = 0, 1, 2, \dots;$
- mesh points (x_i, y_j, t_k);
- $v_{(i,j)}^{(k)}$ denotes the discrete approximation to $u(x_i, y_j, t_k)$;
- notations for some discrete operators:
  - $A_1 v_{(i,j)}^{(k)} = -\left(v_{(i+1,j)}^{(k)} - 2v_{(i,j)}^{(k)} + v_{(i-1,j)}^{(k)} \right) / h_1^2$
  - $A_2 v_{(i,j)}^{(k)} = -\left(v_{(i,j+1)}^{(k)} - 2v_{(i,j)}^{(k)} + v_{(i,j-1)}^{(k)} \right) / h_2^2,$
  - $v_{t,(i,j)}^{(k)} = (v_{(i,j)}^{(k+1)} - v_{(i,j)}^{(k)})/\tau;$ $v_{\bar{t},(i,j)}^{(k)} = \left(v_{(i,j)}^{(k)} - v_{(i,j)}^{(k-1)} \right) / \tau$

Whenever possible the arguments of the mesh functions $\begin{smallmatrix}(k)\\(i,j)\end{smallmatrix}$ are omitted.

Multicomponent ADI Method for BPE

At each time level k we have two discrete approximations $v_{i,j}^{(1)(k)}$, $v_{i,j}^{(2)(k)}$ to the continuous function u. We solve with respect to $v^{(1)(k+1)}$ and $v^{(2)(k+1)}$ the following system of equations

$$
\begin{aligned}
v_{\bar{t}t}^{(1)(k)} &+ \beta_1 A_1 v_{\bar{t}t}^{(1)(k)} + A_1 v^{(1)(k+1)} + \beta_2 A_1^2 v^{(1)(k+1)} + \beta_2 A_1 A_2 v^{(1)(k)} \\
&+ A_2 v^{(2)(k)} + \beta_2 A_2^2 v^{(2)(k)} + \beta_2 A_1 A_2 v^{(2)(k)} + \beta_1 A_2 v_{\bar{t}t}^{(2)(k-1)} \\
&+ A_1 f(v^{(1)(k)}) + A_2 f(v^{(2)(k)}) = 0, \\
v_{\bar{t}t}^{(2)(k)} &+ \beta_1 A_2 v_{\bar{t}t}^{(2)(k)} + A_2 v^{(2)(k+1)} + \beta_2 A_2^2 v^{(2)(k+1)} + A_1 v^{(1)(k+1)} \\
&+ \beta_2 A_1^2 v^{(1)(k+1)} + \beta_2 A_1 A_2 v^{(1)(k+1)} + \beta_2 A_1 A_2 v^{(2)(k)} \\
&+ \beta_1 A_1 v_{\bar{t}t}^{(1)(k)} + A_1 f(v^{(1)(k)}) + A_2 f(v^{(2)(k)}) = 0.
\end{aligned}
$$

- Nonlinearities are taken on the main time level k.

Initial conditions

The multicomponent ADI scheme is a four-level scheme. Thus values of the numerical solution on the three initial time levels are required in order to start the method.

$$v_{(i,j)}^{(m)(0)} = u_0(x_i, y_j), m = 1, 2,$$

$$v_{(i,j)}^{(m)(1)} = u_0(x_i, y_j) + \tau u_1(x_i, y_j) + 0.5\tau^2 \left(I + \beta_1(A_1 + A_2)\right)^{-1}$$
$$\left((A_1 + A_2)u_0 + \beta_2(A_1 + A_2)^2 u_0 + \alpha(A_1 + A_2)f(u_0)\right)(x_i, y_j), m = 1, 2.$$

The third initial value $v^{(m)(-1)}$, $m = 1, 2$ at time level $t = -\tau$ is determined from

$$\left(v_{(i,j)}^{(m)(1)} - 2v_{(i,j)}^{(m)(0)} + v_{(i,j)}^{(m)(-1)}\right)\tau^{-2} = \left(I + \beta_1(A_1 + A_2)\right)^{-1}$$
$$\left((A_1 + A_2)u_0 + \beta_2(A_1 + A_2)^2 u_0 + \alpha(A_1 + A_2)f(u_0)\right)(x_i, y_j), m = 1, 2.$$

Boundary conditions

For approximation of the second boundary condition

$$\Delta u(x,t) \to 0$$

the mesh is extended outside the domain Ω_h by one line at each space boundary and symmetric second-order finite differences $A_i v^{(k)}$, $A_i v^{(k)}$, $k = 1, 2$, $i = 1, 2$ are used.

The numerical implementation of Multicomponent ADI method is based on solving of a set of mesh problems in y direction and another set of mesh problems in x direction.

- Sweep in the x direction for evaluation of $v^{(1)(k+1)}$.
  1D subproblems along the lines $y = const$.
- Sweep in the y direction for evaluation of $v^{(2)(k+1)}$.
  1D subproblems along the lines $x = const$.

These 1D subproblems are five-diagonal linear systems of equations. Thus the numerical method is efficient.

Properties of the multicomponent ADI scheme

- Both finite difference equations approximate the initial equation with $O(|h|^2 + \tau)$ error.
- Both discrete solutions approximate the continuous solution (see the main Theorem below).
- The method is a generalization of classical ADI method as both FDS are absolutely stable for $n \geq 2$.
- The algorithm for evaluation of $v^{(1)(k+1)}$ and $v^{(2)(k+1)}$ is based on solving of 5-diagonal linear systems in each direction. Hence the method is efficient.

Analysis of the linear Multicomponent ADI Scheme

Let $f(u) = 0$; then BPE become linear. In the space of functions, which vanish on infinity, define operators

$$\Lambda_1(u) = -\frac{\partial^2 u}{\partial x_1^2}, \quad \Lambda_2(u) = -\frac{\partial^2 u}{\partial x_2^2}.$$

Define the functional $E(u)(t)$:

$$E(u)(t) = \left\|\Lambda_1^{1/2}\frac{\partial u}{\partial t}(\cdot,t)\right\|^2 + \left\|\Lambda_2^{1/2}\frac{\partial u}{\partial t}(\cdot,t)\right\|^2 + \beta_2\left\|(\Lambda_1+\Lambda_2)\frac{\partial u}{\partial t}(\cdot,t)\right\|^2$$
$$+ \beta_1\left\|\Lambda_1^{1/2}\frac{\partial^2 u}{\partial t^2}(\cdot,t)\right\|^2 + \beta_1\left\|\Lambda_2^{1/2}\frac{\partial^2 u}{\partial t^2}(\cdot,t)\right\|^2 + \left\|\frac{\partial^2 u}{\partial t^2}(\cdot,t)\right\|^2,$$

where $||\cdot||$ is the standard norm in $L_2(R^2)$.

Theorem (Conservation law for the linear problem)

Let $f(u) = 0$. Then $E(u)(t) = E(u)(0)$ for every $t > 0$.

Define $\vec{\mathbf{v}}^{k} = (v^{(1)(k)}, v^{(2)(k)})$ as the couple of solutions. Let $N(\vec{\mathbf{v}}^{(k)})$ be the semi-norm (energy norm):

$$N(\vec{\mathbf{v}}^{(k)}) = \|A_1^{\frac{1}{2}} v_t^{(1)(k)}\|^2 + \|A_2^{\frac{1}{2}} v_t^{(2)(k)}\|^2 + \beta_2 \|A_1 v_t^{(1)(k)} + A_2 v_t^{(2)(k)}\|^2 + \beta_1 \|A_1^{\frac{1}{2}} v_{\bar{t}t}^{(1)(k)}\|^2 + \beta_1 \|A_2^{\frac{1}{2}} v_{\bar{t}t}^{(2)(k)}\|^2 + \|v_{\bar{t}t}^{(2)(k)}\|^2.$$

- The values of $v^{(1)(k)}$, $v^{(2)(k)}$ on three consecutive time levels $(k-1), (k), (k+1)$ are included in this norm.

Theorem (Discrete identity)

The solutions $\vec{\mathbf{v}}^{(K)}$ to BPE with $f(u) = 0$ satisfy the equalities

$$N(\vec{\mathbf{v}}^{(K)}) + \tau \sum_{k=1}^{K} \tau \left(\|A_1^{\frac{1}{2}} v_{\bar{t}t}^{(1)(k)}\|^2 + \beta_2 \|A_1 v_{\bar{t}t}^{(1)(k)}\|^2 + \beta_1 \|A_1^{\frac{1}{2}} v_{\bar{t}t\bar{t}}^{(1)(k)}\|^2 \right)$$

$$+ \tau \sum_{k=1}^{K} \tau \left(\|A_2^{\frac{1}{2}} v_{\bar{t}t}^{(2)(k)}\|^2 + \beta_2 \|A_2 v_{\bar{t}t}^{(2)(k)}\|^2 + \beta_1 \|A_2^{\frac{1}{2}} v_{\bar{t}t\bar{t}}^{(2)(k)}\|^2 \right)$$

$$+ \tau \sum_{k=1}^{K} \tau \|A_1 v_t^{(1)(k)} + \beta_2 A_1^2 v_t^{(1)(k)} + \beta_2 A_1 A_2 v_t^{(2)(k-1)} + \beta_1 A_1 v_{\bar{t}tt}^{(1)(k-1)}\|^2$$

$$+ \tau \sum_{k=1}^{K} \tau \|A_2 v_t^{(2)(k)} + \beta_2 A_2^2 v_t^{(2)(k)} + \beta_2 A_1 A_2 v_t^{(1)(k)} + \beta_1 A_2 v_{\bar{t}tt}^{(2)(k-1)}\|^2$$

$$= N(\vec{\mathbf{v}}^{(0)}), \quad K = 1, 2, \ldots$$

Corollary

$$N(\vec{\mathbf{v}}^{(K)}) - N(\vec{\mathbf{v}}^{(0)}) = O(\tau), K = 1, 2, \cdots .$$

Theorem (Convergence of the Multicomponent ADI Scheme)

Assume that the solution u to BPE obeys $u \in C^{6,6}\left(\mathbb{R}^2 \times (0, T)\right)$ and the solutions $v^{(1)(k)}$, $v^{(2)(k)}$ to the multicomponent ADI scheme are bounded in the maximal norm. Then $v^{(1)(k)}$ and $v^{(2)(k)}$ converge to the exact solution u as $|h|, \tau \to 0$ and the energy norm estimate

$$N(\vec{\mathbf{z}}^{(k)}) \leq C\left(|h|^2 + \tau\right)^2, \quad k = 2, 3, \cdots, K$$

holds with a constant C independent on h and τ, where $z^{(1)(k)} = y^{(1)(k)} - u(\cdot, t^k)$ and $z^{(2)(k)} = y^{(2)(k)} - u(\cdot, t^k)$ are the errors of the method.

Corollaries

According to the main Theorem, the multicomponent ADI scheme has one and the same order of convergence $O(|h|^2 + \tau)$ for the nonlinear problem and for the linear problem.

Corollary

Under the assumptions of the main Theorem the Multicomponent ADI scheme admits the following error estimates for every $k = 2, 3, \cdots, K$

$$\|z^{(1)(k)}\| + \|z^{(2)(k)}\| + \|A_1^{0.5} z^{(1)(k)}\| + \|A_2^{0.5} z^{(2)(k)}\| \leq C\left(|h|^2 + \tau\right),$$
$$\|A_1 z^{(1)(k)} + A_2 z^{(2)(k)}\| \leq C\left(|h|^2 + \tau\right),$$
$$\|z_t^{(m)(k)}\| + \|z_{\bar{t}t}^{(m)(k)}\| \leq C\left(|h|^2 + \tau\right), \ m = 1, 2$$
$$\|z^{(m)(k)}\|_{L_\infty} \leq C\left(|h|^2 + \tau\right), \ m = 1, 2.$$

Numerical results

Parameters: $\alpha = 3$, $\beta_1 = 3$, $\beta_2 = 1$, $p = 2$. Initial conditions (from Chertok, Christov, Kurganov, 2011) correspond to a solitary wave moving along the y-axis with velocity c.

τ	h	R $v^{(1)}$	R $v^{(2)}$	τ	h	R $v^{(1)}$	R $v^{(2)}$
0.08	0.075	-	-	0.02	0.3	-	-
0.04	0.075	0.9384	0.9450	0.02	0.15	2.5502	2.6853
0.02	0.075	-	-	0.02	0.075	-	-

Table: Numerical rate of convergence, dependence on τ (left part) and h (right part); time $T = 8$

The numerical rate of convergence (in the uniform norm) is evaluated by Runge method using three nested meshes.
The calculations confirm that the schemes are of order $O(|h|^2 + \tau)$.

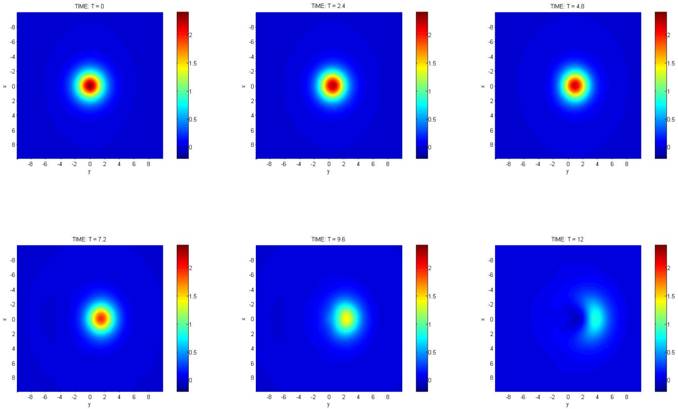

Figure: Evolution of the numerical solution in time

For $t < 5$ the shape of the numerical solution is similar to the initial solution. For larger times the numerical solution changes its initial form and transforms into a diverging propagating wave.

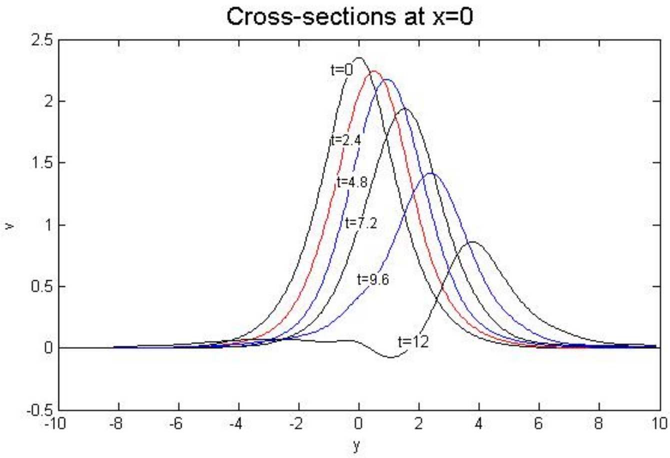

Figure: Cross section $x = 0$ of the solution $v^{(2)}$ with $c = 0.2$ at times $t = 0, 2.4, 4.8, 7.2, 9.6, 12$

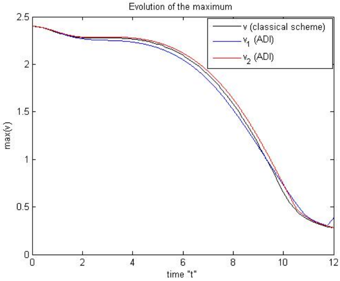

Figure: Evolution of the maximum of the solutions for $c = 0$ (left) and $c = 0.2$ (right)

Comparison with the maximum of the numerical solution obtained by the conservative scheme with accuracy $O(|h|^2 + \tau^2)$, from Christov, Vasileva, Kolkovska (2011)

Concluding remarks

- We develop a multicomponent ADI finite difference scheme for multidimensional BPE. We replace the numerical solution of the original BPE with a solution of a system of two finite difference schemes (FDS).
- The energy norm of the numerical solution to the linear FDS at each time t^k deviates from the energy norm of the initial data by a small term of first order in time step.
- Error estimates in the uniform norm and in the Sobolev mesh norms are obtained.
- Efficient algorithm for evaluation of the numerical solutions is derived. The numerical experiments show good agreement with the theoretical results.