Numerical Study of Traveling Wave Solutions to 2D Boussinesq Equation

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Posing The Problem

Hyperbolic Equation

Boussinesq Paradigm Equation (BPE):

\[ u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \beta_2 \Delta^2 u + \Delta F(u) = 0, \quad F(u) := \alpha u^2 \]

\[ u: \mathbb{R}^2 \times [0,T] \to \mathbb{R} \]

\( (x, y, t) \to u(x, y, t) \)

- Origins\(^1,\(^2\)
- Model
- Properties
  - soliton solution
  - behavior of the soliton

Hyperbolic Equation
Elliptic (Stationary) Equation

- Variable change
- Stationary BPE (S BPE)
  - solutions of type $u(x,y,t) = v(x,y-ct)$:
    \[ \beta c^2 (E - \Delta) v_{yy} - \beta \Delta v + \Delta^2 v + \alpha \beta \Delta (v^2) = 0, \quad (S \ BPE) \]
    with $\beta = \beta_1 / \beta_2$ and $\alpha, \beta > 0$; $c^2 < \text{min}(1, 1/\beta) - 4^{th}$ order elliptic equation

- Equation (S BPE) as a second order system (SYS):
  \[-(1 - c^2 \beta) \Delta v + \beta (1 - c^2) v + \alpha \beta v^2 = w \]
  \[-\Delta w = c^2 \beta (E - \Delta) v_{xx} \]

Elliptic Equation
Solver Algorithm

Simple Iteration Method

- Add artificial time
- Add false time derivatives
- Solve the new pertinent transient equation system
  - Wait for \( \hat{\nu} \) and \( \hat{\omega} \) to converge

\[
\frac{\partial \hat{\nu}}{\partial t} - (1 - c^2 \beta) \Delta \hat{\nu} + \beta (1 - c^2) \hat{\nu} + \alpha \beta \Theta \hat{\nu}^2 = \hat{\omega} \quad (SIM.1)
\]

\[
\frac{\partial \hat{\omega}}{\partial t} - \Delta \hat{\omega} = c^2 \beta (E - \Delta) \hat{\nu}_{xx}. \quad (SIM.2)
\]
## Finite Differences

<table>
<thead>
<tr>
<th>order</th>
<th>finite difference</th>
<th>second derivative approx</th>
</tr>
</thead>
<tbody>
<tr>
<td>p = 2:</td>
<td>$[1 \ -2 \ 1]$</td>
<td>$\frac{\partial^2}{\partial x^2} v = \frac{1}{h^2} \left[ (x-h) - 2v(x) + v(x-h) \right]$</td>
</tr>
<tr>
<td>p = 4:</td>
<td>$\left[ -\frac{1}{12}, \frac{4}{3}, \frac{5}{2}, \frac{4}{3}, \frac{1}{12} \right]$</td>
<td>$\frac{\partial^2}{\partial x^2} v = \frac{1}{h^2} \left[ \frac{1}{12} v(x-2h) + \frac{4}{3} v(x-h) - \frac{5}{2} v(x) + \frac{4}{3} v(x+h) - \frac{1}{12} v(x+2h) \right]$</td>
</tr>
<tr>
<td>p = 6:</td>
<td>$\left[ \frac{1}{90}, -\frac{3}{20}, \frac{3}{18}, -\frac{3}{20}, \frac{3}{90} \right]$</td>
<td>$\frac{\partial^2}{\partial x^2} v = \frac{1}{h^2} \left[ \frac{1}{90} v(x-3h) - \frac{3}{20} v(x-2h) + \frac{3}{2} v(x-h) - \frac{49}{18} v(x) + \frac{3}{2} v(x+h) - \frac{3}{20} v(x+2h) + \frac{3}{90} v(x+3h) \right]$</td>
</tr>
</tbody>
</table>
Additional Tools

- The trivial solution must be avoided
- Fix the value of the function in point \((0, 0)\)
  - \(\nu(0,0) = \theta\)
  - \(\tilde{\nu} = \theta \nu\) and \(\tilde{w} = \theta w\)

\[
-(1 - c^2 \beta) \Delta \hat{\nu} + \beta(1 - c^2) \hat{\nu} + \alpha \beta \Theta \hat{\nu}^2 = \hat{\nu} \quad (SYS.1)
\]

\[
- \Delta \hat{w} = c^2 \beta (E - \Delta) \hat{\nu}_{xx} \quad (SYS.2)
\]

- The value of \(\theta\) is found from the equation (S BPE)

\[
\theta = \left. \frac{(1 - c^2 \beta) \Delta \hat{\nu} - \beta(1 - c^2) \hat{\nu} + \hat{w}}{\alpha \beta} \right|_{x=0, y=0} \quad (TH)
\]

Boundary Condition

Solution Asymptotics

- $1/r^2$ asymptotics decay at infinity\(^4\)

\[
\beta c^2 v_{yy} - \beta \Delta v - \beta c^2 \Delta v_{yy} + \Delta^2 v + \alpha \beta \Delta(v^2) = 0, \quad (S \ BPE)
\]

- Assume that $(\partial^n/\partial r^n)v$ has $(1/r^{n+2})$ asymptotics decay at infinity

\[
\frac{\partial^2}{\partial y^2} = \sin^2 \theta \frac{\partial^2}{\partial r^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\sin 2\theta}{r} \frac{\partial^2}{\partial r \partial \theta} - \frac{\sin 2\theta}{r^2} \frac{\partial}{\partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial}{\partial r}.
\]

\[
c^2 v_{yy} = \Delta v, \quad (\text{inf})
\]

\[
v(x, y), \Delta v(x, y) \xrightarrow{\infty} \text{as} \quad r = \sqrt{x^2 + y^2} \xrightarrow{\infty}
\]

Using the given properties of the equation:

- $1/r^2$ asymptotics decay at infinity
- the symmetry of the solution
- positive/negative domains

the following formula is obtained for the boundary condition:

\[
\begin{align*}
\bar{v}(x, y) &= \mu \frac{(1-c^2)x^2 - y^2}{(1-c^2)x^2 + y^2} \quad (vB) \\
\bar{w}(x, y) &= \bar{\mu} \frac{(1-c^2)x^2 - y^2}{(1-c^2)x^2 + y^2} \quad (wB)
\end{align*}
\]
Validation

New Stop Criterion

- Choose neutral condition - the $1/r^2$ profile of the solution
- better convergence results for all finite difference schemes
- legit results
  - solution
  - boundary function
x-y cross-sections of the solution

Upper panels:
- The absolute value of the function on log-log plots.
- Black line describes \((vB)\) function with the respective \(\mu\) parameter

Lower panels show:
- Plots display \(vr^2\) values along the vertical \(z\)-axis

The solution settles down as the number of points \(N_x, N_y\) per simulation increases!

The effect of the mesh size. Lower panels: the function scaled by \(r^2\). \(N_x, N_y\) – number of mesh-points along \(x, y\) axis.
Validation - Algorithm's Convergence

<table>
<thead>
<tr>
<th>FDS</th>
<th>h</th>
<th>Errors $E_i$ in $L_2$</th>
<th>Conv Rate</th>
<th>Errors $E_i$ in $L_\infty$</th>
<th>Conv Rate</th>
<th>Diff $D_i$ in $L_2$</th>
<th>Diff $D_i$ in $L_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chr O(h^2)</td>
<td>0.2 0.1 0.05</td>
<td>1.4232e-02 3.2384e-03</td>
<td>2.135</td>
<td>1.6732e-02 3.9976e-03</td>
<td>2.065</td>
<td>9.9534e-09 3.9273e-06</td>
<td>6.7328e-08</td>
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<tr>
<td>Nat O(h^2)</td>
<td>0.2 0.1 0.05</td>
<td>1.4228e-02 3.2416e-03</td>
<td>2.134</td>
<td>1.6729e-02 4.0012e-03</td>
<td>2.063</td>
<td>6.3317e-06 1.7911e-08</td>
<td>6.7328e-08</td>
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<tr>
<td>Chr O(h^4)</td>
<td>0.2 0.1 0.05</td>
<td>1.7575e-03 1.1329e-04</td>
<td>3.955</td>
<td>2.4992e-03 1.6753e-04</td>
<td>3.898</td>
<td>1.8764e-08 3.1189e-06</td>
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<tr>
<td>Nat O(h^4)</td>
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<tr>
<td>Chr O(h^6)</td>
<td>0.4 0.2 0.1</td>
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<tr>
<td>Nat O(h^6)</td>
<td>0.4 0.2 0.1</td>
<td>2.0981e-02 3.6134e-04</td>
<td>5.859</td>
<td>2.9345e-02 5.9050e-04</td>
<td>5.635</td>
<td>1.3942e-08 1.4391e-07</td>
<td>4.9035e-08</td>
</tr>
</tbody>
</table>

1. Runge's formula for convergence rate

$$E_1 = \| \hat{v}_{[h]} - \hat{v}_{[h/2]} \|, E_2 = \| \hat{v}_{[h/2]} - \hat{v}_{[h/4]} \|,$$

2. Diff between Chr and Nat solutions

$$D_1 = \| \hat{v}.Chr_{[h]} - \hat{v}.Nat_{[h]} \|, D_2 = \| \hat{v}.Chr_{[h/2]} - \hat{v}.Nat_{[h/2]} \|, D_3 = \| \hat{v}.Chr_{[h/4]} - \hat{v}.Nat_{[h/4]} \|$$
**Derivative Convergence**

<table>
<thead>
<tr>
<th>FDS</th>
<th>$h$</th>
<th>errors in $L_2$</th>
<th>Conv. Rate</th>
<th>errors in $L_{\infty}$</th>
<th>Conv. Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c=0.45$</td>
<td>0.8</td>
<td>2.9698e-01</td>
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<td></td>
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<td>8.6465e-02</td>
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<tr>
<td></td>
<td>0.2</td>
<td>8.7696ee-02</td>
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<tr>
<td>$c=0.1$</td>
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<td></td>
<td>3.0271e-01</td>
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<tr>
<td></td>
<td>0.4</td>
<td>8.7696ee-02</td>
<td>1.9905</td>
<td>7.5691e-02</td>
<td>1.9998</td>
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<tr>
<td></td>
<td>0.2</td>
<td>8.0095e-01</td>
<td>5.6747</td>
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<tr>
<td></td>
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<td></td>
<td>2.1238e-02</td>
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</tr>
<tr>
<td></td>
<td>0.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Errors in $L_2$ and $L_{\infty}$ norms and convergence rate for fourth order $x$-derivative evaluated by the FDS with $O(h^2)$ and $O(h^6)$ approximation order.

Runge's test, evaluating the fourth $x$-derivative of the solution, show that it converges numerically. Tests for other fourth order derivatives are similar and we do not present them here.
Best-Fitt Approximation formulae

\[
\begin{align*}
    w^5(x, y, t; c) &= f(x, y) + c^2[(1 - \beta_1)g_a(x, y) + \beta_1g_b(x, y)] \\
    &\quad + c^2[(1 - \beta_1)h_1(x, y) + \beta_1h_2(x, y)\cos(2\theta)],
\end{align*}
\]

where

\[
\begin{align*}
    f(x, y) &= \frac{2.4(1 + 0.24r^2)}{\cosh(r)(1 + 0.095r^2)^{1.5}}, \\
    g_a(x, y) &= -\frac{1.2(1 - 0.177r^{2.4})}{\cosh(r)|1 + 0.11r^{2.4}|}, \\
    g_b(x, y) &= -\frac{1.2(1 + 0.22r^2)}{\cosh(r)|1 + 0.11r^{2.4}|}, \\
    h_1(x, y) &= \frac{a_tr^2 + b_tr^3 + c_tr^4 + v_tr^6}{1 + d_tr + e_tr^2 + f_tr^3 + g_tr^4 + h_tr^5 + q_tr^6 + w_tr^8}.
\end{align*}
\]

Comparison between the numerical solution $\tilde{\nu}$ and the best fit formulae $c$,$\beta$.
Thank you for your attention