Numerical Study of Traveling Wave Solutions to 2D Boussinesq Equation

Mathematics Days in Sofia

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Posing The Problem

Hyperbolic Equation

Boussinesq Paradigm Equation (BPE):

$$u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \beta_2 \Delta^2 u + \Delta F(u) = 0, \qquad F(u) := \alpha u^2$$

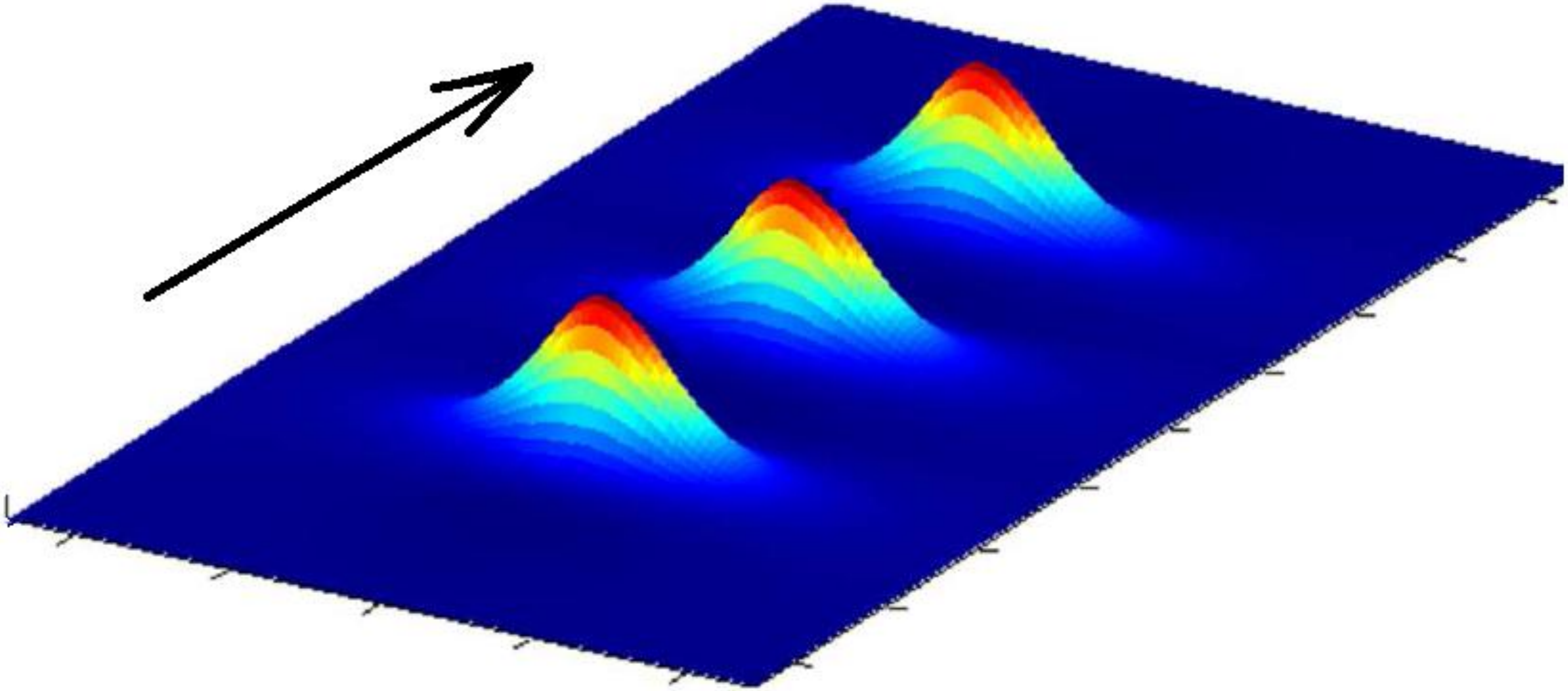
$$u : \Re^2 \times [0,T] \to \Re$$

$$(x,y,t) \to u(x,y,t)$$

- Origins[1,2]
- Model
- Properties
  - soliton solution
  - behavior of the soliton

[1]Christov, C. I. 2001, ‘An energy-consistent Galilean-invariant dispersive shallow-water model’, Wave Motion 34, 161–174.
[2]Christov, C. I. 1995a, Conservative difference scheme for Boussinesq model of surface waves, in, In: ‘Proc. ICFD V’, Oxford University Press, pp. 343–349.

Hyperbolic Equation



Elliptic (Stationary) Equation

- Variable change[3]
- Stationary BPE (S BPE)
  - solutions of type $u(x,y,t) = v(x,y-ct)$:

$$\beta c^2 (E-\Delta) v_{\overline{yy}} - \beta \Delta v + \Delta^2 v + \alpha\beta\Delta(v^2) = 0, \quad (S\ BPE)$$

with $\beta = \beta_1 / \beta_2$ and $\alpha, \beta > 0$; $c^2 < \min(1, 1/\beta)$ – 4^{th} *order elliptic equation*

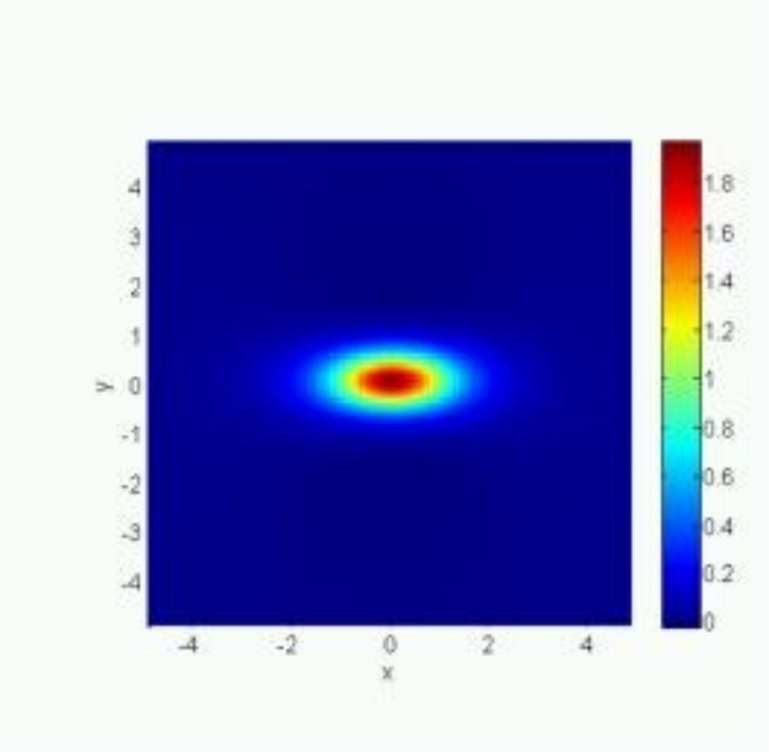
- Equation (S BPE) as a second order system (SYS):

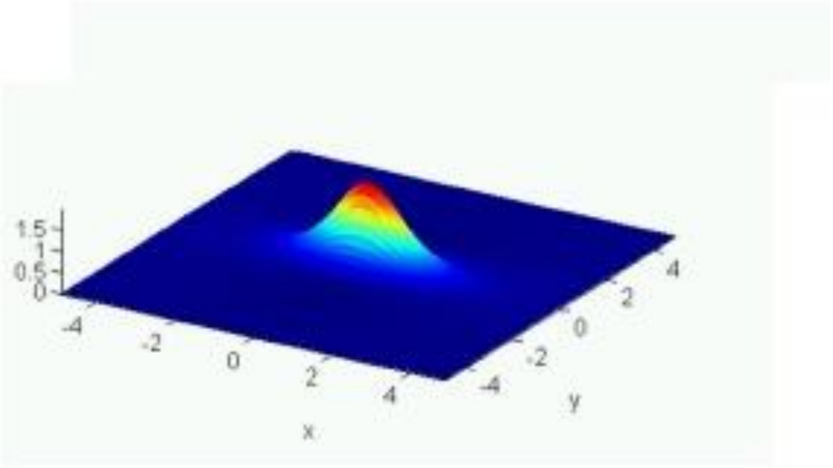
$$-(1-c^2\beta)\Delta v + \beta(1-c^2)v + \alpha\beta v^2 = w$$

$$-\Delta w = c^2\beta(E-\Delta)v_{xx}$$

[3]Kolkovska, N. 2001, ‘Two families of finite difference schemes for multidimensional Boussinesq paradigm equation. In: AIP CP, vol. 1301, pp. 395–403 (2010)

Elliptic Equation





Solver Algorithm

Simple Iteration Method

- Add artifical time
- Add false time derivatives
- Solve the new pertinent transient equation system
    - Wait for $\tilde{v}$ and $\hat{w}$ to converge

$$\frac{\partial \hat{v}}{\partial t} - (1 - c^2\beta)\Delta\hat{v} + \beta(1 - c^2)\hat{v} + \alpha\beta\theta\hat{v}^2 = \hat{w} \quad (SIM.1)$$

$$\frac{\partial \hat{w}}{\partial t} - \Delta\hat{w} = c^2\beta(E - \Delta)\hat{v}_{xx}. \quad (SIM.2)$$

Finite Differences

order	finite difference	second derivative approx
p = 2:	$[1 \quad -2 \quad 1]$	$\frac{\partial^2}{\partial \hat{x}^2} v = \frac{1}{h^2} \left[v(x-h) - 2v(x) + v(x-h) \right]$
p = 4:	$\left[-\frac{1}{12}, \frac{4}{3}, -\frac{5}{2}, \frac{4}{3}, -\frac{1}{12}\right]$	$\frac{\partial^2}{\partial \hat{x}^2} v = \frac{1}{h^2} \left[-\frac{1}{12} v(x-2h) + \frac{4}{3} v(x-h) - \frac{5}{2} v(x) + \frac{4}{3} v(x+h) - \frac{1}{12} v(x+2h) \right]$
p = 6:	$\left[\frac{1}{90}, -\frac{3}{20}, \frac{3}{2}, -\frac{49}{18}, \frac{3}{2}, -\frac{3}{20}, \frac{1}{90}\right]$	$\frac{\partial^2}{\partial \hat{x}^2} v = \frac{1}{h^2} \left[\frac{1}{90} v(x-3h) - \frac{3}{20} v(x-2h) + \frac{3}{2} v(x-h) - \frac{49}{18} v(x) + \frac{3}{2} v(x+h) - \frac{3}{20} v(x+2h) + \frac{3}{90} v(x+3h) \right]$

Additional Tools

- The trivial solution must be avoided
- Fix the value of the function in point (0, 0)[4]
  - $v(0,0) = \theta$
  - $\tilde{v} = \theta v$ and $\hat{w} = \theta w$

$$-(1-c^2\beta)\Delta\hat{v} + \beta(1-c^2)\hat{v} + \alpha\beta\theta\hat{v}^2 = \hat{w} \qquad (SYS.1)$$

$$-\Delta\hat{w} = c^2\beta(E-\Delta)\hat{v}_{xx} \qquad (SYS.2)$$

- The value of θ is found from the equation (S BPE)[4]

$$\theta = \left.\frac{(1-c^2\beta)\Delta\hat{v} - \beta(1-c^2)\hat{v} + \hat{w}}{\alpha\beta}\right|_{x=0,y=0} \qquad (TH)$$

[4]C. I. Christov, Numerical implementation of the asymptotic boundary conditions for steadily propagating 2D solitons of Boussinesq type equation, Math. Computers Simul., 82 (2012) 1079 - 1092.

Boundary Condition

- Solution Asymptotics
  - $1/r^{2}$ asymptotics decay at infinity[4]

$$\beta c^2 v_{\overline{yy}} - \beta\Delta v - \beta c^2 \Delta v_{\overline{yy}} + \Delta^2 v + \alpha\beta\Delta(v^2) = 0, \quad (S\ BPE)$$

  - Assume that $(\partial^n/\partial r^n)v$ has $(1/r^{n+2})$ asymptotics decay at infinity

$$\frac{\partial^2}{\partial y^2} \equiv \sin^2\theta \frac{\partial^2}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\sin 2\theta}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{\sin 2\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r}.$$

$$c^2 v_{yy} = \Delta v, \quad (\inf)$$

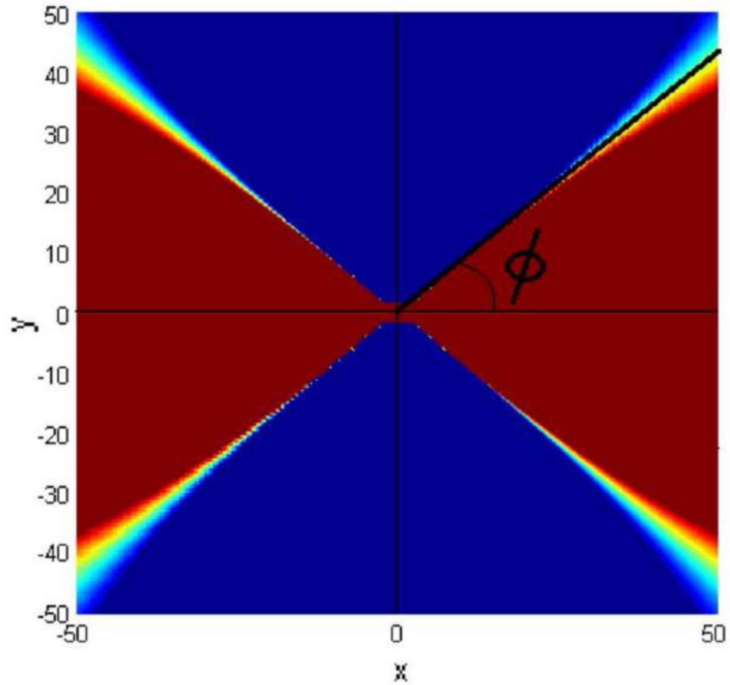
$$v(x,y), \Delta v(x,y) \longrightarrow \infty \quad \text{as} \quad r = \sqrt{x^2+y^2} \longrightarrow \infty$$

[4]C. I. Christov, Numerical implementation of the asymptotic boundary conditions for steadily propagating 2D solitons of Boussinesq type equation, Math. Computers Simul., 82 (2012) 1079 - 1092.

Using the given properties of the equation:

- $1/r^{2}$ asymptotics decay at infinity
- the symmetry of the solution
- positive/negative domains

the following formula is obtained for the boundary condition:



$$\bar{v}(x,y)=\mu\frac{\left((1-c^2)x^2-y^2\right)}{\left((1-c^2)x^2+y^2\right)^2}\qquad (vB)$$

$$\bar{w}(x,y)=\bar{\mu}\frac{\left((1-c^2)x^2-y^2\right)}{\left((1-c^2)x^2+y^2\right)^2}\qquad (wB)$$

Validation

New Stop Criterion

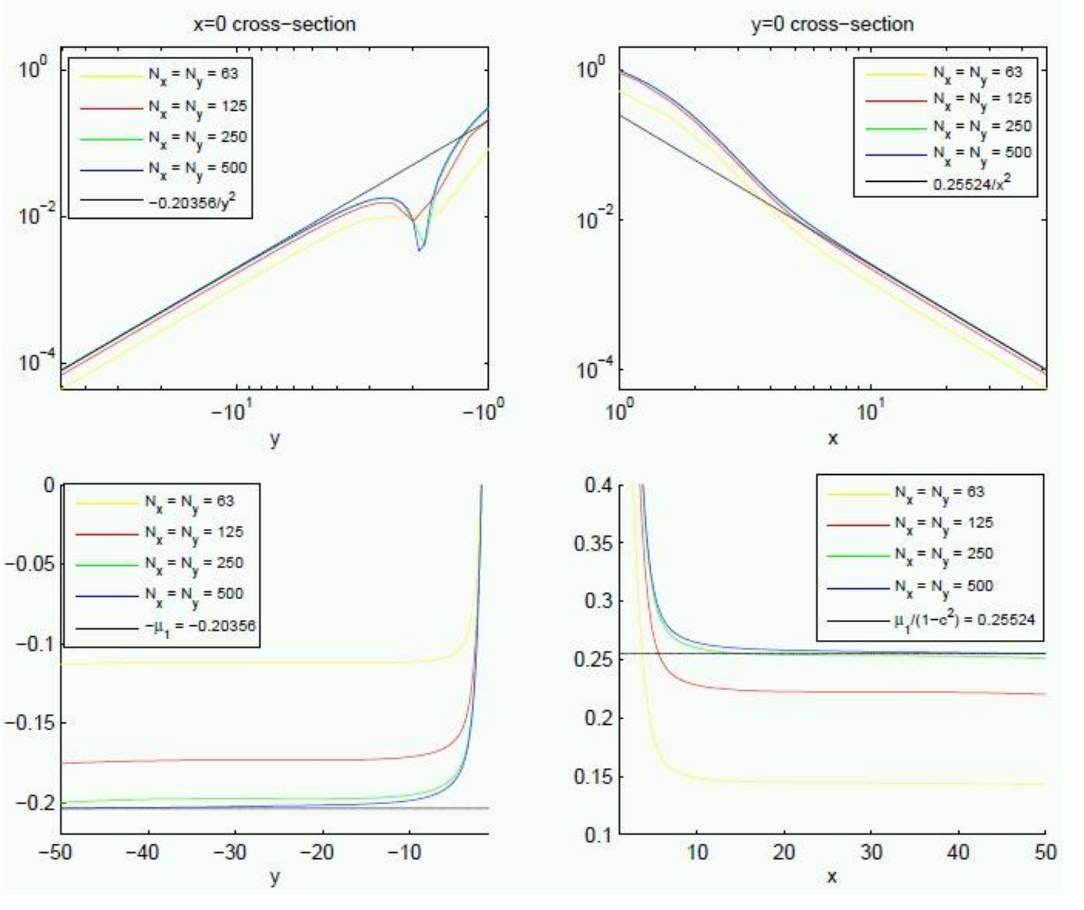
- Choose neutral condition - the $1/r^2$ profile of the solution
- better convergence results for all finite difference schemes
- legit results
  - solution
  - boundary function

x-y cross-sections of the solution

Upper panels:
-The absolute value of the function on log-log plots.
-Black line describes (vB) function with the respective *μ* parameter

Lower panels show:
-Plots display vr^2 values along the vertical z-axis

The solution settles down as the number of points N_x N_y per simulation increases!



The efect of the mesh size. Lower panels: the function scaled by r^2. N_x,N_y – number of mesh-points along x,y axis.

Validation - Algorithm's Convergence

1. Runge's formula for convergence rate

2. Diff between Chr and Nat solutions

FDS	h	Errors E_i in L_2	Conv Rate	Errors E_i in L_∞	Conv Rate	Diff D_i in L_2	Diff D_i in L_∞
Chr $O(h^2)$	0.2 0.1 0.05	1.4232e-02 3.2384e-03	2.135	1.6732e-02 3.9976e-03	2.065	9.9534e-09 3.9273e-06 6.7328e-08	1.5086e-08 6.3317e-06 1.7911e-08
Nat $O(h^2)$	0.2 0.1 0.05	1.4228e-02 3.2416e-03	2.134	1.6729e-02 4.0012e-03	2.063		
Chr $O(h^4)$	0.2 0.1 0.05	1.7575e-03 1.1329e-04	3.955	2.4992e-03 1.6753e-04	3.898	1.8764e-08 3.1189e-06 5.5434e-08	2.7887e-08 5.0020e-06 8.6233e-08
Nat $O(h^4)$	0.2 0.1 0.05	1.7548e-03 1.1584e-04	3.921	2.4957e-03 1.7092e-04	3.868		
Chr $O(h^6)$	0.4 0.2 0.1	2.0981e-02 3.6129e-04	5.859	2.9345e-02 5.9043e-04	5.635	1.0594e-08 9.6980e-08 3.0651e-08	1.3942e-08 1.4391e-07 4.9035e-08
Nat $O(h^6)$	0.4 0.2 0.1	2.0981e-02 3.6134e-04	5.859	2.9345e-02 5.9050e-04	5.635		

1. $(\log E_1 - \log E_2)/\log 2$ $\quad E_1 = \left\| \hat{v}_{[h]} - \hat{v}_{[h/2]} \right\|, E_2 = \left\| \hat{v}_{[h/2]} - \hat{v}_{[h/4]} \right\|,$

2. $D_1 = \left\| \hat{v}.Chr_{[h]} - \hat{v}.Nat_{[h]} \right\|, D_2 = \left\| \hat{v}.Chr_{[h/2]} - \hat{v}.Nat_{[h/2]} \right\|, D_3 = \left\| \hat{v}.Chr_{[h/4]} - \hat{v}.Nat_{[h/4]} \right\|$

Derivative Convergence

FDS	h	errors in L_2	Conv. Rate	errors in L_∞	Conv. Rate
c=0.45	0.8				
$O(h^2)$	0.4	2.9698e-01		4.2497e-01	
	0.2	6.8742e-02	2.1111	8.6465e-02	2.2972
c=0.1	0.8				
$O(h^2)$	0.4	3.4849e-01		3.0271e-01	
	0.2	8.7696ee-02	1.9905	7.5691e-02	1.9998
c=0.45	0.8				
$O(h^6)$	0.4	1.0766e+00		1.2316e+00	
	0.2	3.5768e-02	4.91117	5.8927e-02	4.3855
c=0.1	0.8				
$O(h^6)$	0.4	8.0095e-01		9.8911e-01	
	0.2	1.5680e-02	5.6747	2.1238e-02	5.5414

Errors in L_2 and L_{inf} norms and convergence rate for fourth order x-derivatve evaluated by the FDS with $O(h^2)$ and $O(h^6)$ approximation order

Runge's test, evaluating the fourth x-derivative of the solution, show that it converges numerically. Tests for other fourth order derivatives are similar and we do not present them here.

Best-Fitt Approximation formulae

$$w^s(x,y,t;c) = f(x,y) + c^2[(1-\beta_1)g_a(x,y) + \beta_1 g_b(x,y)] + c^2[(1-\beta_1)h_1(x,y) + \beta_1 h_2(x,y)\cos(2\theta)],$$

where

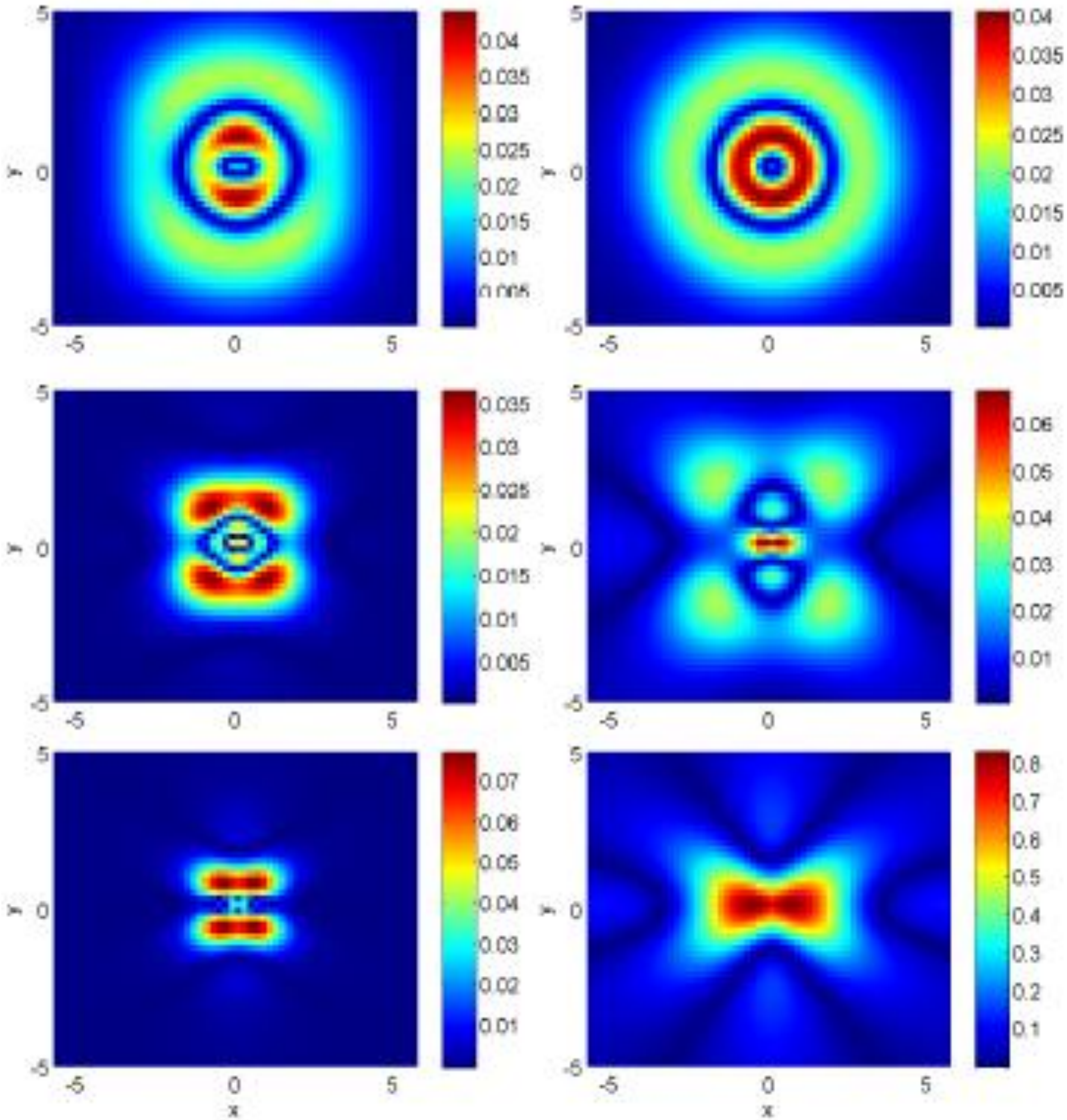
$$f(x,y) = \frac{2.4(1+0.24r^2)}{\cosh(r)(1+0.095r^2)^{1.5}},$$

$$g_a(x,y) = -\frac{1.2(1-0.177r^{2.4})}{\cosh(r)|1+0.11r^{2.1}|}, \quad g_b(x,y) = -\frac{1.2(1+0.22r^2)}{\cosh(r)|1+0.11r^{2.4}|},$$

$$h_l(x,y) = \frac{a_l r^2 + b_l r^3 + c_l r^4 + v_l r^6}{1 + d_l r + e_l r^2 + f_l r^3 + g_l r^4 + h_l r^5 + q_l r^6 + w_l r^8}.$$

[5]C. I. Christov, J. Choudhury, Perturbation solution for the 2D Boussinesq equation, Mech. Res. Commun., 38 (2011) 274 - 281.

Comparison between the numerical solution $\tilde{v}$ and the best fit formulae[3]



c=0.3
$\beta = 1$

c=0.1
$\beta = 1$

c=0.3
$\beta = 3$

c=0.5
$\beta = 1$

c=0.3
$\beta = 5$

c=0.9
$\beta = 1$

Thank you for your attention