Numerical Investigation of Unsteady Solutions for the 2D Boussinesq Paradigm Equation

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An implicit, energy conserving and unconditionally stable difference scheme with second order truncation error in space and time is presented for the solution of the 2D Boussinesq Paradigm Equation (BPE) [1]:

$$u_{tt} = \Delta \left[u - F(u) + \beta_1 u_{tt} - \beta_2 \Delta u \right], \quad F(u) := \alpha u^2, \tag{1}$$

where u is the surface elevation of the wave, $\beta_1,\ \beta_2 > 0$ are two dispersion coefficients, and $\alpha > 0$ is an amplitude parameter. The main difference of (1) from the original Boussinesq Equation is the presence of a term proportional to $\beta_1 \neq 0$ called "rotational inertia".

It has been recently shown that the 2D BPE admits stationary translating localized solutions [2, 3, 4], which can be obtained approximately using finite differences, perturbation technique, or Galerkin spectral method. First results about their time behaviour and structural stability are presented in [5] and [6], and here we continue their investigation, designing an energy conserving numerical method.

Numerical method for solving BPE. We introduce the following new dependent function

$$v(x, y, t) := u - \beta_1 \Delta u \tag{2a}$$

and substituting it in Eq. (1) we get the following equation for v

$$v_{tt} = \frac{\beta_2}{\beta_1} \Delta v + \frac{\beta_1 - \beta_2}{\beta_1^2}(u - v) - \Delta F(u). \tag{2b}$$

Thus we obtain a system consisting of an elliptic equation for u, Eq. (2a), and a hyperbolic equation for v: Eq. (2b).

The following implicit time stepping can be designed for the system (2)

$$\begin{aligned} \frac{v_{ij}^{n+1} - 2v_{ij}^{n} + v_{ij}^{n-1}}{\tau^2} &= \frac{\beta_2}{2\beta_1} \Lambda \left[v_{ij}^{n+1} + v_{ij}^{n-1} \right] + \frac{\beta_1 - \beta_2}{2\beta_1^2} [u_{ij}^{n+1} - v_{ij}^{n+1} + u_{ij}^{n-1} - v_{ij}^{n-1}] \\ &\quad - \Lambda G(u_{ij}^{n+1}, u_{ij}^{n-1}), \end{aligned} \tag{3a}$$

$$u_{ij}^{n+1} - \beta_1 \Lambda u_{ij}^{n+1} = v_{ij}^{n+1}, \quad i = 0, \ldots, N_x + 1, \quad j = 0, \ldots, N_y + 1. \tag{3b}$$

Here τ is the time increment, $G(u_{ij}^{n+1}, u_{ij}^{n-1}) = \left[(u_{ij}^{n+1})^2 + u_{ij}^{n+1}u_{ij}^{n-1} + (u_{ij}^{n-1})^2\right]/3$, and $\Lambda = \Lambda^{xx} + \Lambda^{yy}$ stands for the difference approximation of the Laplace operator Δ on a non-uniform grid, for example

$$\Lambda^{xx}\phi_{ij} = \frac{2\phi_{i-1j}}{h_{i-1}^x(h_i^x + h_{i-1}^x)} - \frac{2\phi_{ij}}{h_i^x h_{i-1}^x} + \frac{2\phi_{i+1j}}{h_i^x(h_i^x + h_{i-1}^x)} = \frac{\partial^2\phi}{\partial x^2}\Big|_{ij} + O(|h_i^x - h_{i-1}^x|).$$

For a smooth distribution of the nonuniform grid (as the one considered here) one has

$$O(|h_i^x - h_{i-1}^x|) \approx \frac{\partial h^x}{\partial x} O(|h_{i-1}|^2) = O(|h_{i-1}|^2).$$

The values of the sought functions at the $(n-1)$-st and n-th time stages are considered as known when computing the $(n+1)$-st stage. The nonlinear term G is linearized using Picard method, i.e., we perform successive iterations for u and v on the $(n+1)$-st stage, starting with initial conditions from the already computed n-th stage.
The unconditional stability of the scheme and the conservation of the energy are shown in [7, 8]. The convergence is investigated in [8].
The following non-uniform grid is used in the $x-$direction

$$x_i = \sinh[\hat{h}_x(i - n_x)],\ x_{N_x+1-i} = -x_i,\ i = n_x + 1, \ldots, N_x + 1,\ x_{n_x} = 0,$$

where N_x is an odd number, $n_x = (N_x + 1)/2$, $\hat{h}_x = D_x/N_x$, and D_x is selected in a manner to have large enough computational region. The grid in the $y-$direction is defined in the same way.
Because of the localization of the wave profile, the boundary conditions can be set equal to zero, when the size of the computational domain is large enough. The initial conditions are created using the best-fit approximation provided in [4]. The coupled system of equations (3) is solved by the Bi-Conjugate Gradient Stabilized Method with ILU preconditioner [9].

Numerical experiments. Denote by $u^s(x, y; c)$ the best-fit approximation of the stationary translating (with speed c) localized solutions, obtained in [4]

$$\begin{aligned} u^s(x,y;c) &= f(x,y) + c^2\left[(1-\beta_1)g_a(x,y) + \beta_1 g_b(x,y)\right] \\ &+ c^2\left[(1-\beta_1)h_1(x,y) + \beta_1 h_2(x,y)\right]\cos\left[2\arctan(y/x)\right], \end{aligned}$$

where the formulas for the functions $f,\ g_a,\ g_b$ may be found in [4]. For $t = 0$, the first initial condition is obvious: $u(x, y, 0) = u^s(x, y; c)$, and the second initial condition may be chosen as

$$u(x, y, -\tau) = u^s(x, y + c\tau; c). \tag{4}$$

The solutions for $\beta_1 = 3,\ \beta_2 = 1,\ \alpha = 1$ are computed on three different grids in the region $x, y \in [-50, 50]$ (with 161×161, 321×321 and 641×641 grid points), and with at least three different time increments ($\tau = 0.2$, 0.1 and 0.05). The results for $c = 0$, 0.25 and 0.3 are in good agreement with those in [6], where the nonlinear term was approximated on the already computed n-th time stage, but the corresponding

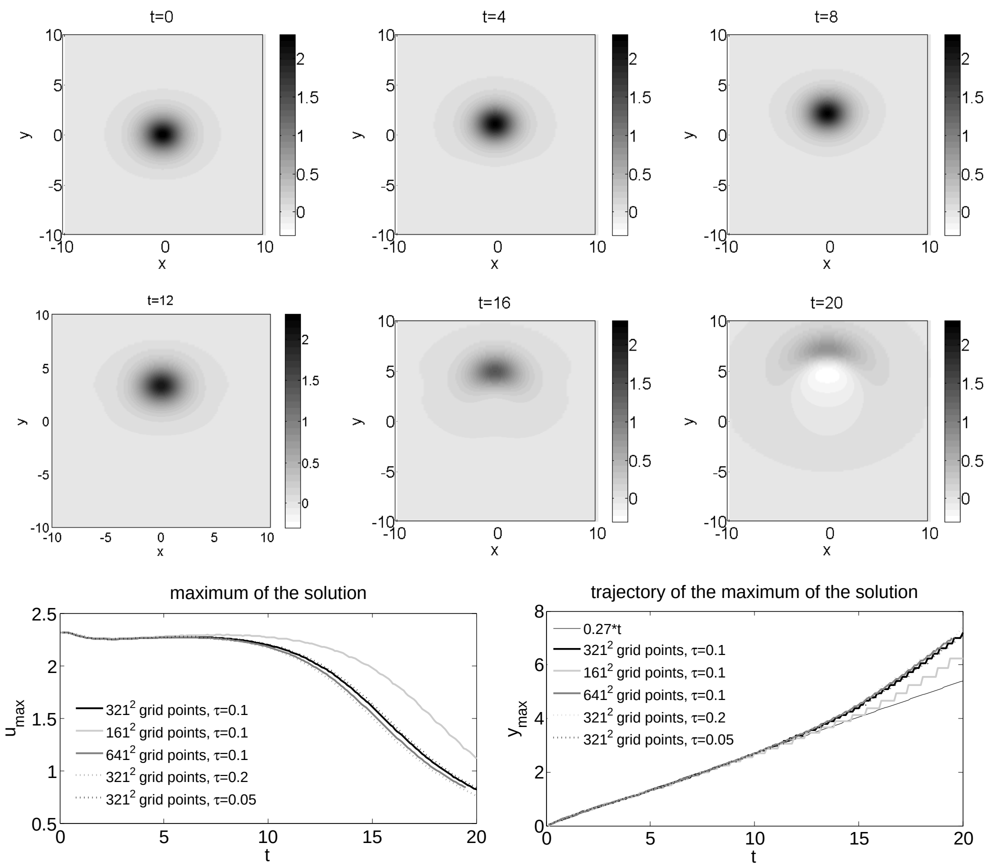

Figure 1: Evolution of the solution for $c = 0.27$, the maximum $u(0, y_{\max})$, and the trajectory of the maximum.

scheme is not energy conserving. That is why here we will present some results for different values of c.

Example 1. We present the evolution of the solution for the case $c = 0.27$ in Fig. 1. The values of the maximum of the solution $u_{\max}$ and its y-coordinate $y_{\max}$ as functions of time are also shown in Fig. 1. The behaviour of the solution is the same on all grids and for all times steps. For $t \leq 10$, the solution not only moves with a speed, close to $c = 0.27$, but also behaves like a soliton, i.e., preserves its shape, albeit its maximum decreases slightly. For larger times, the solution transforms into a diverging propagating wave. The phase speed of the wave increases and reaches the limit for the small linear waves, $c = 1$.

For $t = 4, 8, 12$ the computed maximum of the solution $u_{\max}$, the difference $\Delta u_{\max} := u^{\text{prev}}_{\max} - u_{\max}$ (subscript ‘prev’ denotes the previous row in the table), and the rate of

Table 1: The maximum of the solution, convergence in space and time, $c = 0.27$

		$t=4$			$t=8$			$t=12$			$t \in [0,30]$
τ	N_x+1	$u_{\max}$	$\Delta u_{\max}$	l	$u_{\max}$	$\Delta u_{\max}$	l	$u_{\max}$	$\Delta u_{\max}$	l	energy
0.1	160	2.2683			2.2929			2.2108			9.153520
0.1	320	2.2631	5.23e-3		2.2553	3.76e-2		2.0426	1.68e-1		9.151786
0.1	640	2.2615	1.60e-3	1.7	2.2473	8.06e-3	2.2	2.0047	3.78e-2	2.2	9.151368
0.2	320	2.2627			2.2400			1.9701			9.151562
0.1	320	2.2631	-3.84e-4		2.2553	-1.53e-2		2.0426	-7.25e-2		9.151786
0.05	320	2.2632	-1.15e-4	1.7	2.2597	-4.38e-3	1.8	2.0602	-1.77e-2	2.0	9.151861
0.025	320	2.2633	-3.00e-5	1.9	2.2608	-1.12e-3	2.0	2.0650	-4.79e-3	1.9	9.151877

Table 2: The maximum of the solution, convergence in space and time, $c = 0.28$

		$t=4$			$t=8$			$t=12$			$t \in [0,30]$
τ	N_x+1	$u_{\max}$	$\Delta u_{\max}$	l	$u_{\max}$	$\Delta u_{\max}$	l	$u_{\max}$	$\Delta u_{\max}$	l	energy
0.1	160	2.2767			2.3555			2.6126			9.266297
0.1	320	2.2698	6.97e-3		2.3213	3.42e-2		2.4007	2.11e-1		9.264460
0.1	640	2.2680	1.79e-3	2.0	2.3125	8.80e-3	2.0	2.3510	4.97e-2	2.1	9.264001
0.2	320	2.2700			2.3058			2.3116			9.264181
0.1	320	2.2698	2.34e-4		2.3213	-1.54e-2		2.4007	-8.90e-2		9.264460
0.05	320	2.2697	5.50e-5	2.1	2.3247	-3.40e-3	2.2	2.4220	-2.13e-2	2.1	9.264524

convergence $l = \log_2\left(|u_{\max}^{\mathrm{prev}} - u_{\max}^{\mathrm{prev,prev}}|/|u_{\max} - u_{\max}^{\mathrm{prev}}|\right)$, are shown in Table 1. It is seen that the method has second order numerical accuracy in space and time. The last column in the table is for the energy of the numerical solution, as defined in [8]. The energy is really conserved during the computations and the presented values are for each $t \in [0, 30]$.
Here is to be mentioned that the evolution of the solution for phase speeds $c \leq 0.27$ is qualitatively the same. Quantitatively, the time needed the solution to set on the dispersive track is usually smaller for a smaller phase speed, because of the reduced self-focusing role of the nonlinearity.

Example 2. In Fig. 2, results for $c = 0.28$ are presented. For $t < 10$ the behavior of the solution is similar to that in the previous example, but for larger times and when the time step τ is less than 0.2, it turns to grow and blows-up for $t \approx 20$. The blow-up is connected with the fact that the energy functional is not positive definite for BPE with quadratic nonlinearity (see [10] and the literature cited therein). Even when the amplitude is increasing, the energy is kept constant. A threshold value $c = 0.3$ was the last one for which a non-blowing-up evolution was found in [5] on the coarsest grid, while blow-up was encountered on the finest grid. Here we observe a non-blow-up for large time steps ($\tau = 0.2$) and a smaller value of c, which is probably due to the different numerical method used, namely the different scheme dispersion.

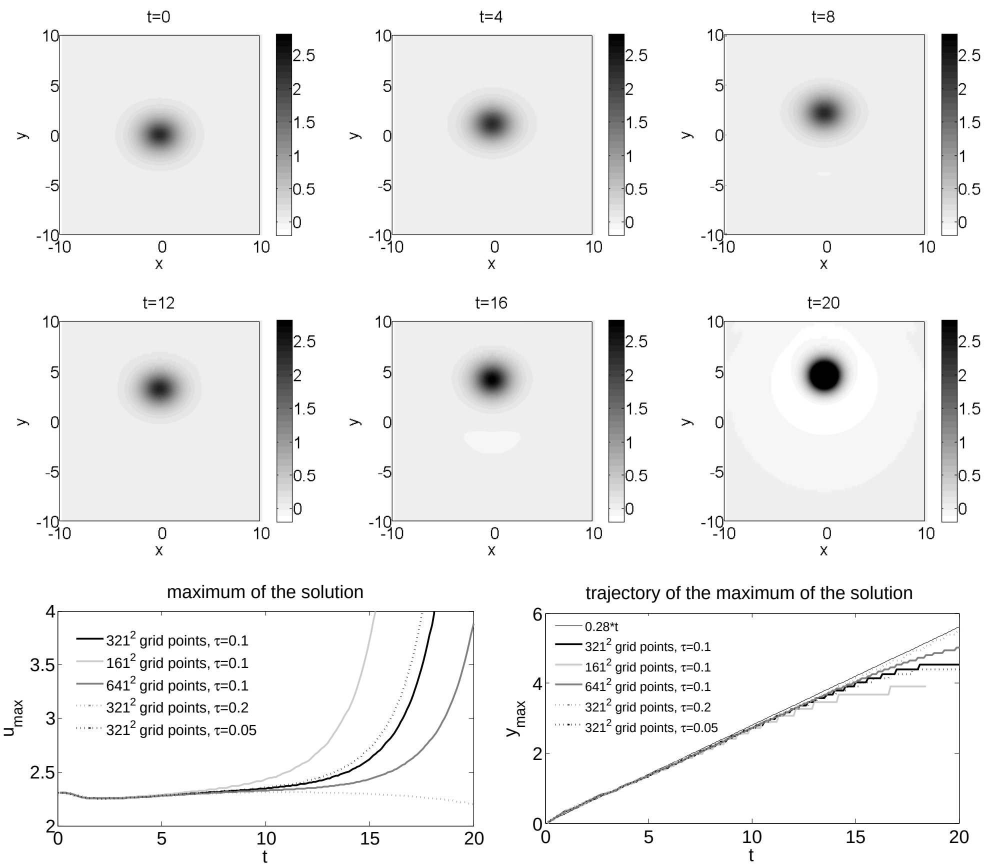

Figure 2: Evolution of the solution for $c = 0.28$, the maximum $u(0, y_{\max})$, and the trajectory of the maximum.

Let us also note that when the energy non-conserving method from [6] is used, the solution blows-up for $\tau = 0.2$, as well.
As can be seen from Table 2 the method has second order numerical accuracy in space and time.

Conclusion. An energy conserving difference scheme for the investigation of the time evolution of the localized solutions of the Boussinesq Paradigm Equation (BPE) in two spatial dimensions is devised. The grid is non-uniform and the truncation error is second order in space and time. The results obtained for the time evolution of supposedly stationary propagating waves for different phase speeds are very similar to those in [6]. We have found that for phase speeds $0 \neq c \leq 0.27$, the initially localized wave disperses in the form of ring-wave expanding to infinity. Respectively, for $c \geq 0.29$ the initial evolution resembles a stationary propagation, but after some period of time a blow-up of the solution takes place. When $c = 0.28$, the asymptotic behavior of the solution depends on the numerical dispersion of the scheme, which is contingent on the value of time increment. Our results are in good agreement with [5], where a similar ($c = 0.3$) threshold is established for the appearance of the blow-up. The fact that for $c \approx 0.28$, an time interval exists in which the solution is virtually reserving its shape while steadily translating means that 2D solitons could be found for the class of BPEs. This means that the nonlinearity is strong enough to balance the dispersion which is now much stronger than in the 1D case. In order to firmly establish this fact, our future plans are to consider also equations with different nonlinearities for which the blow-up is not possible.

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