On the Numerical Investigation of Unsteady Solutions for the 2D Boussinesq Paradigm Equation in a Moving Frame Coordinate System

Daniela Vasileva, Christo I. Christov

Introduction. One of the most important features of the generalized wave equations containing nonlinearity and dispersion, is that they possess solutions of type of permanent waves as shown in the original Boussinesq work [1]. In 1D, a plethora of deep mathematical results have been obtained for solitons, but it is of crucial importance to investigate also the 2D case, because of the different phenomenology and the practical importance. The accurate derivation of the Boussinesq system combined with an approximation, that reduces the full model to a single equation, leads to the Boussinesq Paradigm Equation (BPE) [2]:

$$u_{tt} = \Delta\left[u - F(u) + \beta_1 u_{tt} - \beta_2 \Delta u\right], \quad F(u) := \alpha u^2, \tag{1}$$

where u is the surface elevation of the wave, $\beta_1,\ \beta_2 > 0$ are two dispersion coefficients, and $\alpha > 0$ is an amplitude parameter. The main difference of (1) from the original Boussinesq Equation is the presence of a term proportional to $\beta_1 \neq 0$ called "rotational inertia".

It has been recently shown that the 2D BPE admits stationary translating localized solutions [3, 4, 5], which can be obtained approximately using finite differences, perturbation technique, or Galerkin spectral method. Results about their time behaviour and structural stability are presented in [6, 7, 8], and here we continue their investigation using a moving frame coordinate system. It allows us to keep the localized structure in the center of the coordinate system, reducing the effects of reflection from the boundary.

Numerical method for solving BPE. We introduce the following new dependent function

$$v(x, y, t) := u - \beta_1 \Delta u \tag{2a}$$

and substituting it in Eq. (1) we get the following equation for v

$$v_{tt} = \frac{\beta_2}{\beta_1}\Delta v + \frac{\beta_1 - \beta_2}{\beta_1^2}(u - v) - \Delta F(u).$$

We set $z := y - ct$, where c is the velocity of the stationary propagating soliton and obtain the following equation for $w(x,z,t) := v(x, z+ct, t)$

$$w_{tt} - 2cw_{tz} + c^2 w_{zz} = \frac{\beta_2}{\beta_1}\Delta w + \frac{\beta_1 - \beta_2}{\beta_1^2}(u - w) - \alpha \Delta F(u). \tag{2b}$$

Thus we obtain a system consisting of an equation for u, Eq. (2a), and an equation for w: Eq. (2b).
The following implicit time stepping can be designed for the system (2)

$$\begin{aligned} &\frac{w_{ij}^{n+1} - 2w_{ij}^{n} + w_{ij}^{n-1}}{\tau^2} - c\frac{V^z[w_{ij}^{n+1} - w_{ij}^{n-1}]}{\tau} + \frac{c^2}{2}\Lambda^{zz}[w_{ij}^{n+1} + w_{ij}^{n-1}] \\ &= \frac{\beta_2}{2\beta_1}\Lambda\left[w_{ij}^{n+1} + w_{ij}^{n-1}\right] + \frac{\beta_1 - \beta_2}{2\beta_1^2}[u_{ij}^{n+1} - w_{ij}^{n+1} + u_{ij}^{n-1} - w_{ij}^{n-1}] \\ &\qquad - \Lambda G(u_{ij}^{n+1}, u_{ij}^{n-1}), \end{aligned} \tag{3a}$$

$$u_{ij}^{n+1} - \beta_1 \Lambda u_{ij}^{n+1} = w_{ij}^{n+1}, \quad i = 0, \ldots, N_x + 1, \quad j = 0, \ldots, N_y + 1. \tag{3b}$$

Here τ is the time increment, $G(u_{ij}^{n+1}, u_{ij}^{n-1}) = \left[(u_{ij}^{n+1})^2 + u_{ij}^{n+1}u_{ij}^{n-1} + (u_{ij}^{n-1})^2\right]/3$, $\Lambda = \Lambda^{xx} + \Lambda^{zz}$ stands for the difference approximation of the Laplace operator Δ on a non-uniform grid, for example

$$\Lambda^{xx}\phi_{ij} = \frac{2\phi_{i-1j}}{h_{i-1}^x(h_i^x + h_{i-1}^x)} - \frac{2\phi_{ij}}{h_i^x h_{i-1}^x} + \frac{2\phi_{i+1j}}{h_i^x(h_i^x + h_{i-1}^x)} = \frac{\partial^2 \phi}{\partial x^2}\Big|_{ij} + O(|h_i^x - h_{i-1}^x|),$$

and V^z is a central difference approximation of $\frac{\partial}{\partial z}$

$$V_z\phi_{ij} = \frac{h_{j-1}^z \phi_{ij+1}}{h_j^z(h_j^z + h_{j-1}^z)} - \frac{h_i^z \phi_{ij-1}}{h_{j-1}^z(h_j^z + h_{j-1}^z)} - \frac{(h_j^z - h_{j-1}^z)\phi_{ij}}{h_i^z h_{j-1}^z} = \frac{\partial \phi}{\partial z}\Big|_{ij} + O(|h_j^z - h_{j-1}^z|).$$

Another way to approximate w_{zt} for $c > 0$ is by the following "upwind" approximation

$$w_{zt} = \frac{w_{ij+1}^{n+1} - w_{ij}^{n+1} - w_{ij+1}^{n} + w_{ij}^{n}}{2\tau h_j^z} + \frac{w_{ij}^{n} - w_{ij-1}^{n} - w_{ij}^{n-1} + w_{ij-1}^{n-1}}{2\tau h_{j-1}^z} + O(|h_j^z - h_{j-1}^z| + \tau^2).$$

The values of the sought functions at the $(n-1)$-st and n-th time stages are considered as known when computing the $(n+1)$-st stage. The nonlinear term G is linearized using what we call internal iterations (translating the Picard's idea to the case of differential equations), i.e., we perform successive iterations for u and w on the $(n+1)$-st stage, starting with initial conditions from the already computed n-th stage.
The following non-uniform grid is used in the $x-$direction

$$x_i = \sinh[\hat{h}_x(i - n_x)], \ x_{N_x+1-i} = -x_i, \ i = n_x + 1, \ldots, N_x + 1, \ x_{n_x} = 0,$$

where N_x is an odd number, $n_x = (N_x + 1)/2$, $\hat{h}_x = D_x/N_x$, and D_x is selected in a manner to have large enough computational region. The grid in the $z-$direction is defined in the same way.

Because of the localization of the wave profile, the boundary conditions can be set equal to zero, when the size of the computational domain is large enough. The initial conditions are created using the best-fit approximation provided in [5]. The coupled system of equations (3) is solved by the Bi-Conjugate Gradient Stabilized Method with ILU preconditioner [9].

Numerical experiments. Denote by $u^s(x,y;c)$ the best-fit approximation of the stationary translating (with speed c) localized solutions, obtained in [5]

$$\begin{aligned} u^s(x,z;c) &= f(x,z) + c^2\left[(1-\beta_1)g_a(x,z) + \beta_1 g_b(x,z)\right] \\ &+ c^2\left[(1-\beta_1)h_1(x,z) + \beta_1 h_2(x,z)\right]\cos\left[2\arctan(z/x)\right], \end{aligned}$$

where the formulas for the functions $f,\ g_a,\ g_b$ may be found in [5]. For $t=0$, the first initial condition is obvious: $u(x,z,0) = u^s(x,z;c)$, and the second initial condition may be chosen as $u(x,z,-\tau) = u^s(x,z;c)$.
In the next examples solutions for $\beta_1 = 3,\ \beta_2 = 1,\ \alpha = 1$ are presented.

Example 1. The first example is for a phase speed $c = 0.27$. The basic grid has 161×161 points in the region $[-20,20]^2$, $\tau = 0.1$. The results are for computations in fixed coordinates, for the moving frame coordinate system with upwind approximation of w_{tz}, for the moving frame coordinate system with central differences approximation of w_{tz}, for finer grid with 321×321 points and $\tau = 0.05$, and for a larger computational region with 641×641 points in $[-200,200]^2$, $\tau = 0.1$. The behaviour of the solution is almost the same in all cases.
For $t < 10$ the solution stays near the center of the moving coordinate system and behaves like a soliton, i.e., preserves its shape, although its maximum slightly decreases. For larger times the solution transforms into a diverging propagating wave. As the structure is moving the weaves are not concentric – just like when we throw a stone in a pond at an angle. The evolution of the solution, as well as values of the maximum of the solution $u_{\max}$ and the trajectory of the maximum $z_{\max}$ ($y_{\max}$ for fixed coordinates) are shown in Fig.1.

Example 2. In Fig.2 results for $c = 0.28$ are presented. For $t < 10$ the solution stays near the center of the moving frame coordinate system and behaves like a soliton, i.e., preserves its shape, although its maximum slightly varies. For larger times the solution turns to grow and blows-up for $t \approx 20$. The results for the fixed and moving frame coordinate system are very similar.
The results from the Experiments 1 and 2 show that the mechanism for having a balance between the nonlinearity and dispersion is present, but the solution is not robust (even when it is stable as a time stepping process) and eventually takes the path to the attractor presented by the propagating wave.

Example 3. In order to show that the lack of robustness is a intrinsically 2D effect rather than due to the imperfections of the scheme, we use the exact solution for the 1D case [2] as initial data:

$$u(x,z,0) := u^{\text{sech}}(x) = (1-c^2)\frac{1.5}{\alpha}\text{sech}^2\big(0.5x\sqrt{(1-c^2)/(\beta_2-\beta_1 c^2)}\big).$$

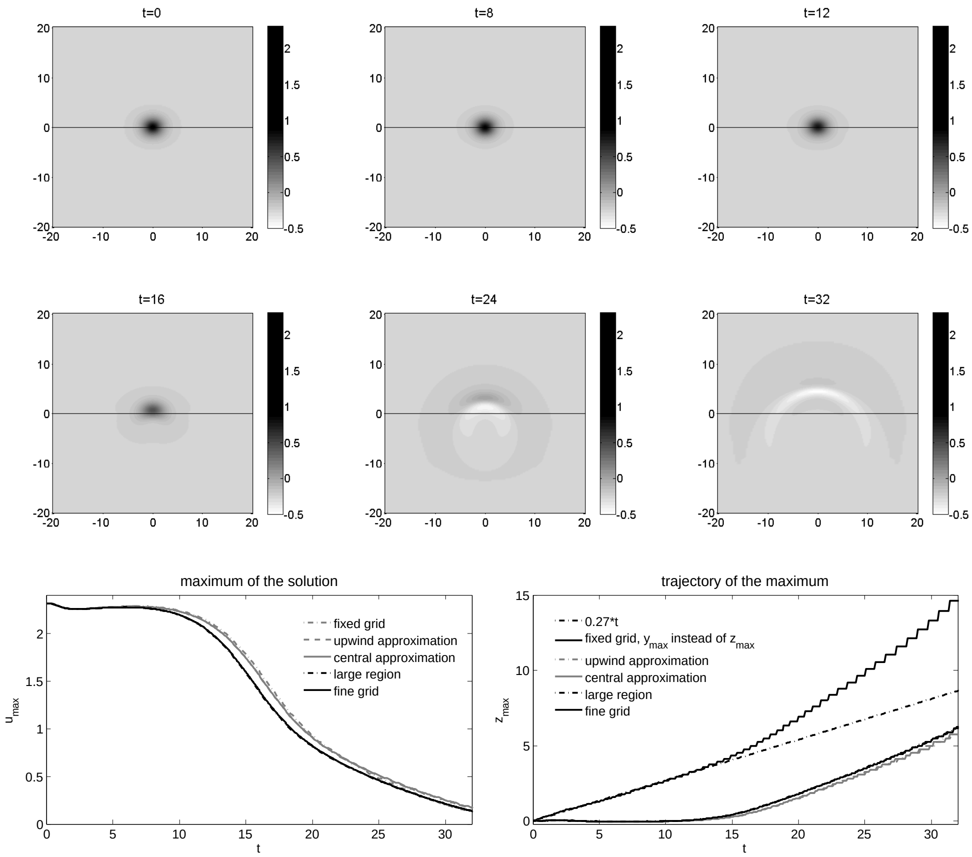

Figure 1: Evolution of the solution for $c = 0.27$, the maximum $u(0, z_{\max})$, and the trajectory of the maximum.

The boundary conditions on $z = -20$ and $z = 20$ are $u(x, \pm 20, t) := u^{\text{sech}}(x)$. The maximum of the difference between the numerical and the exact solution $\Delta u := \max |u - u^{\text{sech}}|$ and the order of convergence l are shown in Table 1. As it is seen, the results confirm both the solitonic behaviour of the 1D solution and the second order convergence of the difference scheme (3). The central difference and upwind approximations of u_{tz} lead to practically the same values in the numerical solution. The comparison between the moving frame and fixed grid computations shows that the latter produces larger errors on non-uniform grids, but smaller errors on uniform grids.

Conclusion. A difference scheme in a moving frame coordinate system is designed for the investigation of the time evolution of the localized solutions of the 2D Boussinesq Paradigm Equation (BPE). The grid is non-uniform and the truncation error is second order in space and time. The results obtained for the time evolution of

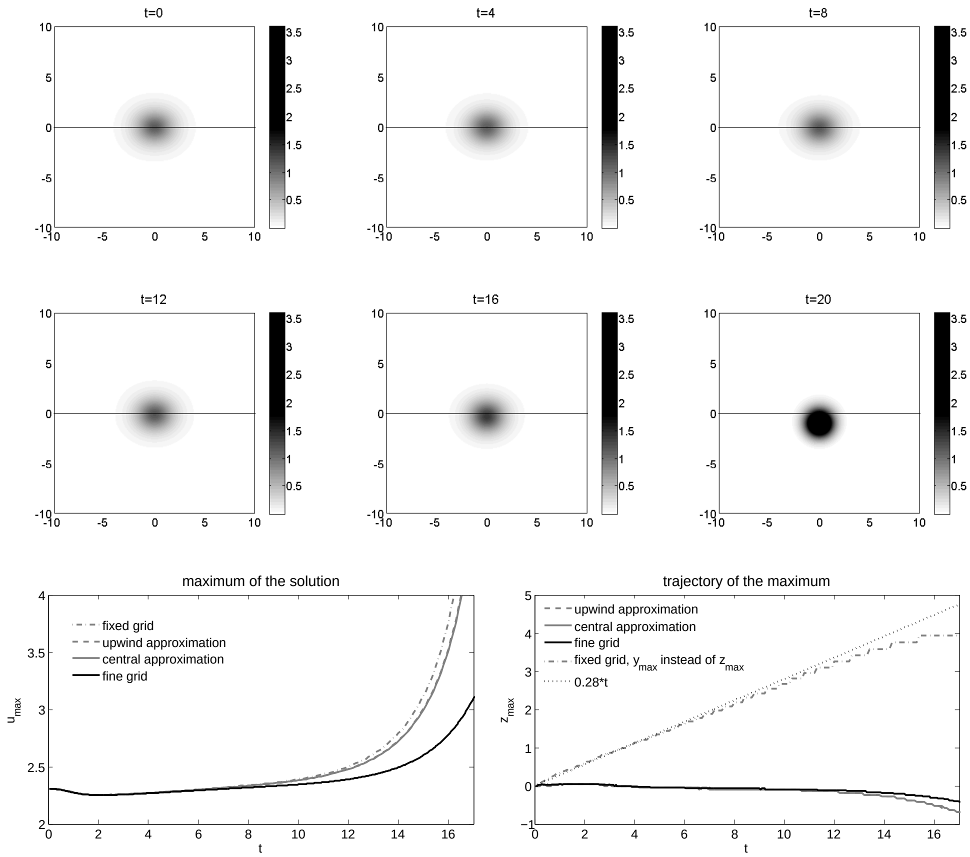

Figure 2: Evolution of the solution for $c = 0.28$, the maximum $u(0, z_{\max})$, and the trajectory of the maximum.

supposedly stationary propagating waves for different phase speeds are very similar to those in [7, 8] – for phase speeds $0 \neq c \leq 0.27$, the initially localized wave disperses in the form of ring-wave expanding to infinity. Respectively, for $c \geq 0.28$ the initial evolution resembles a stationary propagation, but after some period of time a blow-up of the solution takes place. The results are in good agreement with [6], where a similar ($c = 0.3$) threshold is established for the appearance of the blow-up.

The moving frame coordinate system helps us to keep the localized structure in the center of the coordinate system, where the grid is much finer. It also reduces the effects of the reflection from the boundaries, and thus allows us to use a smaller computational box.

Acknowledgment. This work has been partially supported by Grant DDVU02/71 from the National Science Fund by Ministry of Education, Youth, and Science of Republic of Bulgaria.

Table 1: Convergence in space and time for $c = 0.27$

		$t = 4$		$t = 8$		$t = 12$	
τ	N_x+1	Δu	l	Δu	l	Δu	l
moving frame, non-uniform grid							
0.1	160	1.36e-3		5.04e-3		1.70e-2	
0.05	320	3.56e-4	1.93	1.32e-3	1.93	4.44e-3	1.94
fixed grid							
0.1	160	1.81e-3		6.78e-3		2.42e-2	
0.05	320	4.69e-4	1.95	1.75e-3	1.95	6.21e-3	1.96
moving frame, uniform grid							
0.1	160	1.05e-2		3.36e-2		1.13e-1	
0.05	320	2.66e-3	1.98	8.41e-3	2.00	2.74e-2	2.04
fixed uniform grid							
0.1	160	1.00e-2		2.82e-2		8.61e-2	
0.05	320	2.56e-3	1.97	7.18e-3	1.97	2.15e-2	2.00

Literature cited

[1] Boussinesq, J.V.: Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide Journal de Mathématiques Pures et Appliquées **17** (1872) 55–108

[2] Christov, C.I.: An energy-consistent Galilean-invariant dispersive shallow-water model. *Wave Motion*, **34** (2001) 161–174

[3] Christou, M.A., Christov, C.I.: Fourier-Galerkin method for 2D solitons of Boussinesq equation. *Math. Comput. Simul.*, **74** (2007) 82–92

[4] Christov, C.I.: Numerical implementation of the asymptotic boundary conditions for steadily propagating 2D solitons of Boussinesq type equations, *Math. Comp. Simul.*, Appeared online August 10, 2010 Doi:10.1016/j.matcom.2010.07.030

[5] Christov, C.I., Choudhury, J.: Perturbation solution for the 2D Boussinesq Equation. *Mech. Res. Commun.*, **38** (2010), 274–281

[6] Chertock, A., Christov, C.I., Kurganov, A.: Central-upwind schemes for the Boussinesq paradigm equation. *Proc. 4th Russian-German Advanced Research Workshop on Comp. Science and High Performance Computing, NNFM*, **113** (2011), 267–281

[7] Christov, C.I., Kolkovska, N., Vasileva, D.: On the Numerical Simulation of Unsteady Solutions for the 2D Boussinesq Paradigm Equation, *Lecture Notes Computer Science*, **6046** (2011) 386–394

[8] Christov, C.I., Kolkovska, N., Vasileva, D.: Numerical Investigation of Unsteady Solutions for the 2D Boussinesq Paradigm Equation, 5th Annual Meeting of the Bulgarian Section of SIAM, Sofia, Bulgaria, *BGSIAM'10 Proceedings*, 2011, 11-16

[9] van der Vorst, H.: Iterative Krylov methods for large linear systems. *Cambridge Monographs on Appl. and Comp. Math.*, **13** (2009)