On the use of numerical data for evaluating the parameters of the a-priori estimates for the onset of blow up in Boussinesq Paradigm Equation

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I. Statement of the problem

$$u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \beta_2 \Delta^2 u = \Delta f(u) \quad \text{for} \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+$$
$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad \text{for} \quad x \in \mathbb{R}^n \tag{1}$$

$$f(u) = \alpha |u|^p, \qquad \alpha > 0, \quad \beta_1 \geq 0, \quad \beta_2 > 0 \qquad \text{– real constants} \tag{2}$$

$$1 < p < \infty \quad \text{for} \quad n = 1, 2 \quad \text{and}$$
$$\tfrac{n+2}{n} \leq p \leq \tfrac{n+2}{n-2} \quad \text{for} \quad n \geq 3 \tag{3}$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \quad - \quad \text{Laplace operator}$$

Notations:

$$u_0 \in \mathrm{H} = \{u \in \mathrm{H}^1; (-\Delta)^{-1/2} u \in \mathrm{L}^2\}; \quad \|u\|^2_{\mathrm{H}} = \|u\|^2_{\mathrm{H}^1} + \|(-\Delta)^{-1/2} u\|^2_{\mathrm{L}^2}$$

$$u_1 \in \mathrm{L} = \{u \in \mathrm{L}^2; (-\Delta)^{-1/2} u \in \mathrm{L}^2\}; \quad \|u\|^2_{\mathrm{L}} = \|u\|^2_{\mathrm{L}^2} + \|(-\Delta)^{-1/2} u\|^2_{\mathrm{L}^2}$$

$$\|u\|^2_{\mathrm{H}^1} = \|u\|^2_{\mathrm{L}^2} + \|\nabla u\|^2_{\mathrm{L}^2}, \qquad \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right)$$

$$\|u\|_{\mathrm{L}^s} = \left[\iint_{\mathbb{R}^n} |u|^s \, dx\right]^{1/s}, \quad s > 0 \qquad (-\Delta)^{-s} u = F^{-1}\left(|\xi|^{-2s} F(u)\right)$$

$$F(u) = \frac{1}{(2\pi)^{\frac{n}{2}}} \iint e^{-ix.\xi} u(x) \, dx \quad - \quad \text{Fourier transformation}$$

$$F^{-1}(u) = \frac{1}{(2\pi)^{\frac{n}{2}}} \iint e^{ix.\xi} u(x) \, dx \quad - \quad \text{inverse Fourier transformation}$$

$$\xi.x = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n, \quad \xi = (\xi_1, \xi_2, \dots, \xi_n), \quad x = (x_1, x_2, \dots, x_n)$$

$$
\begin{aligned}
&u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \beta_2 \Delta^2 u = \Delta f(u) \quad \text{for} \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+ \\
&u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad \text{for} \quad x \in \mathbb{R}^n
\end{aligned}
\tag{4}
$$

Definition: Weak solution of (4) in $[0,T) \times \mathbb{R}^n$ is the function u,

$$
u \in \mathrm{L}^\infty([0,T); \mathrm{H}^1), \quad u_t \in \mathrm{L}^\infty([0,T); \mathrm{L}).
$$

Problem: When the weak solution of (4) are globally defined or blow up for a finite time?

II. History of the problem

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  $n = 1,\ \beta_1 = 0,\ \beta_2 = 1$

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  $n \geq 1,\ \beta_1 = 1,\ \beta_2 = 1$

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- Yue Liu, *SIAM J. Math. Anal.*, 26, (1995), 1527–1546.
  $n = 1,\ \beta_1 = 0,\ \beta_2 = 1,\ f(u) = -|u|^{p-1}u$

- Ying Yang, Chunlai Mu, Jing Deng, *Nonl. Anal.*, 69, (2008), 1599–1614.

Change of the variables

$$\overline{x} = x/\sqrt{\beta_2}, \qquad \overline{t} = t$$

$$\begin{aligned} &\beta_2 u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \Delta^2 u = \Delta f(u) \\ &u(x,0) = u_0(\sqrt{\beta_2}x), \qquad u_t(x,0) = u_1(\sqrt{\beta_2}x) \end{aligned} \tag{5}$$

Conservation of the full energy

$$E(t) = \frac{1}{2}\left[\beta_2 \left\|(-\Delta)^{-1/2}u_t\right\|^2_{\mathrm{L}^2} + \beta_1\|u_t\|^2_{\mathrm{L}^2} + \|u\|^2_{\mathrm{H}^1}\right] + \frac{\alpha}{p+1}\iint_{\mathbb{R}^n} |u|^p u\,dx \equiv E(0)$$

$$J(u) = \frac{1}{2}\|u\|^2_{\mathrm{H}^1} + \frac{\alpha}{p+1}\iint_{\mathbb{R}^n} |u|^p u\,dx$$

Sign preserving energy

$$I(u) = \|u\|^2_{\mathrm{H}^1} + \alpha\iint_{\mathbb{R}^n} |u|^p u\,dx = (p+1)\left[J(u) - \frac{p-1}{2}\|u\|^2_{\mathrm{H}^1}\right]$$

Theorem 1:([X-L])

If $0 < E(0) < d$ then the sign of $I(u)$ is invariant under the flow of equation (5).

$$d = \inf_{u \in \mathrm{N}} J(u), \qquad \mathrm{N} = \{u \in \mathrm{H}^1; I(u) = 0, \|u\|_{\mathrm{H}^1} \neq 0\}$$

Theorem 2:([X-L])

If $E(0) < 0$ then every weak solution of (5) blows up for a finite time.

If $E(0) = 0$ then every weak solution of (5), exept the trivial one, blows up for a finite time.

When $0 < E(0) < d$ then:

- if $I(u_0) < 0$ then the weak solution of (5) blows up for a finite time;
- if $I(u_0) > 0$ then the weak solution of (5) is globally defined for $t \in [0, \infty)$.

Lemma:([X-L])

$$d \geq \frac{p-1}{2(p+1)} \left[\alpha C^{p+1}\right]^{-\frac{2(p+1)}{p-1}}, \quad \text{where} \quad C = C(p) = \sup_{\substack{u \in \mathrm{H}^1 \\ u \neq 0}} \frac{\|u\|_{\mathrm{L}^{p+1}}}{\|u\|_{\mathrm{H}^1}}.$$

Theorem 3:(Liu)

$$d = E(\varphi) = \frac{1}{2}\|\varphi\|^2_{\mathrm{H}^1} - \frac{1}{p+1}\,\|\varphi\|^{p+1}_{\mathrm{L}^{p+1}}\,,$$

where φ is the positive radial $\mathrm{H}^1(\mathbb{R}^n)$ solution of

$$-\Delta\varphi + \varphi - |\varphi|^{p-1}\varphi = 0$$

and $1 < p < \infty$ for $n = 1, 2$; $1 < p < \frac{n+2}{n-2}$ for $n \geq 3$.

For $n = 1$

$$\frac{(p-1)^{\frac{1}{2}}}{2(p+1)}\left[(2(p+1))^{-\frac{1}{2}}\,(p+3)^{\frac{p+3}{4(p+1)}}\right]^{-\frac{2(p+1)}{p-1}} < d <$$

$$\frac{(p-1)^{\frac{1}{2}}}{2(p+1)}\left[(2(p+1))^{-\frac{1}{2}}\,(p+3)^{\frac{p+3}{4(p+1)}}\,e^{-\frac{p-1}{2(p+1)}}\right]^{-\frac{2(p+1)}{p-1}}$$

III. Main result

Theorem 4: The equality

$$d = \frac{p-1}{2(p+1)}\left[\alpha C^{p+1}\right]^{-\frac{2(p+1)}{p-1}} \qquad \text{holds.}$$

Moreover, for $\quad n = 1 \qquad d = \frac{1}{p+3}\left[\frac{2(p+1)}{\alpha}\right]^{\frac{2}{p-1}} \frac{\Gamma\left(\frac{2}{p-1}\right)^2}{\Gamma\left(\frac{4}{p-1}\right)}$

Corollary: For $n = 1$, $p = 2$ i.e. for equation

$$\beta_2 u_{tt} - u_{xx} - \beta_1 u_{ttxx} + u_{xxxx} = \alpha \Delta u^2, \qquad d = \frac{6}{5\alpha^2}.$$

If $\alpha = 3$ then $d = \frac{2}{15} = 0.133\ldots.$

Idea of the proof:

E.Lieb, *Annals of Math.*, 118, (1983), 349–374.

For $n = 1$, $p > 1$

$$C = C(p) = \sup_{\substack{u \in \mathrm{H}^1 \\ u \neq 0}} \frac{\|u\|_{\mathrm{L}^{p+1}}}{\|u\|_{\mathrm{H}^1}} = \left\{ \frac{1}{2(p+1)} \left[(p-1)(p+3) \frac{\Gamma(\frac{4}{p-1})}{\Gamma(\frac{2}{p-1})^2} \right]^{\frac{p-1}{p+1}} \right\}^{\frac{1}{2}}$$

Extremal functions $Aw = A\left(\cosh(\frac{p-1}{2})x\right)^{-\frac{2}{p-1}}, A = const \neq 0$

$$u_* = A_* w, \qquad A_* = -\left[\frac{\|w\|^2_{\mathrm{H}^1}}{\alpha \iint_{\mathbb{R}^n} w^{p+1}\, dx} \right]^{\frac{1}{p-1}}$$

$$I(u_*) = 0, \qquad \inf_{u \in \mathrm{N}} J(u) = J(u_*) = d$$

IV. Applications

$n = 1$, $p = 2$, $f(u) = \alpha u^2$, $\beta_1 \geq 0$, $\beta_2 > 0$

$$\begin{aligned} &\beta_2 u_{tt} - u_{xx} - \beta_1 u_{ttxx} + u_{xxxx} = \alpha (u^2)_{xx} \\ &u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \end{aligned} \tag{6}$$

$$w^c(x,t) = \frac{3}{2}\frac{(c^2-1)}{\alpha}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2 - \beta_2}}\left(\sqrt{\beta_2}x - ct\right)\right)$$

for

$$|c| < \min\{1, \sqrt{\frac{\beta_2}{\beta_1}}\}, \qquad |c| > \max\{1, \sqrt{\frac{\beta_2}{\beta_1}}\}$$

$$E(w^c) = \frac{6}{5\alpha^2}\left|(1-c^2)(-5\beta_1 c^4 + 4\beta_2 c^2 + \beta_2)\right|\left|\frac{c^2-1}{\beta_2(\beta_1 c^2 - \beta_2)}\right|^{\frac{1}{2}}$$

Theorem 5: Suppose $E(w^c) < d$ and

$$\|u_0 - w^c(x,0)\|^2_{\mathrm{H}^1} < \varepsilon, \|u_1 - w^c_t(x,0)\|^2_{\mathrm{L}^2} + \left\|(-\Delta)^{-1/2}(u_1 - w^c_t(x,0))\right\|^2_{\mathrm{L}^2} < \varepsilon,$$

$\varepsilon > 0$ sufficiently small, $u_0 \in \mathrm{H}$, $u_1 \in \mathrm{L}$.

Then the solution $u(x,t)$ of (6) is defined and bounded for every $t \in [0, \infty)$.

Example 1: $n = 1$, $\beta_2/\beta_1 = 1$,

$$E(w^c) = \frac{6}{5\alpha^2}(1 - c^2)(-5c^4 + 4c^2 + 1)$$

For $c_-^2 < c^2 < 1$, $1 < c^2 < c_+^2$, $c_- \approx 0.6646$, $c_+ \approx 1.1654$ the solution of (6) is defined and bounded in $[0, \infty)$

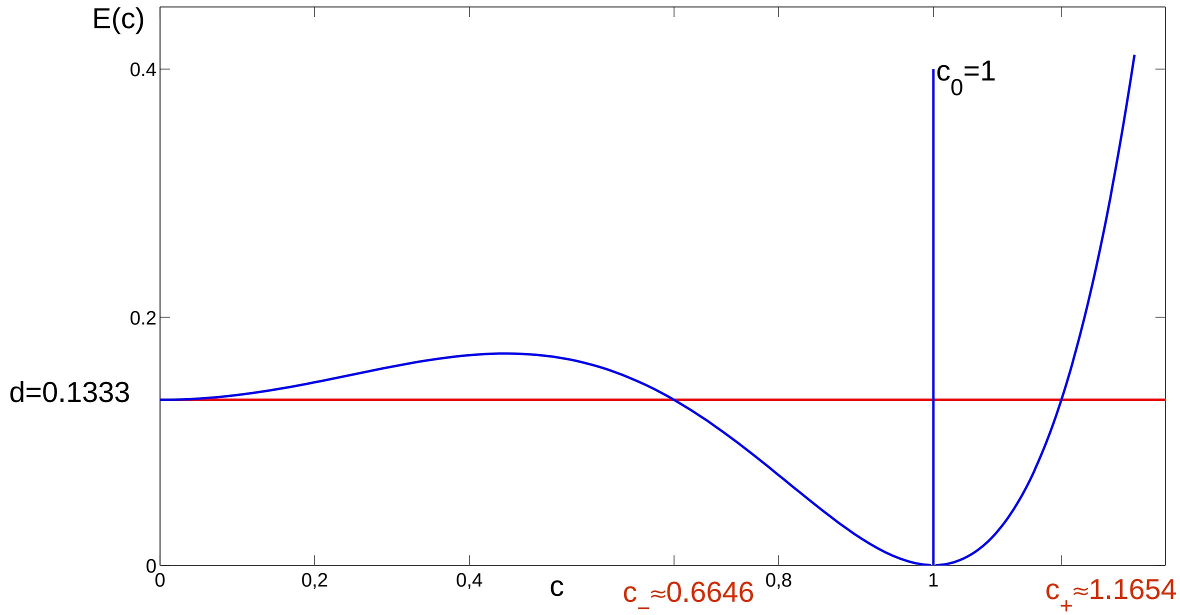


Figure 1: $\beta_2/\beta_1 = 1$, $\alpha = 3$.

Ying Yang, Chunlai Mu, Jing Deng, *Nonl. Anal.*, 69, (2008), 1599–1614
For $0 < c^2 < c_\delta^2 < 1$, $\beta_2/\beta_1 = 1$ the solution $u(x,t)$ of (6) blows up for a finite time.

Example 2: $n = 1$, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_2/\beta_1 < 1$, $|c| > 1$

$$E(w^c) = \frac{6}{5\alpha^2}(c^2-1)(5\beta_1 c^4 - 4\beta_2 c^2 - \beta_2)\left|\frac{(c^2-1)}{\beta_2(\beta_1 c^2 - \beta_2)}\right|^{\frac{1}{2}}$$

For $1 < c^2 < c_*^2, \quad c_* = c_*(\beta_1, \beta_2)$,

$$(c_*^2-1)(5\beta_1 c_*^4 - 4\beta_2 c_*^2 - \beta_2)\left|\frac{(c_*^2-1)}{\beta_2(\beta_1 c_*^2 - \beta_2)}\right|^{\frac{1}{2}} < 1$$

the solution $u(x,t)$ of (6) is defined and bounded in $[0,\infty)$.

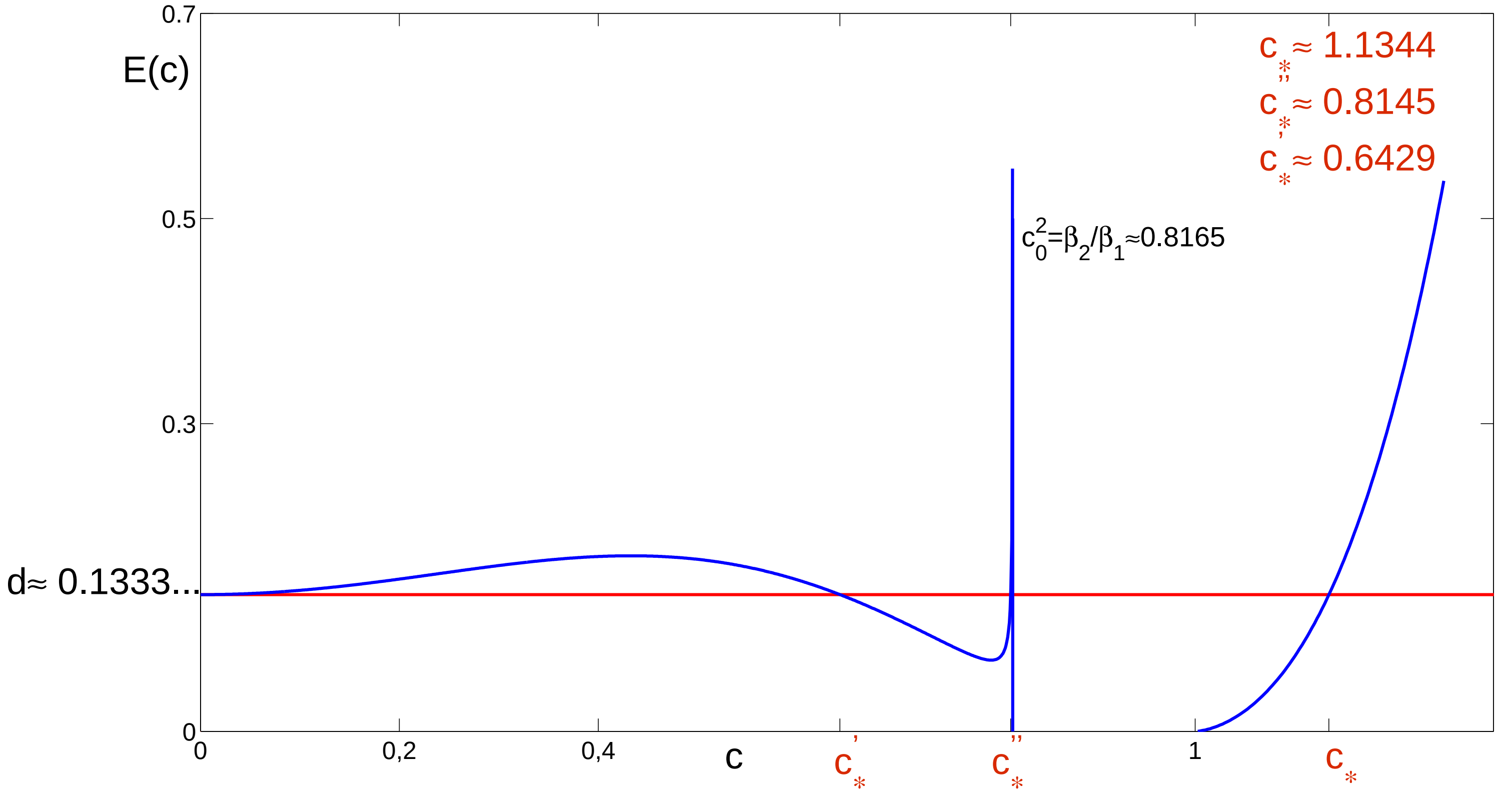


Figure 2: $\beta_2/\beta_1 = 2/3$, $\alpha = 3$.

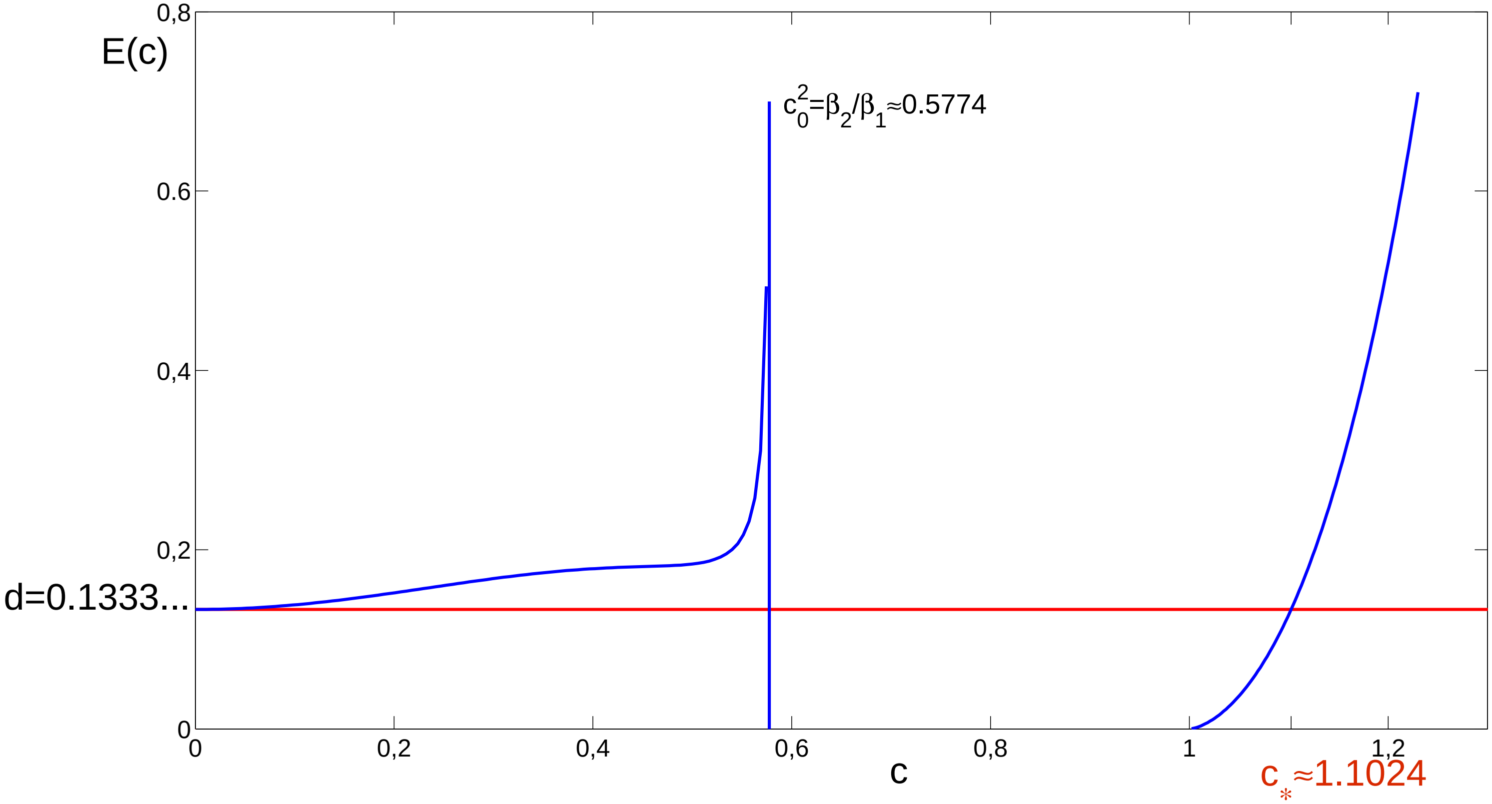


Figure 3: $\beta_2/\beta_1 = 1/3$, $\alpha = 3$.

Example 3: $n = 1,\ \beta_2/\beta_1 = 1$

$$u_0(x) = \left(w^c(x + x_0, t) + w^{-c}(x - x_0, t)\right)\big|_{t=0}\,,$$
$$u_1(x) = \left(w_t^c(x + x_0, t) + w_t^{-c}(x - x_0, t)\right)\big|_{t=0}$$

$$E(0) \approx 2\frac{6}{5\alpha^2}(1 - c^2)(-5c^4 + 4c^2 + 1)$$

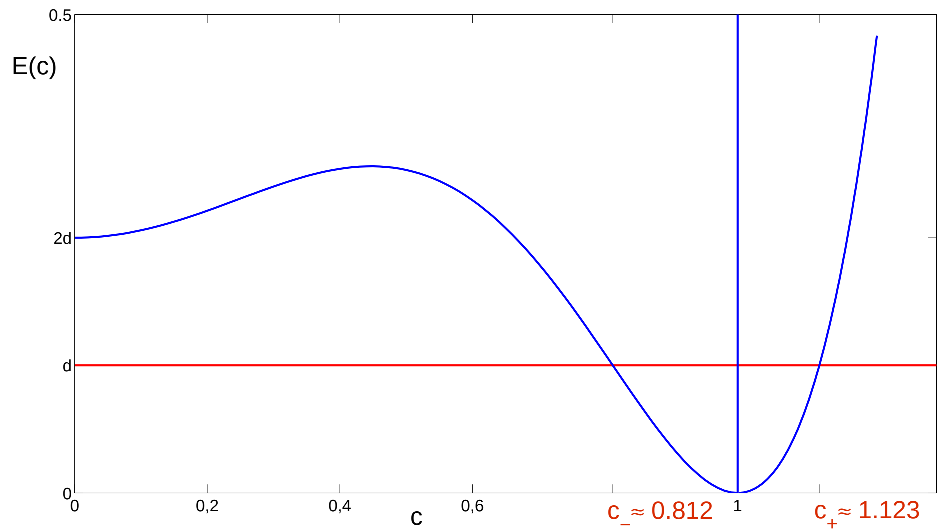


Figure 4: $\beta_1 = 1$, $\beta_2 = 1$, $\alpha = 3$, $c_{blowup} \approx 2.4$.

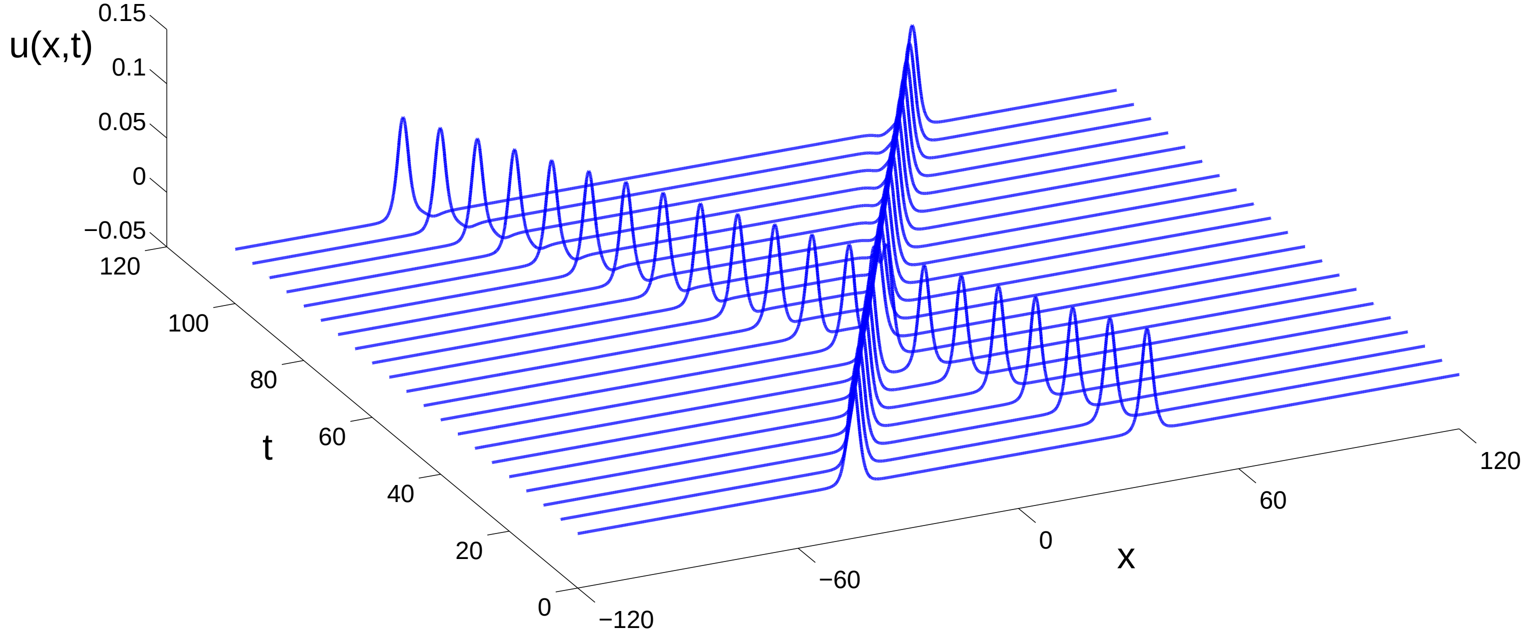


Figure 5: $\beta_1 = 1$, $\beta_2 = 1$, $\alpha = 3$, $c = 1.09$.

V. Numerical experiments

Problem: What is the optimal bound d for the energy E for global solvability or finite blow up of the solutions of Boussinesq Paradigm Equation?

$n = 1$, $p = 2$, $\alpha = 3$, $f(u) = 3u^2$, $\beta_1 = 3/2$, $\beta_2 = 1/2$

$$\begin{aligned} &\tfrac{1}{2}u_{tt} - u_{xx} - \tfrac{3}{2}u_{ttxx} + u_{xxxx} = 3(u^2)_{xx} \\ &u(x,0) = u_0(x) = -A\cosh^{-2}(\tfrac{x}{2}), \qquad u_t(x,0) = u_1(x) = \varepsilon u_0(x) \end{aligned} \tag{7}$$

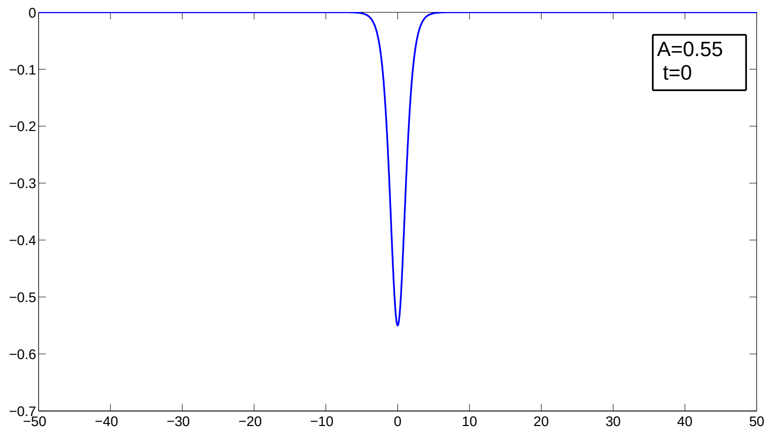
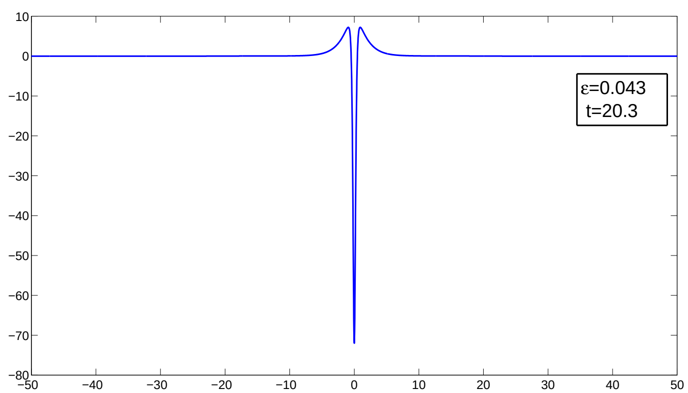


Table 1: $A = 0.55$, $I(0) = -0.0968 < 0$, $x \in [-50, 50]$, $0 \le t \le 50$, t^* - blow up time.

ε	$E(0)$	existence time
0	0.12906	$t^* \approx 7.4$
0.01	0.13328	$t^* \approx 7.1$
0.1	0.55081	$t^* \approx 5.6$
-0.01	0.13328	$t^* \approx 7.85$
-0.1	0.55081	t= 50

ε	$E(0)$	existence time
-0.0400	0.19654	$t^* \approx 12.15$
-0.0410	0.19996	$t^* \approx 12.95$
-0.0420	0.20346	$t^* \approx 14.20$
-0.0430	0.20704	$t^* \approx 20.45$
-0.0431	0.20740	t=50



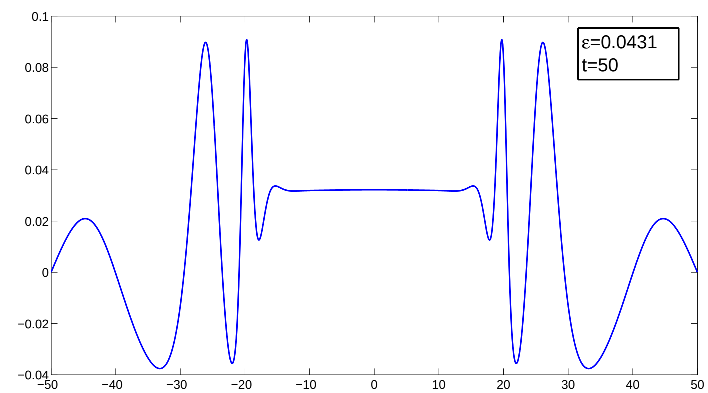
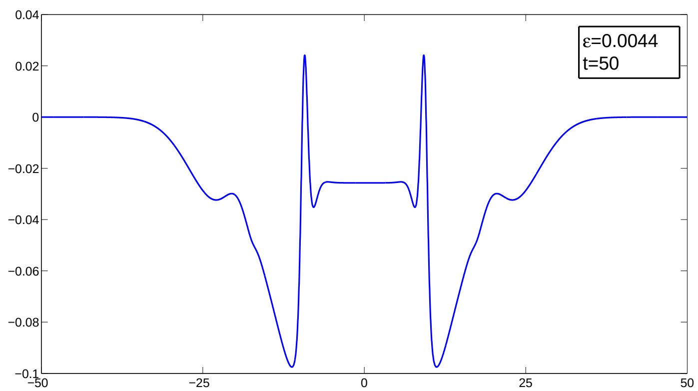
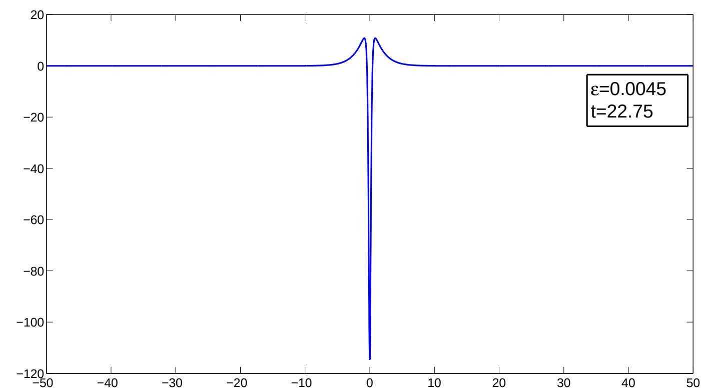


Table 2: $A = 0.495$, $I(0) = 0.0078 > 0$, $x \in [-50, 50]$, $0 \le t \le 50$, t^* - blow up time.

ε	$E(0)$	existence time
0	0.13329	t=50
0.001	0.13332	t=50
0.01	0.13670	$t^* \approx 12.25$
0.1	0.47491	$t^* \approx 6.75$
-0.001	0.13332	t=50

ε	$E(0)$	existence time
0.0040	0.13384	t=50
0.0043	0.13392	t=50
0.0044	0.13395	t=50
0.0045	0.13398	$t^* \approx 23$
0.0050	0.13414	$t^* \approx 17.25$





VI. Conclusions

- The constant d, explicitly found in Theorem 5, is the best possible one for the validity of Theorem 2.

- Another conclusion from the numerical experiments is that the global solvability and finite blow up of the solutions depend on the "angle" between the initial data u_0 and u_1. More precisely in [9, 10] for $n \geq 1$, $\beta_1 = \beta_2 = 1$ there is a partial answer of this question.