A New Conservative Finite Difference Scheme for Boussinesq Paradigm Equation

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Introduction

In the present work we study the Cauchy problem for the Boussinesq Paradigm Equation (BPE)

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \alpha \Delta f(u), \quad x \in \mathbb{R}^n, \; t > 0,$$
$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

on the unbounded region $\mathbb{R}^n$ with asymptotic boundary conditions

$u(x,t) \to 0$, $\Delta u(x,t) \to 0$ as $|x| \to \infty$,

where Δ is the Laplace operator, α, β_1 and β_2 are positive constants.

This is a 4-th order equation in x and t on unbounded region with non-linearity contained in the term $f(u) = u^2$.

Referencies

BPE appears in the modeling of surface waves in shallow waters.

For $\beta_2 > 0$ the problem is *well-posed in the sense of Hadamar*

- the derivation of BPE- Christov C.I., Wave motion, 34, 2001
- Xu&Liu (2009) – existence of a global weak solution; sufficient conditions for both the existence and the lack of a global solution.
- Polat&Ertas (2009) – local and global solution, blow-up of solutions – under different conditions for the nonlinear function $f(u)$.

We assume that the functions u_0, u_1 and $f(u)$ satisfy some regularity conditions so that a unique solution for BPE exists and is smooth enough.

theoretical study of numerical methods for 'good'BE (BPE with $\beta_1 = 0$)

- finite difference method- Ortega, Sanz-Serna, Numerische Math., 1990, 58
- finite element method, optimal error estimates- A. Pani, Saranga, Nonlinear Analysis, 29, 1997;
- pseudospectral method- Ortega, Sanz-Serna , Math. Comp., 1991, 57; for the damped BE- S. Choo, Comm. Korean Math. Soc., 13 , 1998;

numerical simulations and physical interpretations - 1D, 2D:

- Christov, C.I., Wave motion, 34, 2001; Christov, Velarde, Intern. J Bifurcation Chaos, 4, 1994;
- Chertock, A., Christov, C., Kurganov, A.
- Christou, M., Christov, C., AIP, 1186, 2009
- Christov, C., Kolkovska, N., Vasileva, D., LNCS, 6046, 2011;
- Kolkovska, N., LNCS, 6046, 2011; AIP, 2010

Properties to the BPE

Let $\|\cdot\|$ denote the standard norm in $L_2(R^n)$.
Define the energy functional

$$E\left(u(t)\right) = \left\|(-\Delta)^{-1/2}\frac{\partial u}{\partial t}\right\|^2 + \beta_1\left\|\frac{\partial u}{\partial t}\right\|^2 + \|u\|^2 + \beta_2\|\Delta u\|^2 + 2\int_{R^n} F(u)dx$$

with

$$F(u) = \alpha\int_0^u f(s)ds$$

Theorem (Conservation law)

The solution u to Boussinesq problem satisfies the following energy identity

$$E\left(u(t)\right) = E\left(u(0)\right).$$

We obtain similar energy identities for the solutions of the FDS employed in the discretization of BPE.

Notations

- Domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$, L_1, L_2 – sufficiently large;
- a uniform mesh with steps h_1, h_2 in Ω:
  $x_i = ih_1, i = -M_1, M_1;\ y_j = jh_2, j = -M_2, M_2;$
- τ - the time step, $t_k = k\tau, k = 0, 1, 2, \ldots;$
- mesh points (x_i, y_j, t_k);
- $v^k_{(i,j)}$ denotes the discrete approximation $u(x_i, y_j, t_k)$;
- notations for some discrete derivatives of mesh functions:
  - $v^k_{x,(i,j)} = (v^k_{(i+1,j)} - v^k_{(i,j)})/h_1;\qquad v^k_{\bar{x},(i,j)} = (v^k_{(i,j)} - v^k_{(i-1,j)})/h_1;$
  - $v^k_{\bar{x}x,(i,j)} = \left(v^k_{(i+1,j)} - 2v^k_{(i,j)} + v^k_{(i-1,j)}\right)/h_1^2;$
  - $v^k_{\bar{t}t,(i,j)} = \left(v^{k+1}_{(i,j)} - 2v^k_{(i,j)} + v^{k-1}_{(i,j)}\right)/\tau^2;$
  - $\Delta_h v = v_{\bar{x}x} + v_{\bar{y}y}$ – the 5-point discrete Laplacian.
  - $(\Delta_h)^2 v = v_{\bar{x}x\bar{x}x} + v_{\bar{y}y\bar{y}y} + 2v_{\bar{x}x\bar{y}y}$ – the discrete biLaplacian

Whenever possible the arguments of the mesh functions ${}^k_{(i,j)}$ are omitted.

Finite Difference Schemes

In approximation of $\Delta_h v$ and $(\Delta_h)^2 v$ we use v^θ – the symmetric θ-weighted approximation to $v^k_{(i,j)}$:

$v^{\theta,k}_{(i,j)} = \theta v^{k+1}_{(i,j)} + (1-2\theta) v^k_{(i,j)} + \theta v^{k-1}_{(i,j)}$, $\theta \in R$.

for approximation of non-linear term $f(u(x_i, y_j, t_k))$ we use

-
$$f_2(v^k) = 2\frac{F(0.5(v^{k+1}+v^k)) - F(0.5(v^k+v^{k-1}))}{v^{k+1}-v^{k-1}}, \qquad (1)$$

- in 2010:
$$f_1(v^k) = \frac{F(v^{k+1}) - F(v^{k-1})}{v^{k+1}-v^{k-1}}, \quad F(u) = \alpha \int_0^u f(s)ds. \qquad (2)$$

Note that in the case under consideration function $f(v)$ is a polynomial of v, thus the integral $F(v)$ used in f_1, f_2 is explicitly evaluated!

Implicit (with respect to the nonlinearity) scheme

$$v^k_{\bar{t}t} - \beta_1 \Delta_h v^k_{\bar{t}t} - \Delta_h v^{\theta,k} + \beta_2 (\Delta_h)^2 v^{\theta,k} = \Delta_h f_2(v^k). \qquad (3)$$

Initial conditions

$$v^0_{(i,j)} = u_0(x_i, y_j),$$
$$v^1_{(i,j)} = u_0(x_i, y_j) + \tau u_1(x_i, y_j)$$
$$+ 0.5\tau^2 (I - \beta_1 \Delta_h)^{-1} \left(\Delta_h u_0 - \beta_2 (\Delta_h)^2 u_0 + \alpha \Delta_h f(u_0)\right)(x_i, y_j).$$

The equations, boundary and initial conditions form a family of finite difference schemes.

Algorithm

$$\left(v^{k+1} - 2v^k + v^{k-1}\right)/\tau^2 - \beta_1 \Delta_h \left(v^{k+1} - 2v^k + v^{k-1}\right)/\tau^2$$
$$- \theta \Delta_h v^{k+1} - (1-2\theta)\Delta_h v^k - \theta \Delta_h v^{k-1}$$
$$+ \beta_2 \theta (\Delta_h)^2 v^{k+1} + \beta_2 (1-\theta)(\Delta_h)^2 v^k + \beta_2 \theta (\Delta_h)^2 v^{k-1}$$
$$= 2\Delta_h \frac{F(0.5(v^{k+1}+v^k)) - F(0.5(v^k+v^{k-1}))}{v^{k+1}-v^{k-1}}$$

The inner iterations for evaluation of v^{k+1} start from v^k. They stop when the relative error between two successive iterations is less than a given threshold $\epsilon = 10^{-13}$.

- 1D case – 5-diagonal linear system of equations
- 2D case – splitting procedure and 5-diagonal linear systems in each direction

Analysis of the nonlinear schemes

Preliminaries:

the space of mesh functions which vanish on ω;

the scalar product at time t^k with respect to the spatial variables $\langle v, w\rangle = \sum_{i,j} h_1 h_2 v_{(i,j)}^{(k)} w_{(i,j)}^{(k)}$;

operators $A = -\Delta_h$

$$B = I - \beta_1 \Delta_h + \tau^2\theta(-\Delta_h + \beta_2(\Delta_h)^2);$$

A, B are self-adjoint positive definite operators.

Operator form of the schemes:

$$B v_{\bar{t}t} + Av + \beta_2 A^2 v = -A f_2,$$

$$A^{-1} B v_{\bar{t}t} + v + \beta_2 A v + f_2 = 0$$

(derived after applying A^{-1})

The energy functional E_h^L (obtained from the linear part of the equation) at the k-th time level is

$$E_h^L(v^{(k)}) =$$
$$\left\langle A^{-1} v_t^{(k)}, v_t^{(k)} \right\rangle + \beta_1 \left\langle v_t^{(k)}, v_t^{(k)} \right\rangle + \tau^2(\theta - 1/4) \left\langle (I + \beta_2 A) v_t^{(k)}, v_t^{(k)} \right\rangle$$
$$+ 1/4 \left\langle v^{(k)} + v^{(k+1)} + \beta_2 A(v^{(k)} + v^{(k+1)}), v^{(k)} + v^{(k+1)} \right\rangle$$

The *full discrete energy functional* is (including the non-linearity)

$$E_h(v^{(k)}) = E_h^L(v^{(k)}) + 2 \left\langle F(0.5(v^{(k+1)} + v^{(k)})), 1 \right\rangle$$

Theorem (Discrete conservation law)

The solution to the implicit scheme satisfies the energy equalities

$$E_h(v^{(k)}) = E_h(v^{(0)}), \qquad k = 1, 2, \ldots .$$

i.e. the discrete energy is conserved in time.

$$\theta > \frac{1}{4} - \frac{\beta_1}{\tau^2 ||I + \beta_2 A||}. \tag{4}$$

Note that if parameter θ satisfies (4), then functional $E_h^L(v^k)$ is nonnegative and can be viewed as a norm. Such combined norms depending on the values of solution on several layers are typical for three-layer schemes.

The local truncation error of implicit scheme is $O(|h|^2 + \tau^2)$.

Theorem (Convergence of the Implicit Scheme)

Let $f(u) = u^2$ and the parameter θ satisfies (4). Assume that the solution u to BPE obeys $u \in C^{4,4}\left(\mathbb{R}^2 \times (0, T)\right)$ and the solution v to the finite difference scheme (3) is bounded in the maximal norm. Let M be a constant such that

$$M \geq \max_{i,j,s \leq k} \left(|u(x_i, y_j, t_s)|, \left| \frac{\partial^2 u}{\partial t^2}(x_i, y_j, t_s) \right|, |v_{i,j}^{(s)}| \right)$$

and τ be sufficiently small, $\tau < (2C_2M)^{-1}$. Then v converges to the exact solution u as $|h|, \tau \to 0$ and the following estimate holds for the error $z = y - u$:

$$\left(z^{(k)}, z^{(k)}\right) + \left(Az^{(k)}, z^{(k)}\right) \leq Ce^{Mt_k}\left(|h|^2 + \tau^2\right)^2. \qquad (5)$$

The main feature of Theorem is the established second order of convergence in discrete W_2^1 norm, which is compatible with the rate of convergence of the similar linear problem.

Corollary

(i) The convergence of the solution to FDS with $\theta \geq 0.25$ *to the exact solution is of second order when* $|h|$ *and* τ *go independently to zero.*
(ii) For the scheme with $\theta = 0$ *the convergence of the numerical solution to the exact solution is of second order when* $|h|$ *and* τ *go to* 0 *provided* $\tau^2 < \frac{4}{9}\frac{\beta_1}{\beta_2}h^2$.

Corollary

Under the assumptions of the main Theorem the FDS admits the following error estimate in the uniform norm $(z = y - u)$*:*

$$\max_{i} |z_i^{(k)}| < Ce^{Mt_k} \left(|h|^2 + \tau^2\right), \qquad d = 1;$$

$$\max_{i,j} |z_{i,j}^{(k)}| < Ce^{Mt_k} \sqrt{\ln N} \left(|h|^2 + \tau^2\right), \qquad d = 2.$$

The above estimates are optimal for the 1D case and *almost* optimal (up to a logarithmic factor) for the 2D case.

- The boundedness of the exact solution u to the BPE on the time interval $[0, T]$ is a main assumption in the convergence theorems.
- BPE may have both bounded on the time interval $[0, \infty)$ solutions or blowing up solutions
- the L_∞ norm of the exact solution is included in the exponent in the right-hand sides of the error estimates
- if u blows up at a moment T_0, $T_0 > T$, then: $\|u\|_{L_\infty[0,T]}$ will be big ; the term e^{MT} will be big ; the convergence will slow up!
- additional restriction on the time step in the convergence theorem is

$$\tau < (2C_2M)^{-1}, \quad M \geq \|u\|_{L_\infty[0,T]}.$$

In any case the FDS should be applied with very small τ's if one would like to evaluate the solution in a neighborhood of the blow up moment.

Preliminaries

- An analytical solution of the 1D equation (one solitary wave):

$$u(x,t;x_0,c) = \frac{3}{2}\frac{c^2-1}{\alpha}\operatorname{sech}^2\left(\frac{x-x_0-ct}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}\right),$$

  where x_0 is the initial position of the peak of the solitary wave,
- Parameters: $\alpha = 3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, c is the wave speed.
- Initial conditions for one solitary wave or two solitary waves:

$$u(x,0) = u(x,0;-40,2) + u(x,0;50,-1.5)$$

$$\frac{du}{dt}(x,0) = u(x,0;-40,2)_t + u(x,0;50,-1.5)_t$$

- Two conservative implicit schemes with $\theta = 0.5$; inner iterations until relative error $< \epsilon$, $\epsilon = 10^{-13}$.
  - 'old' (2010) , f_1
  - 'new' (2011) , f_2

Rate of convergence and errors, case of one solitary wave

Table: $\beta_1 = 1.5,\ \beta_2 = 0.5,\ \alpha = 3,\ c = 2,\ x \in [-40, 120],\ T = 40.$

$h = \tau$	Rate 'old'	Rate 'new'	Er. 'old'	Er. 'new'	'old'/'new'
0.2	–	–	0.265115	0.144106	1.83
0.1	1.8836	1.9411	0.071849	0.037527	1.91
0.05	1.9720	1.9852	0.018315	0.009478	1.93
0.025	1.9929	1.9961	0.004601	0.002376	1.94
0.0125	1.9966	1.9961	0.001153	0.000596	1.93

$$E_1 = ||\tilde{u} - u_{[h]}||, \quad E_2 = ||\tilde{u} - u_{[h/2]}|| \quad \text{Rate} = \log_2(E_1/E_2)$$

$$\text{Error} = \max_{0 \le i \le N} |\tilde{u}_i - u_{[h],i}|$$

Rate of convergence and errors, case of two solitary waves

Table: $\beta_1 = 1.5,\ \beta_2 = 0.5,\ \alpha = 3,\ c_1 = 2,\ c_2 = -1.5,\ x \in [-160, 170]$, $T = 80$.

$h = \tau$	Rate 'old'	Rate 'new'	Er. 'old'	Er. 'new'	'old'/'new'
0.1	–	–	–	–	–
0.05	1.9634	1.9819	0.126497	0.066214	1.91
0.025	1.9931	2.0000	0.032210	0.016692	1.93
0.0125	2.1730	2.1789	0.007785	0.004034	1.93

$$\text{Error} = E_1^2/(E_1 - E_2), \quad E_1 = ||u_{[h]} - u_{[h/2]}||, \quad E_2 = ||u_{[h/2]} - u_{[h/4]}||$$

- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.
- For one soliton and two solitary waves the 'new' scheme is about 2 times more precise than the 'old' implicit scheme.

With respect to the error magnitude the 'new' scheme with RHS f_2 performs twice better than the 'old' scheme with RHS f_1!

Justification: Consider the right-hand side of the FDS. We expand f_1, f_2 in Taylor series about the point (x_i, t^k) and get

$$f_1(u(x_i,t^k)) = f(u(x_i,t^k)) + \tau^2 R_1 + O(\tau^3),$$
$$f_2(u(x_i,t^k)) = f(u(x_i,t^k)) + \tau^2 R_2 + O(\tau^3),$$
$$R_1 = \frac{1}{2}\alpha \frac{\partial f}{\partial u}(x_i,t^k)\frac{\partial^2 u}{\partial t^2}(x_i,t^k),$$
$$R_2 = \frac{1}{4}\alpha \frac{\partial f}{\partial u}(x_i,t^k)\frac{\partial^2 u}{\partial t^2}(x_i,t^k).$$

Thus, $R_1 = 2R_2$. This has essential impact on the error, when the solution has large derivatives $(f(u) = u^3)$!

parameteres: $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$

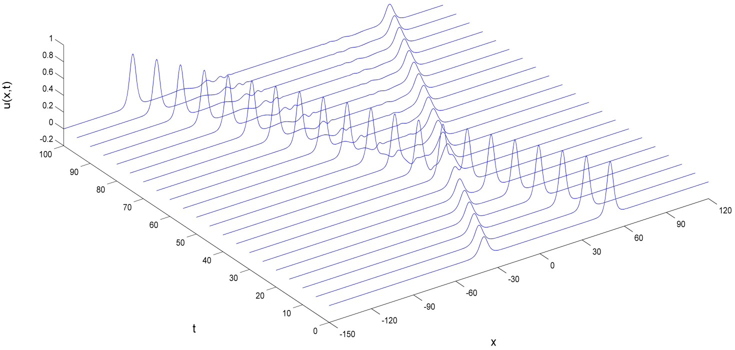

Figure: Iteraction of two solitons: $c_1 = 1.2$, $c_2 = -1.5$.

parameters: $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$

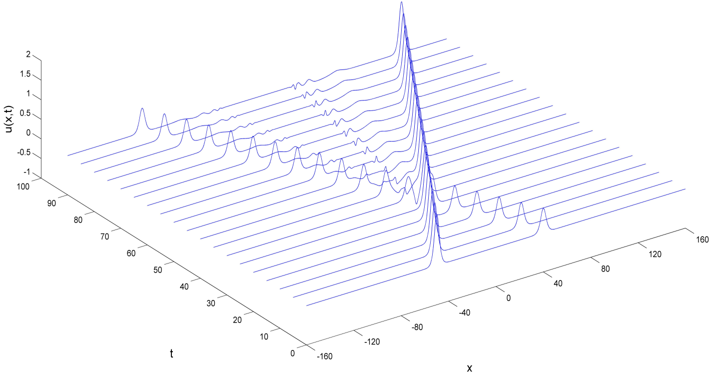

Figure: Iteraction of two solitons: $c_1 = 1.9$, $c_2 = -1.5$.

$$\beta_1 = 1.5,\ \beta_2 = 0.5,\ \alpha = 3$$

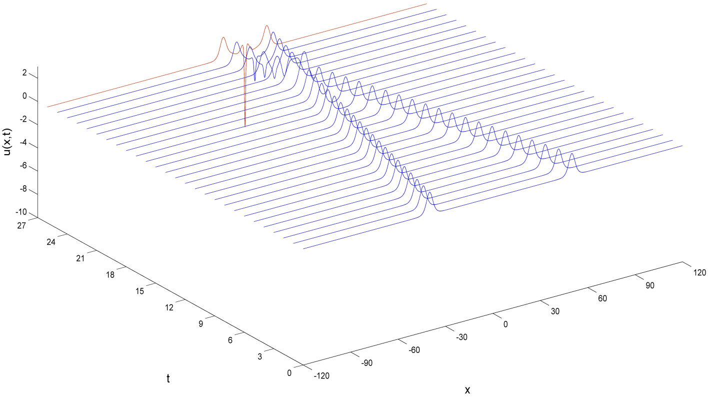

Figure: Iteraction of two solitons: $c_1 = -c_2 = -2.2$,

$t^* \approx 27$, t^* - blow up time

Thank you for your attention!