NEW CLASS OF METASTABLE BLOW-UP SOLUTIONS OF THE MATHEMATICAL MODEL OF HEAT STRUCTURES

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"Advances in Numerical Methods for Computational Science":
a special session dedicated to Professor Raytcho Lazarov
on the occasion of his 70th birthday

AMiTaNS'2013, Albena, Bulgaria, June 24-28, 2013

$$u_t = \sum_{i=1}^{N} (k_i(u) u_{x_i})_{x_i} + Q(u), \ t > 0, \ x \in \mathbb{R}^N$$

$$k_i(u) \geq 0, \quad Q(u) \geq 0, \quad u(t,x) \geq 0$$

$$k_i(u) = u^{\sigma_i}, \ \sigma_i > 0, \quad Q(u) = u^{\beta}, \ \beta > 1$$

A.A. Samarskii, M.I. Sobol (1963), S.P. Kurdyumov (1974), M.I. Bakirova, V.A. Galaktionov, V.A. Dorodnicyn, G.G. Elenin, N.V. Zmitrenko, E.S. Kurkina, A.P. Mihailov, Y.P. Popov, A.B. Potapov, M.N. LeRoux, S. Svirshchevskii, H. Wilhelmsson,...

$$2D, \sigma_1 \neq \sigma_2, \ \sigma_i > 0, \quad Q(u) = u^{\beta}, \ \beta > 1 :$$

blow-up self-similar solution, describing directed heat diffusion (Dorodnicyn, Knyazeva)

numerical implementation:

Bakirova, M.I., Dimova, S.N., Dorodnicyn, V.A., Kurdyumov, S.P., Samarskii, A.A., Svirshchevskii, S.: Invariant solutions of heat-transfer equation, describing directed heat diffusion and spiral waves in nonlinear medium, Soviet Phys. Dokl. **33**(3), 187–189 (1988)

2D Isotropic medium

$$u_t = \frac{1}{r}(r u^{\sigma} u_r)_r + \frac{1}{r^2}(u^{\sigma} u_{\varphi})_{\varphi} + u^{\beta},\ 0 < r < \infty,\ 0 \le \varphi < 2\pi.$$

Свирщевский, 1985: $u_s(t, r, \varphi) = (1 - \frac{t}{T_0})^{-\frac{1}{\beta-1}} \theta(\xi, \phi),$

$$\xi = r(1 - \frac{t}{T_0})^{-\frac{m}{\beta-1}},\ \phi = \varphi + \frac{C_0}{\beta - 1} \ln(1 - \frac{t}{T_0}),\ m = \frac{\beta - \sigma - 1}{2}$$

C_0 - parameter of the family of solutions. For $C_0 \neq 0 :$

$$r(t)e^{s\varphi(t)} = r(0)e^{s\varphi(0)} = \xi e^{s\phi} = \mathrm{const},\ \ s = (\beta - \sigma - 1)/(2C_0).$$

$$-\frac{1 \partial}{\xi \partial \xi}(\xi \theta^{\sigma} \frac{\partial \theta}{\partial \xi}) - \frac{1}{\xi^2} \frac{\partial}{\partial \phi}(\theta^{\sigma} \frac{\partial \theta}{\partial \phi}) + \frac{m}{(\beta - 1)T_0} \xi \frac{\partial \theta}{\partial \xi} - \frac{C_0}{(\beta - 1)T_0} \frac{\partial \theta}{\partial \phi} + \theta - \theta^{\beta} = 0.$$

Two homogenious solutions for $T_0 = \frac{1}{(\beta-1)}$: $\theta_H^0 \equiv 0$, $\theta_H \equiv 1$

$$C_0 = 0, \quad \lim_{\xi\to\infty} \theta(\xi,\phi) = \theta_H^0 \equiv 0:$$

$\beta > \sigma + 1:$ two classes radially nonsymmetric solutions of complex symmetry were found and investigated (Kurdyumov, Kurkina, Potapov, Samarskii, 1984, 1986.)

Koleva, M.G., Dimova, S.N., Kaschiev M.S.: Analisys of the eigen functions of combustion of a nonlinear medium in polar coordinates. Math. Modeling **3**, 76–83 (1992)

$\beta \leq \sigma + 1:$ only simple finite support radially symmetric solutions.

$$\lim_{\xi\to\infty} \theta(\xi,\phi) = \theta_H \equiv 1:$$

two new classes of solutions in $HS-$regime:

– complex symmetry solutions for $C_0 = 0$,

– spiral wave-solutions for $C_0 \neq 0$.

Dimova S.N., Kastchiev M.S., Koleva M.G., Vasileva D.P., Numerical analysis of radially nonsymmetric blow-up solutions of a nonlinear parabolic problem, J. Comp. Appl. Math., 1998, 97, 81-97

Dimova M.G., Dimova S.N., Numerical investigation of spiral structure solutions of a nonlinear elliptic problems, NMA 2010, LNCS 6046, 395–403, 2011

2D radially nonsymmetric case, $C_0 = 0$, $\beta = \sigma + 1$

$$u(t, r, \varphi) = g(t)\theta(\xi, \phi),\; g(t) = (1 - t/T_0)^{-1/\sigma}, \quad \xi = r,\; \phi = \varphi,$$

$$\mathcal{L}(\theta) \equiv -\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \theta^{\sigma} \frac{\partial \theta}{\partial \xi}\right) - \frac{1}{\xi^2}\frac{\partial}{\partial \phi}\left(\theta^{\sigma} \frac{\partial \theta}{\partial \phi}\right) + \theta - \theta^{\sigma+1} = 0.$$

Initial Approximations

$$\tilde{\theta}(\xi, \phi) = 1 + \alpha y(\xi, \phi),\; \alpha = \text{const},\; |\alpha y| \ll 1,$$

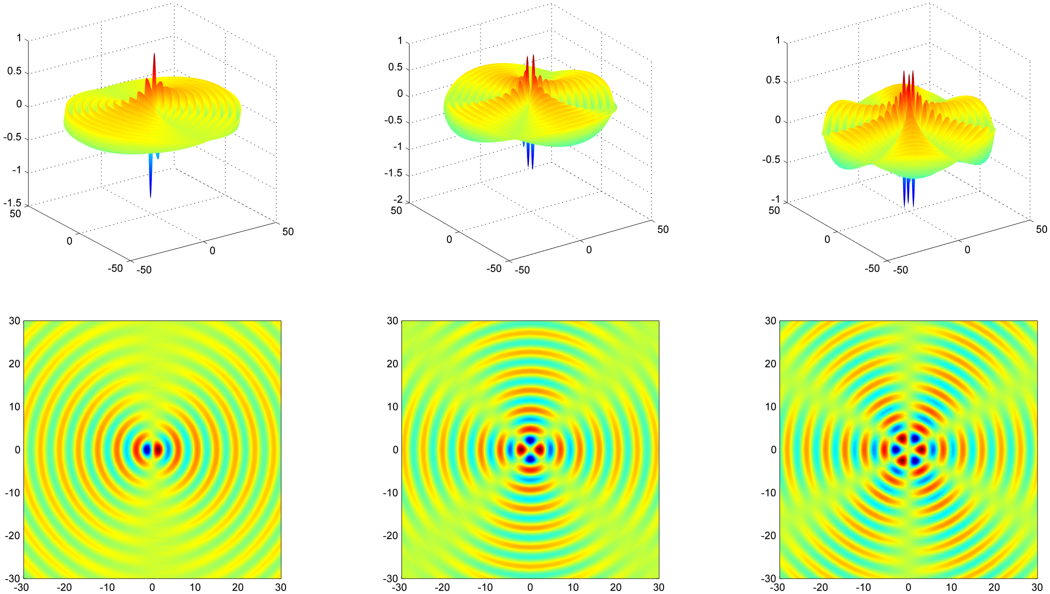
$y(\xi, \phi)$ is a bounded at $\xi = 0$ solution of the linear equation:

$$-\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial y}{\partial \xi}\right) - \frac{1}{\xi^2}\frac{\partial^2 y}{\partial \phi^2} + \sigma y = 0.$$

Particular solutions

$$y(\xi, \phi) = J_k(\sqrt{\sigma}\xi)\cos(k\phi),\; k \in N.$$

$y(\xi, \phi)$ are $2\pi/k$ periodic and have k axes of symmetry:



$k = 1$ $k = 2$ $k = 3$

Fig. 1: Solutions $y(\xi, \phi) = J_k(\sqrt{\sigma}\xi)\cos(k\phi)$ to the linearized equation, $k = 1, 2, 3$.

$$J_k(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{k\pi}{2} - \frac{\pi}{4}\right), \; z \to \infty,$$

$$y(\xi, \phi) \sim \sqrt{\frac{2}{\pi\sqrt{\sigma}\xi}} \cos\left(\sqrt{\sigma}\xi - \frac{k\pi}{2} - \frac{\pi}{4}\right) \cos(k\phi), \; \xi \to \infty.$$

asymptotic for $\theta(\xi, \phi)$

$$\theta(\xi, \phi) \sim 1 + \gamma\sqrt{\frac{2}{\pi\sqrt{\sigma}\xi}} \cos\left(\sqrt{\sigma}\xi - \frac{k\pi}{2} - \frac{\pi}{4}\right) \cos(k\phi),$$

$$\gamma = const, \; \xi \to \infty.$$

Back
Close

The self-similar problem is closed:

$$L(\theta) \equiv -\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\theta^{\sigma}\frac{\partial\theta}{\partial\xi}\right) - \frac{1}{\xi^2}\frac{\partial}{\partial\phi}\left(\theta^{\sigma}\frac{\partial\theta}{\partial\phi}\right) + \theta - \theta^{\sigma+1} = 0,$$

$$(\xi, \phi) \in \Omega, \quad \Omega = (0, l) \times (0, \omega), \quad \omega = \pi/k.$$

boundary conditions

$$\lim_{\xi\to 0} \xi\theta^{\sigma}\frac{\partial\theta}{\partial\xi} = 0, \ \phi \in [0, \omega],$$

$$\frac{\partial\theta}{\partial\xi} + \frac{\theta - 1}{2\xi} = -\gamma\sqrt{\frac{2}{\pi\sqrt{\sigma}\xi}}\sin\left(\sqrt{\sigma}\xi - \frac{k\pi}{2} - \frac{\pi}{4}\right)\cos(k\phi),$$

$$\xi = l \gg 1, \phi \in [0, \omega],$$

$$\frac{\partial\theta}{\partial\phi}(\xi, 0) = \frac{\partial\theta}{\partial\phi}(\xi, \omega) = 0, \quad 0 \leq \xi \leq l.$$

Numerical method for the self-similar problem

Continuous analog of the Newton's method (Gavurin, 1958)

$$0 \leq t < \infty, \qquad \theta = \theta(\xi, \phi, t):$$

$$L'(\theta)\frac{\partial \theta}{\partial t} = -L(\theta), \qquad \theta(\xi, \phi, 0) = \theta_0(\xi, \phi)$$

$$\frac{\partial \theta}{\partial t} = v(\xi, \phi, t)$$

Iteration process

$$L'(\theta_n)v_n = -L(\theta_n)$$

$$\theta_{n+1} = \theta_n + \tau_n v_n, \quad 0 < \tau_n \leq 1, \quad n = 0, 1, \ldots,$$

$$\theta_n = \theta_n(\xi, \phi) = \theta(\xi, \phi, t_n), \quad v_n = v_n(\xi, \phi) = v(\xi, \phi, t_n),$$

$$\theta_0(\xi, \phi) = \theta_k(\xi, \phi) = 1 + \alpha J_k(\sqrt{\sigma}\xi)\cos(k\phi), \ |\alpha y| \ll 1$$

Galerkin FEM for the linear problems at every iteration

Bilinear finite elements

$$A(\theta)\bar{V} = -B(\theta)\bar{\Theta}$$

$$V = \{v_n(\xi_j)\}_{j=1}^{N}, \quad A = LU$$

$$\tau_n = \min\left(1, \tau_{n-1}\frac{\delta_{n-1}}{\delta_n}\right), \text{if } \delta_n < \delta_{n-1},$$

$$\tau_n = \max\left(\tau_0, \tau_{n-1}\frac{\delta_{n-1}}{\delta_n}\right), \text{if } \delta_n \geq \delta_{n-1},$$

$$\delta_n = \max_{\eta\in\bar{\omega}_h} |B(\theta_n)\bar{\Theta}_n|, [Puzynin, ..., 1974]$$

Numerical method for the parabolic problem

$$
\begin{aligned}
&u_t = \frac{1}{r}\left(ru^\sigma u_r\right)_r + \frac{1}{r^2}\left(u^\sigma u_\varphi\right)_\varphi + u^{\sigma+1},\ 0 < t < T_0,\\
&(r,\varphi) \in \Omega,\quad \Omega = (0,R)\times(0,\omega),\ \omega = \pi/k,\\
&ru^\sigma u_r(t,0,\varphi) = 0,\quad t\in[0,T_0),\quad \varphi\in[0,\omega],\\
&u(t,R,\varphi) = \frac{u(t,r^*,\varphi)}{\theta(r^*,\varphi)}\theta(R,\varphi),\quad t\in[0,T_0),\quad \varphi\in[0,\omega],\\
&u(0,r,\varphi) = u_0(r,\varphi) = \theta_k(r,\varphi) \geq 0,\quad (r,\varphi)\in\bar{\Omega},\\
&u^\sigma(t,r,0)u_\varphi(t,r,0) = u^\sigma(t,r,\omega)u_\varphi(t,r,\omega) = 0,\quad t\in[0,T_0),\quad r\in[0,R],\\
&u_0(0,\varphi) = \text{const},\quad u_0(r,\varphi) = u_0(r,2\omega-\varphi).
\end{aligned}
$$

GFEM, based on

- Kirchhoff transformation of the nonlinear heat-conductivity coefficient

$$G(u) = \int_0^u s^\sigma\,ds = u^{\sigma+1}/(\sigma+1);$$

- interpolation of the nonlinear coefficients;
- lumped mass matrix.

semidiscrete problem

$$\dot{U} = -\tilde{M}^{-1}KG(U) + q(U)$$
$$U(0) = U_0$$

Modification of an explicit second order Runge-Kutta method.

Numerical investigations

Aims of the numerical investigations:

- to analyze the evolution in time of the new class of 2D self-similar solutions,
- to analyze their stability.

In the examples bellow: $\sigma = 2,\ T_0 = 0.5,\ l = R = 50,$
$h_r = 0.1,\ h_\varphi = \pi/30,\ \alpha = 0.1,\ \gamma = 0.2.$

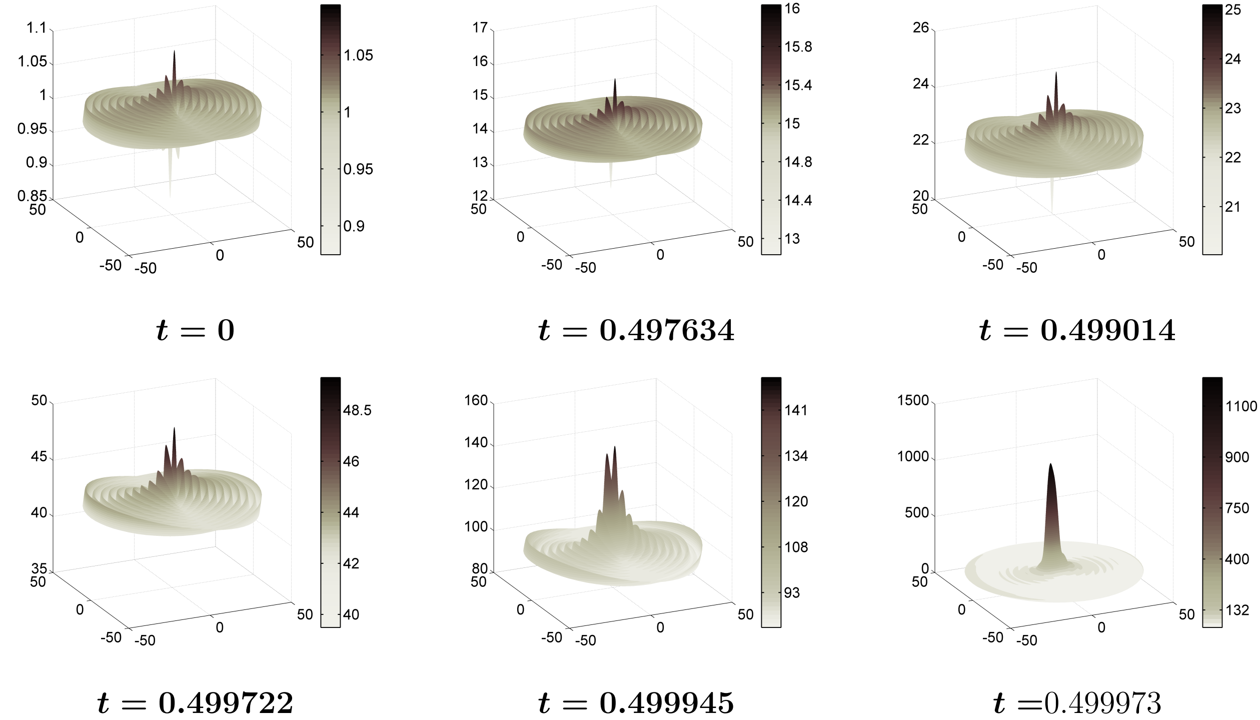


Fig. 2: Evolution of a complex wave in S-regime: $u(t, r, \varphi)$, $\sigma = 2$, $k = 1$, $T_0 = 0.5$.

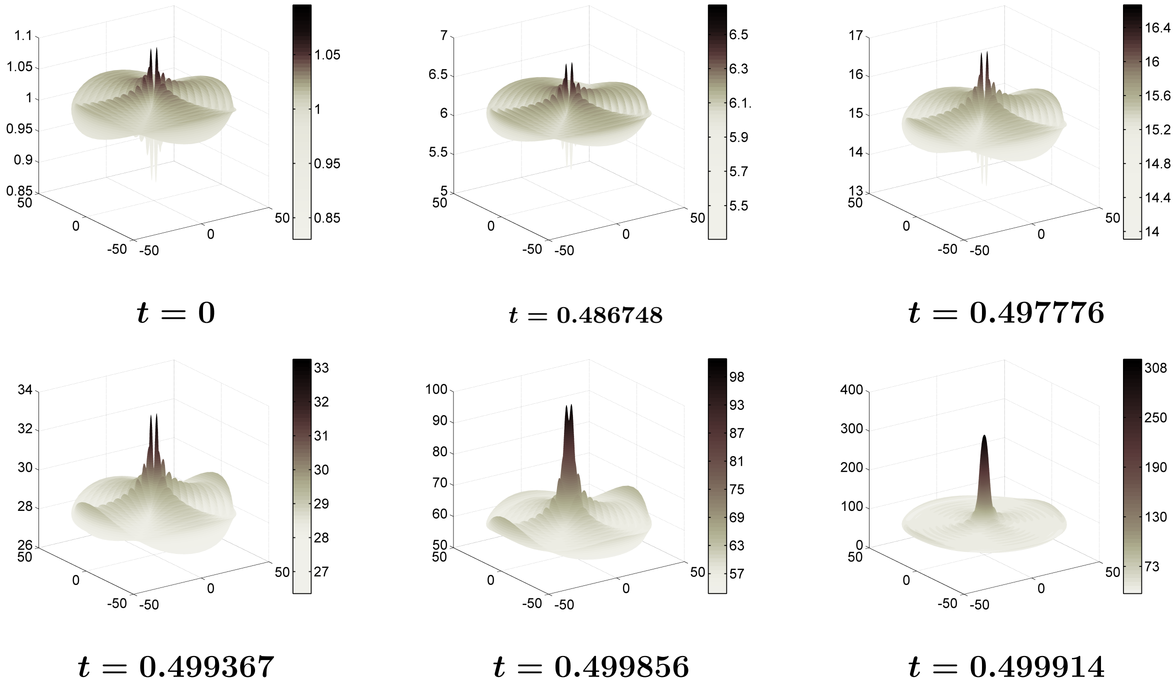


Fig. 3: Evolution of a complex wave in S-regime: $\boldsymbol{\sigma} = \mathbf{2}$, $\boldsymbol{k} = \mathbf{2}$, $\boldsymbol{T_0} = \mathbf{0.5}$.

Back
Close

2D radially symmetric case

$$u_t = \frac{1}{r}(r u^\sigma u_r)_r + u^{\sigma+1}, \quad r \in \mathbb{R}^1_+, \quad t > 0, \quad \sigma > 0.$$

self-similar blow-up solutions

$$u(t,r) = \psi(t)\theta(\xi) = \left(1 - \frac{t}{T_0}\right)^{-\frac{1}{\sigma}} \theta(\xi),\ \xi = r \text{ for } \beta = \sigma + 1.$$

$$\mathcal{L}(\theta) \equiv -\frac{1}{\xi}(\xi \theta^\sigma \theta')' + \theta - \theta^{\sigma+1} = 0, \ \ 0 < \xi < \infty.$$

$$u(0,r) = u_0(r) = \theta(r), \ \ u_r(t,0) = 0.$$

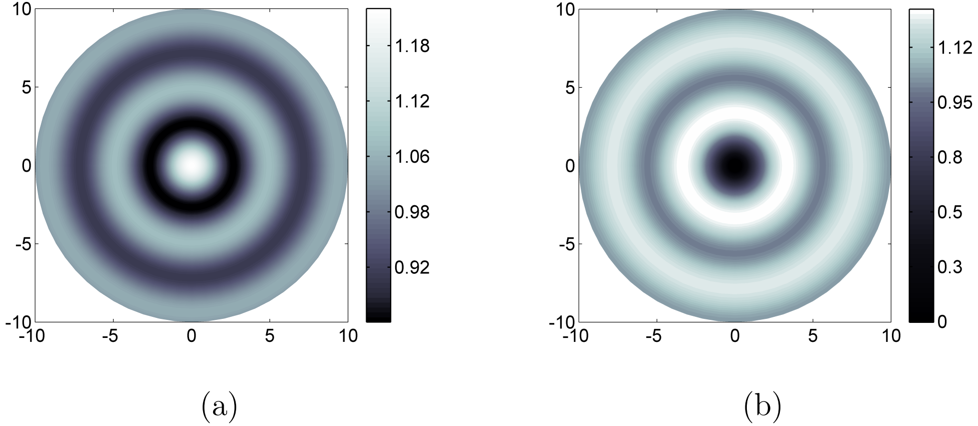


Fig. 4: Self-similar function $\theta(\xi)$ computed for $\sigma = 2$: (a) $\alpha = 0.1$; (b) $\alpha = -0.5$.

We seek for bounded at infinity solutions, which for a finite $R \gg 1,\ R < \infty$, satisfy the self-similar law:

$$u(t, R) = \frac{u(t, r^*)}{\theta(r^*)}\theta(R) \quad t > 0$$

for an appropriate choice of the point $r^* \ll R$.

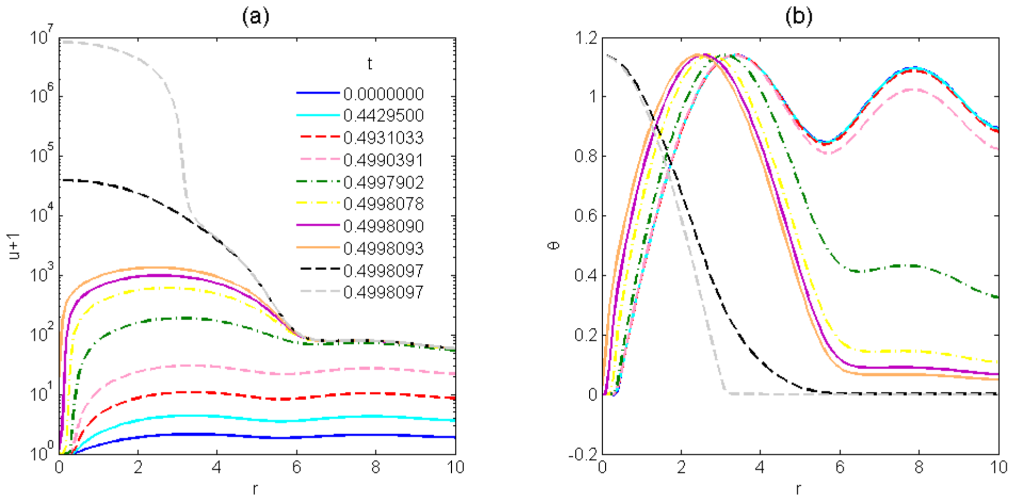


Fig. 5: Evolution in time of the self-similar solution $\boldsymbol{u(t,r)}$ and its self-similar representations. (Zoom in the region near the origin.)