STRUCTURES AND WAVES IN A NONLINEAR HEAT-CONDUCTING MEDIUM

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Symposium in honor of Raytcho Lazarov's 40 years research in Computational Methods and Applied Mathematics,

Sozopol, Bulgaria, June 7-8, 2013

$$\boldsymbol{u_t = \sum_{i=1}^{N}(k_i(u)u_{x_i})_{x_i} + Q(u),\ t > 0,\ x \in \mathbb{R}^N}$$

$k_i(u) \equiv \mathbf{const}$, $Q(u) = \lambda e^u$ (Frank-Kamenetzki equation),

$Q(u) = u^\beta,\ \beta > 1$: H. Fujita (1966), J. Bebernes, A. Bressan, H. Brezis, D. Eberly, A. Friedman, V.A. Galaktionov, I.M. Gelfand, M.A. Herrero, R. Kohn, L.A. Lepin, S.A. Posashkov, A.A. Samarskii, J.L. Vazquez, L.J.L. Velazquez,...

$k_i(u) \neq \mathbf{const}$: D.G. Aronson, A. Friedman, I.M. Gelfand, A.S. Kalashnikov, S. Kaplan, O.A. Ladyjenskaya, H.A. Levine, O.A. Oleinik, L.A. Pelitier, N.N. Uralceva...

$$k_i(u) = u^{\sigma_i},\ \sigma_i > 0, \quad Q(u) = u^\beta,\ \beta > 1$$

A.A. Samarskii, M.I. Sobol(1963), S.P. Kurdyumov (1974), M.I. Bakirova, V.A. Galaktionov, V.A. Dorodnicyn, G.G. Elenin, N.V. Zmitrenko, E.S. Kurkina, A.P. Mihailov, Y.P. Popov, A.B. Potapov, M.N. LeRoux, S. Svirshchevskii, H. Wilhelmsson,...

2D: Anisotropic medium. Directed heat diffusion

$$u_t = (u^{\sigma_1} u_{x_1})_{x_1} + (u^{\sigma_2} u_{x_2})_{x_2} + u^{\beta}, \ (x_1, x_2) \in \mathbb{R}^2, \ \sigma_i > 0, \ \beta > 1$$

blow-up self-similar solution (Dorodnicyn, Knyazeva):

$$u_s(t, x_1, x_2) = (1 - t/T_0)^{-\frac{1}{\beta-1}} \theta(\xi), \ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

$$\xi_i = x_i (1 - t/T_0)^{-m_i/(\beta-1)} \quad m_i = (\beta - \sigma_i - 1)/2, \ i = 1, 2.$$

$$L(\theta) \equiv \sum_{i=1}^{2} \left(-\frac{\partial}{\partial \xi_i} \left(\theta^{\sigma_i} \frac{\partial \theta}{\partial \xi_i} \right) + \frac{\beta - \sigma_i - 1}{2} \xi_i \frac{\partial \theta}{\partial \xi_i} \right) + \theta - \theta^{\beta} = 0.$$

$HS(1 < \beta < \sigma_i + 1)$; $S(1 < \beta = \sigma_i + 1)$; $LS(\sigma_i + 1 < \beta)$

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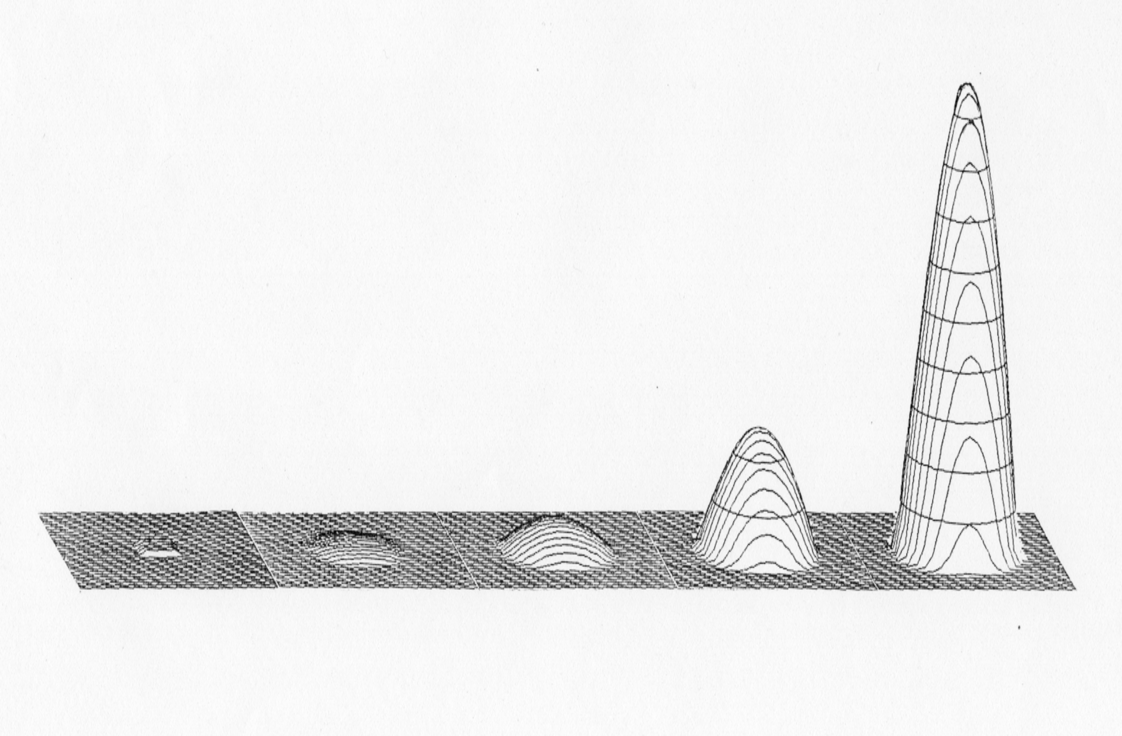

Fig. 1: $\boldsymbol{S}$ evolution: $\boldsymbol{\sigma_1 = 2,\ \sigma_2 = 2,\ \beta = 3}$.

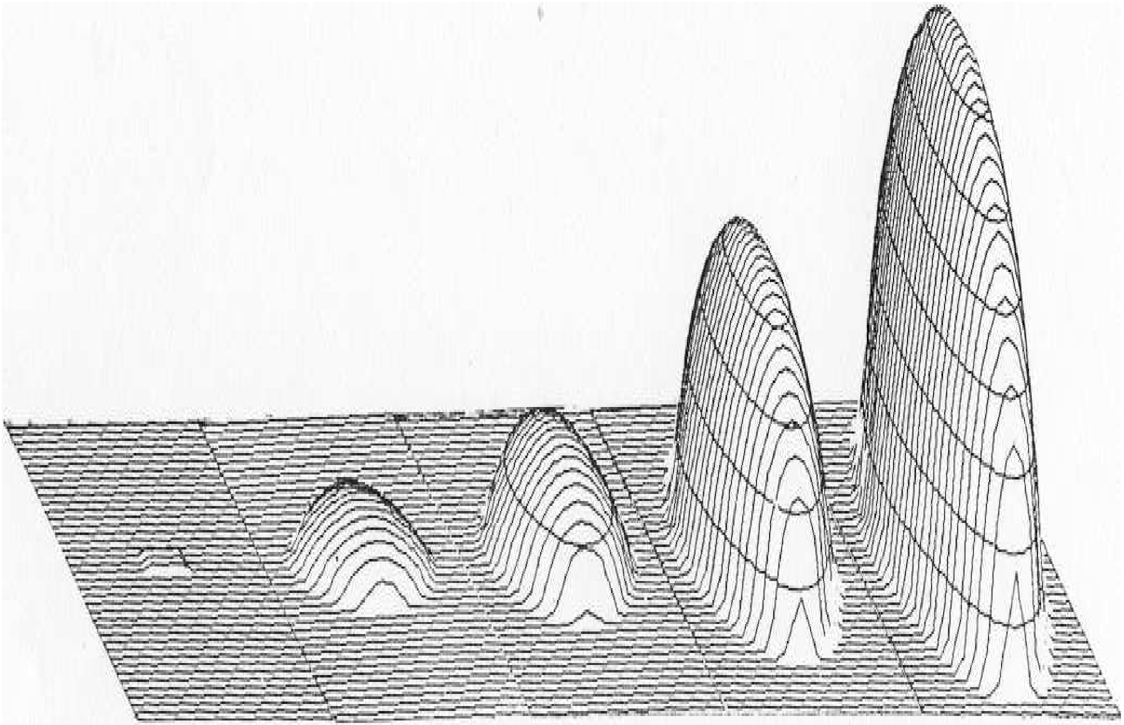

Fig. 2: $\boldsymbol{HS - S}$ evolution: $\boldsymbol{\sigma_1 = 3,\ \sigma_2 = 2,\ \beta = 3}$.

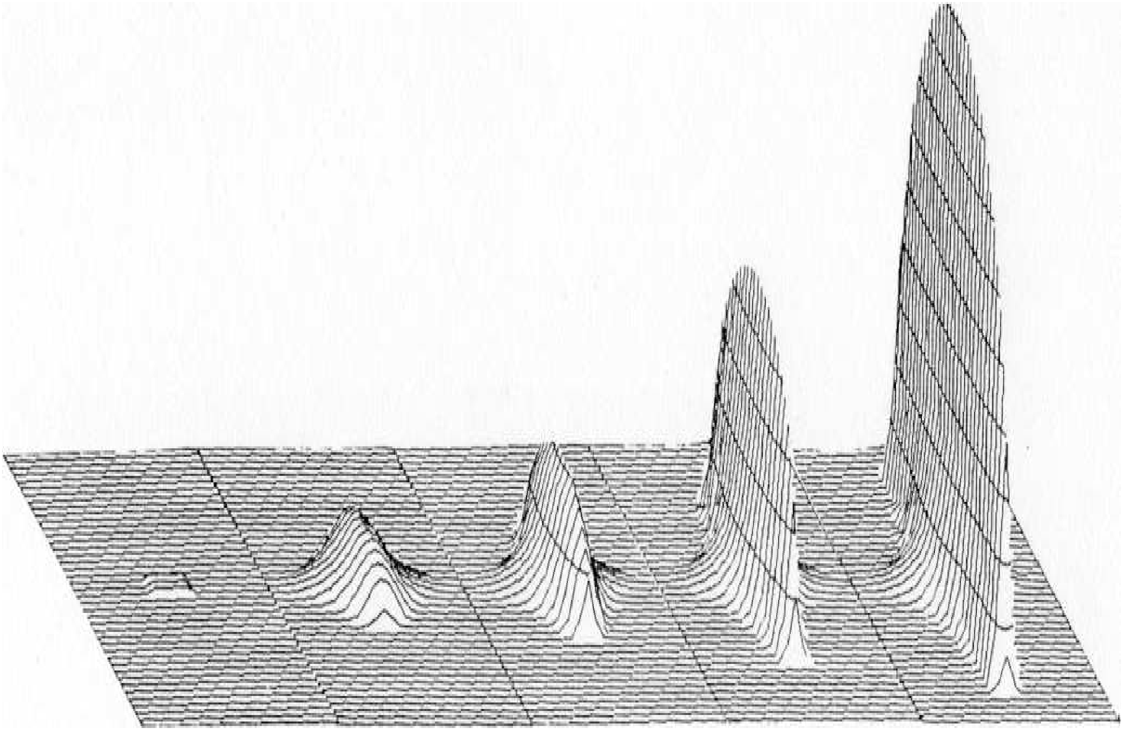

Fig. 3: $\boldsymbol{HS-LS}$ evolution: $\boldsymbol{\sigma_1 = 3,\ \sigma_2 = 1,\ \beta = 3}$.

Bakirova, M.I., Dimova, S.N., Dorodnicyn, V.A., Kurdyumov, S.P., Samarskii, A.A., Svirshchevskii, S.: Invariant solutions of heat-transfer equation, describing directed heat diffusion and spiral waves in nonlinear medium, Soviet Phys. Dokl. **33**(3), 187–189 (1988)

$$u_t = \frac{1}{x^{N-1}}(x^{N-1}u^\sigma u_x)_x + u^\beta,\ x \in \mathbb{R}_+,\ t > 0,$$

$$u_x(t,0) = 0, \quad u(t,\infty) = 0, \quad u^\sigma u_x = 0 \text{ if } u = 0.$$

$$u_s(t,x) = \varphi(t)\theta(\xi) = (1 - t/T_0)^{-\frac{1}{\beta-1}}\theta(\xi),$$

$$\xi = x/\psi(t) = x(1 - t/T_0)^{-\frac{m}{\beta-1}},\quad m = \frac{\beta - \sigma - 1}{2},$$

$$u_s(0,x) = \theta(x).$$

$$-\frac{1}{\xi^{N-1}}(\xi^{N-1}\theta^\sigma\theta')' + \frac{\beta-\sigma-1}{2(\beta-1)T_0}\xi\theta' + \frac{1}{(\beta-1)T_0}\theta - \theta^\beta = 0,$$

$$\theta'(0) = 0,\ \ \theta(\infty) = 0,\ \ \theta^\sigma\theta' = 0, \text{ if } \ \theta = 0.$$

$$\theta(\xi) \equiv 0,\ \theta(\xi) \equiv \theta_H = 1,\ \ T_0 = 1/(\beta - 1).$$

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✓ $1 < \beta \leq \sigma + 1$ ∃ a finite support solution $\theta(\xi) \geq 0$,
$N > 1$ - the solution is unique!!!

Zmitrenko, Kurdyumov, Samarskii, 1976

Samarskii, A.A., Galaktionov, V.A., Kurdyumov, S.P., Mikhailov, A.P.: Blowup in Problems for Quasilinear Parabolic Equations. Nauka, Moscow (1987); Valter de Gruyter&Co., Berlin (1995)

✓ $\beta > \sigma + 1, N \geq 1$ no finite support solutions, $\theta(\xi) > 0$ in $\mathbb{R}_+$

$$\theta(\xi) = C_a \xi^{-2/(\beta-\sigma-1)}[1 + \omega(\xi)], \omega(\xi) \to 0, \ \xi \to \infty$$

Adyutov, Klokov Michailov, 1983; Kurkina (2004)

✓ $N = 1, \ \ \beta > \sigma + 1:$

∃ $K = [a]$: $a \notin \mathbb{N}$; $\quad K = a - 1: \ a \in \mathbb{N}$

different solutions $\theta_i(\xi), \ i = 1, 2, \ldots K$,

$$a = \frac{\beta - 1}{\beta - \sigma - 1} > 1, \qquad a \to \infty, \quad \beta \to \sigma + 1 + 0.$$

✓ $N > 1, \ \ \beta > \sigma + 1:$ not known

Numerical Methods for the self-similar problems

$$L(\theta) \equiv -\frac{1}{\xi^{N-1}}(\xi^{N-1}\theta^{\sigma}\theta')' + m\xi\theta' + \theta - \theta^{\beta} = 0.$$

$$\theta'(0) = 0, \quad \theta(l) = 0, \quad l \gg \xi_0, \quad \beta \leq \sigma + 1,$$

$$\theta'(0) = 0, \quad \theta'(l) + p\frac{\theta(l)}{l} = 0, \; p = \frac{1}{m}, \; \beta > \sigma + 1$$

Continuous analog of the Newton's method (Gavurin, 1958)

$$0 \leq t < \infty, \qquad \theta = \theta(\xi, t):$$

$$L'(\theta)\frac{\partial\theta}{\partial t} = -L(\theta), \qquad \theta(\xi, 0) = \theta_0(\xi)$$

$$\frac{\partial\theta}{\partial t} = v(\xi, t)$$

Dimova, S.N., Kaschiev, M.S., Kurdyumov, S.P.: Numerical analysis of the eigenfunctions of combustion of a nonlinear medium in the radial-symmetric case, USSR Comp. Math. Math. Phys. **29**(6), 61–73 (1989)

Iteration process

$$L'(\theta_k)v_k = -L(\theta_k)$$

$$\theta_{k+1} = \theta_k + \tau_k v_k, \quad 0 < \tau_k \le 1, \quad k = 0, 1, \dots,$$

$$\theta_k = \theta_k(\xi) = \theta(\xi, t_k), \quad v_k = v_k(\xi) = v(\xi, t_k),$$

$\theta_0(\xi)$ – initial approximation,

$$\tau_k = \min\left(1, \tau_{k-1}\frac{\delta_{k-1}}{\delta_k}\right), \text{if } \delta_k < \delta_{k-1}, \quad \delta_k = \max_{\eta\in\bar\omega_h} |B(\theta_k)\bar\Theta_k|,$$

$$\tau_k = \max\left(\tau_0, \tau_{k-1}\frac{\delta_{k-1}}{\delta_k}\right), \text{if } \delta_k \ge \delta_{k-1}, \quad \text{KFKI-74-34, 1974}$$

Galerkin FEM Lagrangian (linear and quadratic) finite elements

$$A(\theta)\bar V = -B(\theta)\bar\Theta$$

$$V = \{v_k(\xi_j)\}_{j=1}^n, \quad A = LU$$

Initial approximations for the iteration process

$$\beta > \sigma + 1 : \quad \theta(\xi) = 1 + \varepsilon y(\xi), \ \mid \varepsilon y(\xi) \mid \ll 1,$$

$$-\frac{1}{\xi^{N-1}}(\xi^{N-1} y')' + \frac{\beta - \sigma - 1}{2}\xi y' + (\beta - 1) y = 0,$$

$$y(\xi) = {}_1F_1(-(\beta - 1)/(\beta - \sigma - 1), \ N/2; \ (\beta - \sigma - 1)\xi^2/4)$$

$$\tilde{\theta}_{0,i}(\xi) = \begin{cases} 1 + \alpha_i y(\xi), & 0 \le \xi < \bar{\xi}_i \\ C_i \xi^{-p}, & \bar{\xi}_i \le \xi \le l. \end{cases},$$

$\tilde{\theta}_{0,i}(\xi) \in C^2[0, l]$ (C^2 – approximations),

$\tilde{\theta}_{0,i}(\xi) \in C^1[0, l]$ (C^1 – approximations), $y(\xi_i) = 0.$

$O(h^2)$ (linear FE), $O(h^4)$ (quadratic FE)

Nonsymmetric GFEM

$$\tilde{L}(\theta) = -(\xi^{\gamma}\theta^{\sigma}\theta')' - \gamma(N-2)\theta^{\sigma}\theta' + m\xi^{1+\gamma}\theta' + \xi^{\gamma}\theta(1-\theta^{\beta-1}) = 0,$$

$$\gamma = 0 \text{ for } N = 1, \qquad \gamma = 1 \text{ for } N > 1.$$

$\beta = \sigma + 1 = 3,\ N = 3,\ \theta_h(\xi)$

	h	$\xi = 0.0$	α	$\xi = 0.8$	α	$\xi = 1.6$	α	$\xi = 2.4$	α	$\xi = 2.8$	α
I sch.	0.40	**1.747684**		**1.605931**		**1.261843**		**0.7624175**		**0.4792797**	
	0.20	**1.726220**	1.68	**1.599599**	1.98	**1.260792**	2.04	**0.7634072**	2.33	**0.4808580**	2.86
	0.10	**1.719496**	1.74	**1.597994**	1.97	**1.260536**	1.91	**0.7636043**	1.91	**0.4810750**	1.65
	0.05	1.717476		1.597584		1.260468		0.7636569		0.4811441	
II sch.	0.40	**1.721136**		**1.602726**		**1.267529**		**0.7725607**		**0.4902303**	
	0.20	**1.717725**	2.10	**1.598714**	2.08	**1.262206**	2.02	**0.7659756**	1.93	**0.4836629**	1.80
	0.10	**1.716929**	1.96	**1.597768**	1.97	**1.260889**	1.98	**0.7642477**	2.01	**0.4817792**	2.04
	0.05	1.716725		1.597527		1.260556		0.7638178		0.4813203	

Eriksson K., V. Thomee. Galerkin methods for singular boundary value problems in one space dimension. *Math. Comp.*, **42**,1984, pp. 345-367.

Eriksson K., Y. Y. Nie. Convergence analysis for a nonsymmetric Galerkin method for a class of singular boundary value problems in one space dimension. *Math. Comp.*, **49**, 1987, pp.167-186.

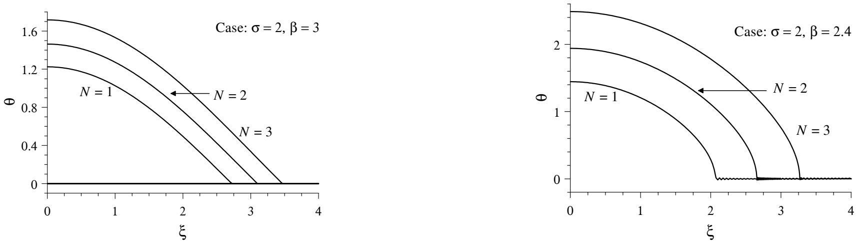


Fig. 4: $\boldsymbol{S}$-evolution, $\boldsymbol{\sigma = 2, \beta = 3;}$ $\boldsymbol{HS}$-evolution, $\boldsymbol{\sigma = 2, \beta = 2.4;}$ $\boldsymbol{N = 1, 2, 3}$

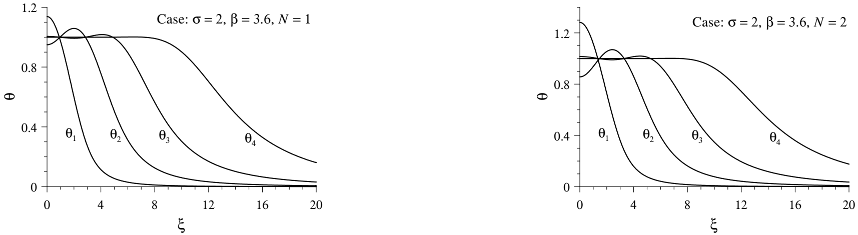


Fig. 5: $\boldsymbol{LS}$-evolution, $\boldsymbol{\sigma = 2, \beta = 3.6;}$ $\boldsymbol{N = 1, 2}$

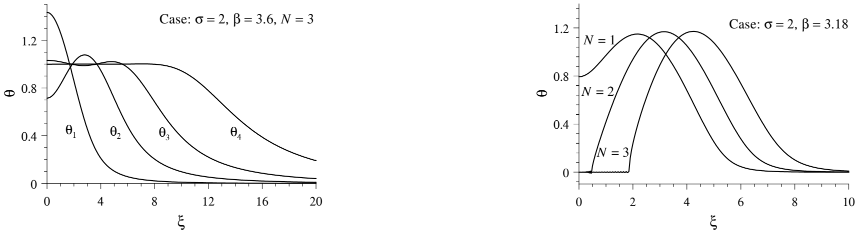
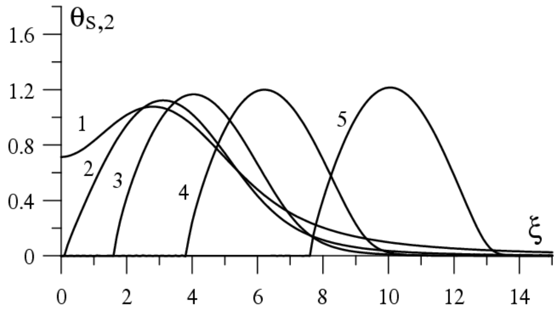
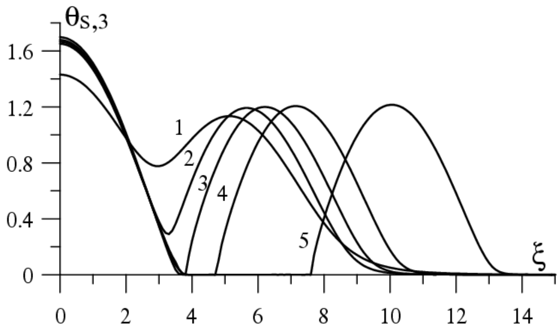


Fig. 6: $\boldsymbol{LS}$-evolution: $\boldsymbol{\sigma = 2, \beta = 3.6; N = 3;}$ $\boldsymbol{\sigma = 2, \beta = 3.18; N = 1, 2, 3}$



$LS \to S$: $N = 3$, $\sigma = 2$, $\beta = \{3.6(1);\ 3.38(2);\ 3.2(3);\ 3.08(4);\ 3.03(5)\}$



$LS \to S$: $N = 3$, $\sigma = 2$, $\beta = \{3.2(1);\ 3.1(2);\ 3.08(3);\ 3.06(4);\ 3.03(5)\}$

Other critical exponents

$\beta_f = \sigma + 1 + 2/N$ (Fudjita's exponent);

$\beta_s = (\sigma + 1)(N + 2)/(N - 2), \quad N \geq 3$ (Sobolev's exponent

$\beta_u = (\sigma + 1)(1 + 4/(N - 4 - 2\sqrt{N - 1})), \quad N \geq 11; \quad \beta_u = \infty, \quad N < 11;$

$\beta_p = 1 + \frac{3(\sigma+1)+(\sigma^2(N-10)^2+2\sigma(5\sigma+1)(N-10)+9(\sigma+1)^2)^{1/2}}{N-10}, \quad N \geq 11.$

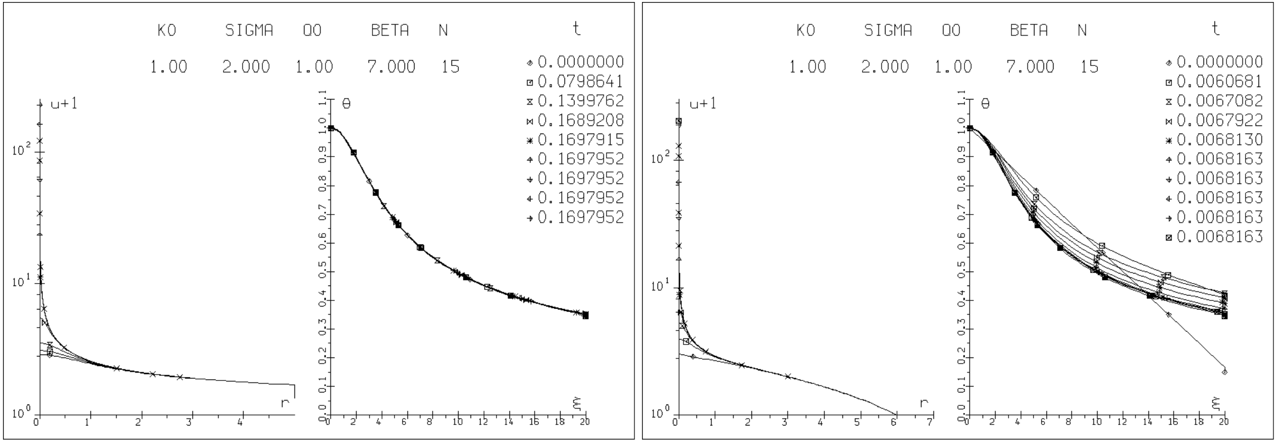


Fig. 8: $\sigma = 2$, $N = 15$, $\beta = 7 > \beta_p$; $T_0 - \tilde{T}_0 = 0.0031$

Dimova, S.N., Chernogorova, T.P.: Nonsymmetric Galerkin finite element method with dynamic mesh refinement for singular nonlinear problems. J. Comp. Appl. Math. (Kiev University) **92**, 3–16 (2005)

Numerical methods for the parabolic problems

$$u_t = \frac{1}{r^{N-1}}(r^{N-1}u^\sigma u_r)_r + u^\beta, \quad 0 < r < R, \quad 0 < t < T_0,$$

$$u_r(t,0) = 0, \quad 0 < t < T_0, \qquad u(t,R) = 0, \quad 0 < t < T_0,$$

$$u(0,r) = u_0(r) \geq 0, \quad 0 \leq r \leq R,$$

$$p(u) = \int_0^u w^\sigma dw = \frac{u^{\sigma+1}}{\sigma+1}, \qquad q(u) = u^\beta, \qquad \text{FEM}$$

$$u_h(t,r) = \sum_{i=1}^n u_i(t)\varphi_i(r), \qquad u_0(r) = \sum_{i=1}^n u_0(r_i)\varphi_i(r),$$

$$p(u) \sim p_I = \sum_{i=1}^n p(u_i)\varphi_i(r), \qquad q(u) \sim q_I = \sum_{i=1}^n q(u_i)\varphi_i(r).$$

$$\dot{\mathbf{u}} = -\tilde{\mathbf{M}}^{-1}\mathbf{K}\mathbf{p}(\mathbf{u}) + \mathbf{q}(\mathbf{u}), \qquad \mathbf{u}(0) = \mathbf{u}_0.$$

Adaptive meshes, based on the invariant properties of the continuous model

$$u_t = Lu, \quad u_a(t,r) = \varphi(t)\theta(\xi), \quad \xi = r/\psi(t),$$

$$\xi = r\Gamma(t)^m, \quad \Delta\xi = \Delta r\Gamma(t)^m, \quad \Gamma(t) = \frac{\max_r u(t,r)}{\max_r u_0(r)}$$

$$m = \frac{\beta - \sigma - 1}{2}.$$

Strategy for adaptation:

$LS-$evolution, $m > 0: \; \Delta\xi^{(k)} = \Delta r^{(k)}\Gamma(t)^m \le \lambda h_0.$

$HS-$evolution, $m < 0: \; h_0/\lambda \le \Delta\xi^{(k)} = \Delta r^{(k)}\Gamma(t)^m.$

Dimova, S.N., Vasileva,D.P.: Lumped-mass finite element method with interpolation of the nonlinear coefficients for a quasilinear heat transfer equation. *Numerical Heat Transfer, Part B* **28**, 199–215 (1995)

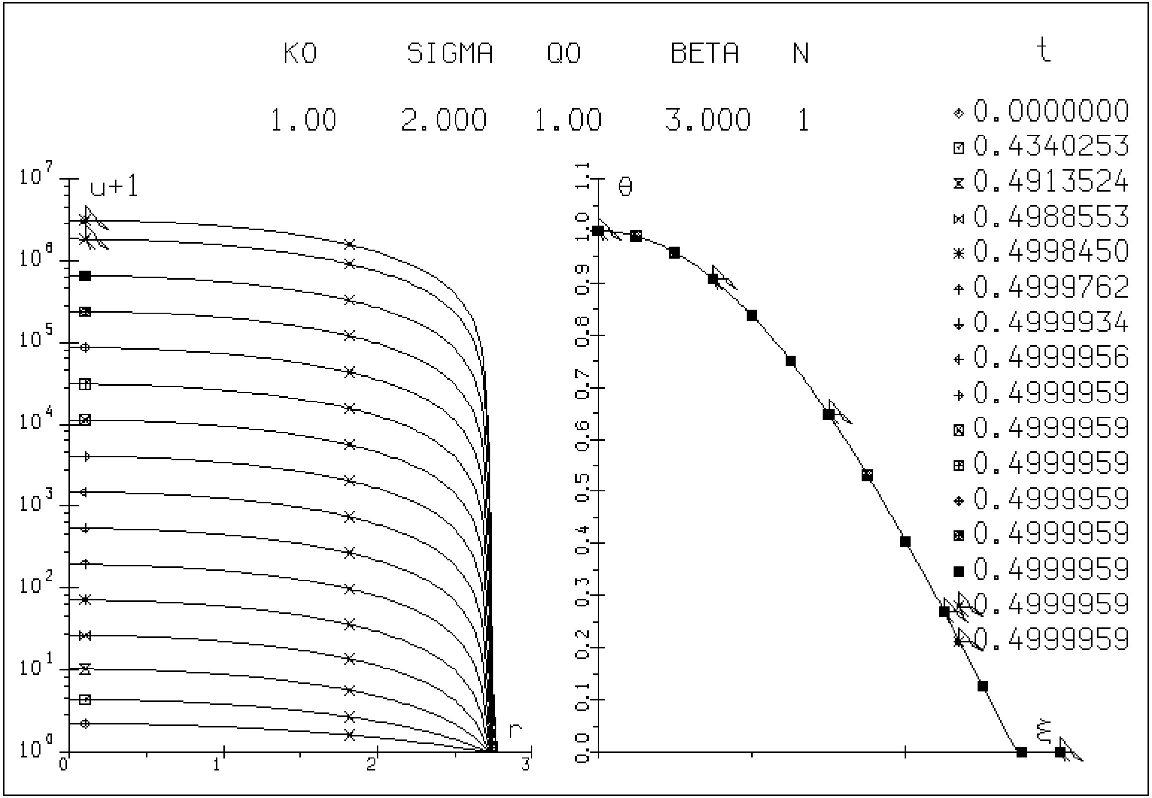


Fig. 9: S-evolution, $u_0(x) = \theta(x)$, $T_0 = 0.5$

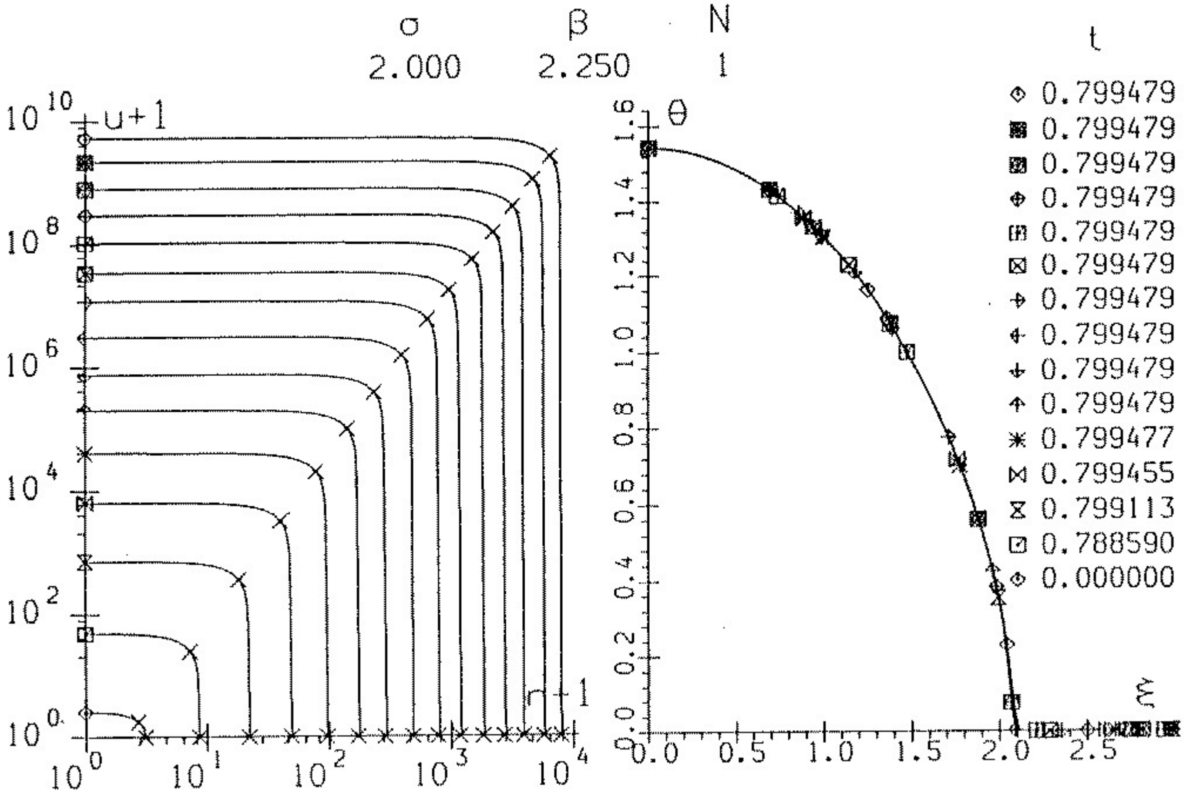


Fig. 10: $\boldsymbol{HS}$-evolution $\boldsymbol{u_0(x) = \theta(x),\ T_0 = 0.8}$

2D Isotropic medium

$$u_t = \frac{1}{r}(r u^{\sigma} u_r)_r + \frac{1}{r^2}(u^{\sigma} u_{\varphi})_{\varphi} + u^{\beta},\ 0 < r < \infty,\ 0 \le \varphi < 2\pi.$$

Свирщевский, 1985: $u_a(t, r, \varphi) = (1 - \frac{t}{T_0})^{-\frac{1}{\beta-1}} \theta_a(\xi, \phi),$

$$\xi = r(1 - \frac{t}{T_0})^{-\frac{m}{\beta-1}},\ \phi = \varphi + \frac{C_0}{\beta - 1} \ln(1 - \frac{t}{T_0}),\ m = \frac{\beta - \sigma - 1}{2}$$

C_0 - parameter of the family of solutions.

$$-\frac{1\partial}{\xi\partial\xi}(\xi\theta^{\sigma}\frac{\partial\theta}{\partial\xi}) - \frac{1}{\xi^2}\frac{\partial}{\partial\phi}(\theta^{\sigma}\frac{\partial\theta}{\partial\phi}) + m\xi\frac{\partial\theta}{\partial\xi} - C_0\frac{\partial\theta}{\partial\phi} + \theta - \theta^{\beta} = 0.$$

$C_0 = 0, LS$-evolution: Kurdyumov, Kurkina, Potapov, Samarskii, 1984, 1986

Koleva, M.G., Dimova, S.N., Kaschiev M.S.: Analisys of the eigen functions of combustion of a nonlinear medium in polar coordinates. Math. Modeling **3**, 76–83 (1992)

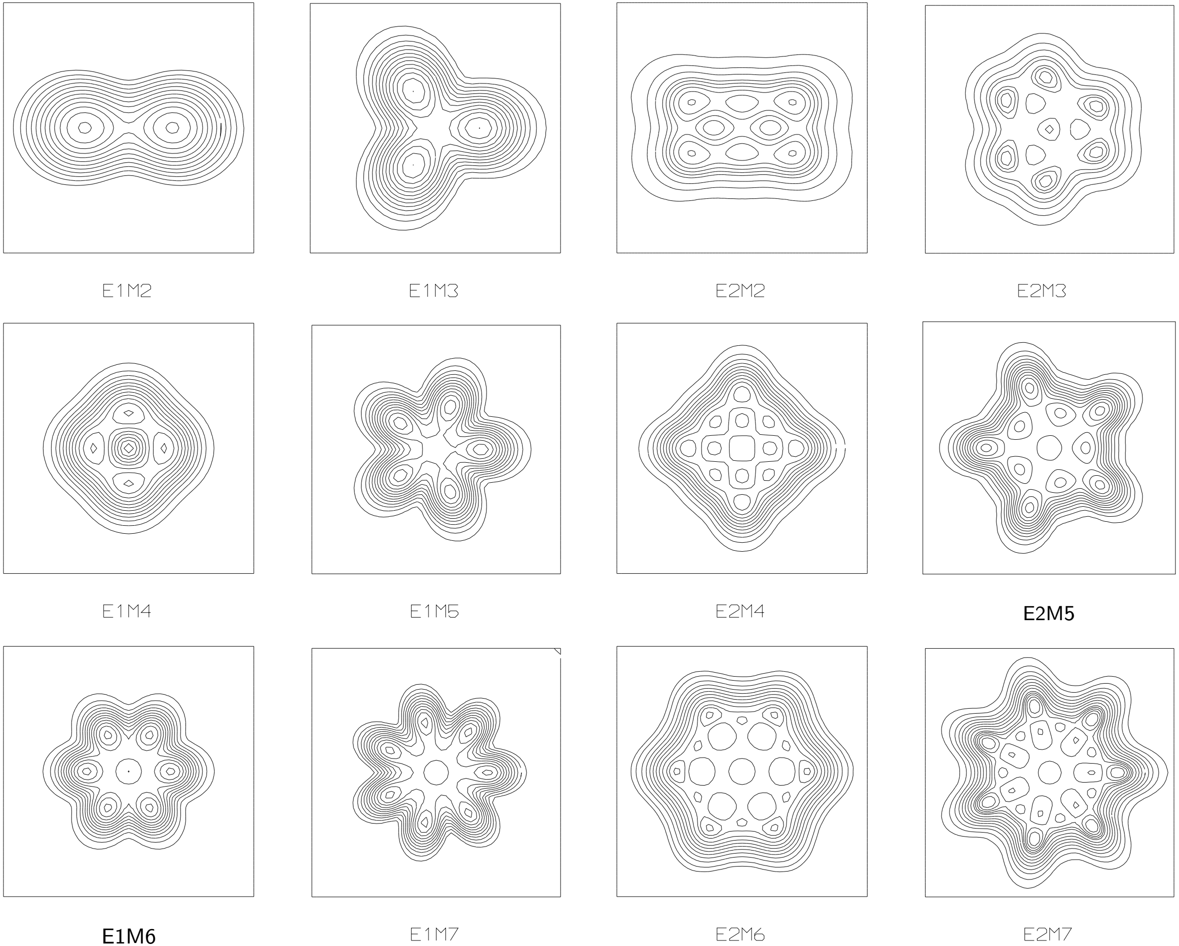


Fig. 12: $\boldsymbol{\sigma = 2,\ \beta = 3.25 : E1Mm,\ E2Mm, m = 2, \ldots, 7}$

2D Isotropic medium, $C_0 \neq 0$

$$\xi e^{s\phi} = r e^{s\varphi} = \text{const}, \quad s = \frac{\beta - \sigma - 1}{2C_0}$$

$$\lim_{\xi \to \infty} \theta(\xi, \phi) = \theta_H^0 \equiv 0 :$$

only radially symmetric functions in $HS-$ and $S-$regimes.

$$\lim_{\xi \to \infty} \theta(\xi, \phi) = \theta_H \equiv 1 :$$

two new classes of solutions in $HS-$regime:
- complex symmetry solutions for $C_0 = 0$,
- spiral wave-solutions for $C_0 \neq 0$.

$$\frac{1}{\xi}(\xi y_\xi)_\xi + \frac{1}{\xi^2} y_{\phi\phi} - \frac{\beta - \sigma - 1}{2} \xi y_\xi + C_0 y_\phi + (\beta - 1) y = 0$$

$$Y_k(\xi, \phi) = R_k(\xi) e^{ik\phi}, \quad k \neq 0, \ k - \text{intiger (for periodicity)}$$

$$R_k'' + \left(\frac{1}{\xi} - \frac{\beta - \sigma - 1}{2}\xi\right) R_k' + \left(-\frac{k^2}{\xi^2} + C_0 k i + \beta - 1\right) R_k = 0$$

$$\beta = \sigma + 1: \; R_k(\xi) = J_k(z), \quad z = (\sigma + C_0 k i)^{1/2}\xi,$$

$$\beta \neq \sigma + 1: \; R_k(\xi) = \xi^k \, {}_1F_1(a, b; z), \quad z = \frac{\beta - \sigma - 1}{4}\xi^2$$

$$a = -\frac{\beta - 1 + C_0 k i}{\beta - \sigma - 1} + \frac{k}{2}, \quad b = 1 + k$$

$\beta < \sigma + 1: \; |R_k(\xi)| \to 0$, when $\xi \to \infty$.

$$y_k(\xi, \varphi) = |R_k(r)| \cos(\arg(R_k(\xi) + k\varphi))$$

Dimova, S.N., Vasileva, D.P.: Numerical realization of blow-up spiral wave solutions of a nonlinear heat-transfer equation. Int. J. Num. Meth. Heat Fluid Flow 4(6), 497–511 (1994)

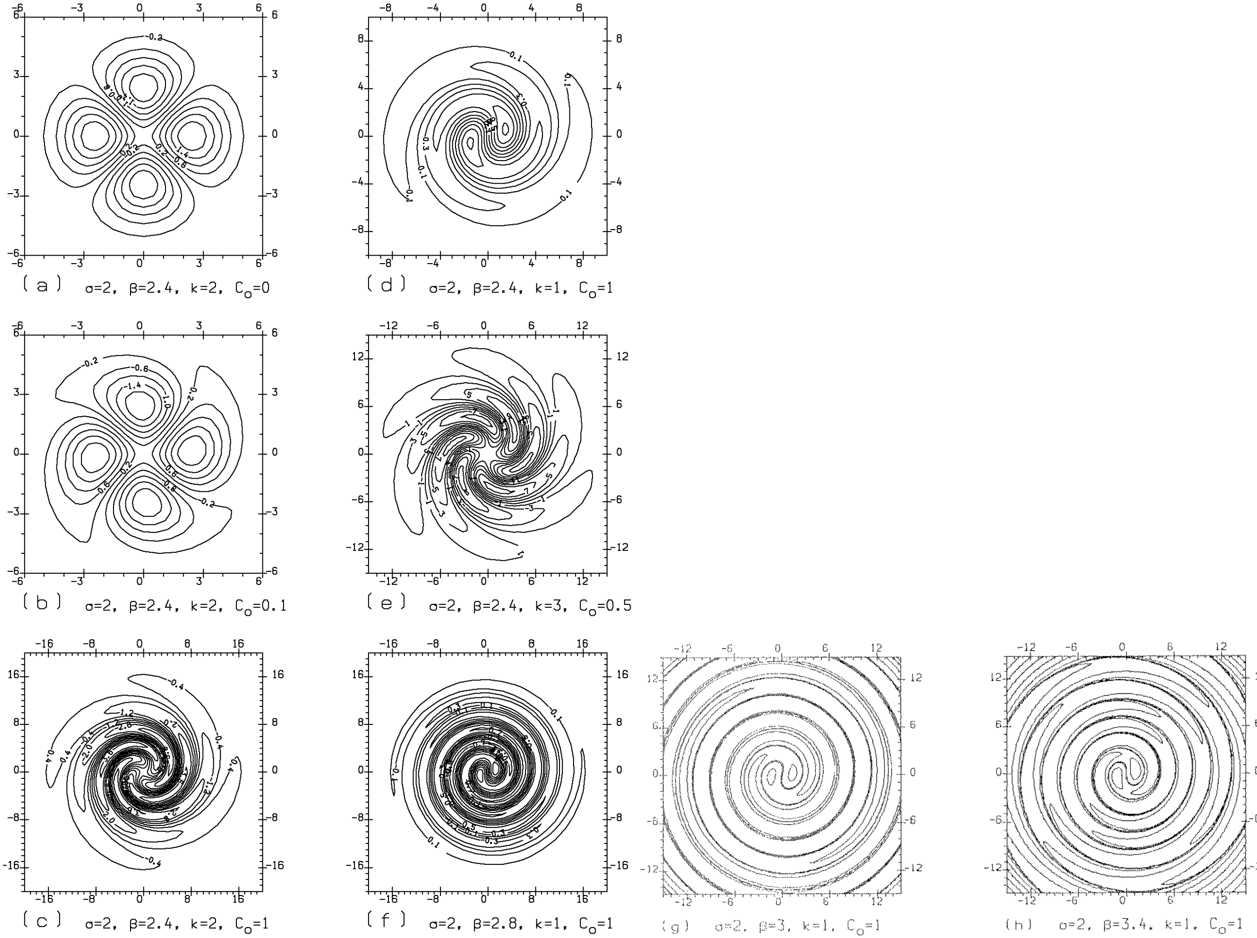


Fig. 13: $y_k(\xi, \varphi)$, HS-regime, S-regime, LS-regime

Asymptotics of the linearized approximations:

$$y_k(\xi,\phi) \sim c\xi^{(\beta-1)/m}\cos k\left(\phi + \frac{1}{s}\ln\xi\right), \quad \xi \to \infty$$

Asymptotics of the s.-s. functions $\theta_k(\xi,\phi)$, $k = 1, 2, \ldots :$

$$\theta_k(\xi,\phi) \sim 1 + \gamma\xi^{(\beta-1)/m}\cos k\left(\phi + \frac{1}{s}\ln\xi\right), \quad \xi \to \infty, \quad \gamma = \alpha c$$

and approximations

$$\tilde{\theta}_k(\xi,\phi) = 1 + \alpha y_k(\xi,\phi), \quad \alpha = \text{const}$$

Third kind boundary conditions at $\xi = l \gg 1$:

$$\frac{\partial\theta_k}{\partial\xi} = \frac{\theta_k - 1}{m^*\xi} - \frac{\gamma k}{s}\xi^{(m^*-1)/m^*}\sin k\left(\phi + \frac{1}{s}\ln\xi\right), \quad m^* = \frac{m}{\beta - 1}.$$

$$\frac{\partial\theta_k}{\partial\xi} = \frac{\theta_k - 1}{m^*\xi}, \quad m^* = \frac{m}{\beta - 1} \quad \text{for } C_0 = 0.$$

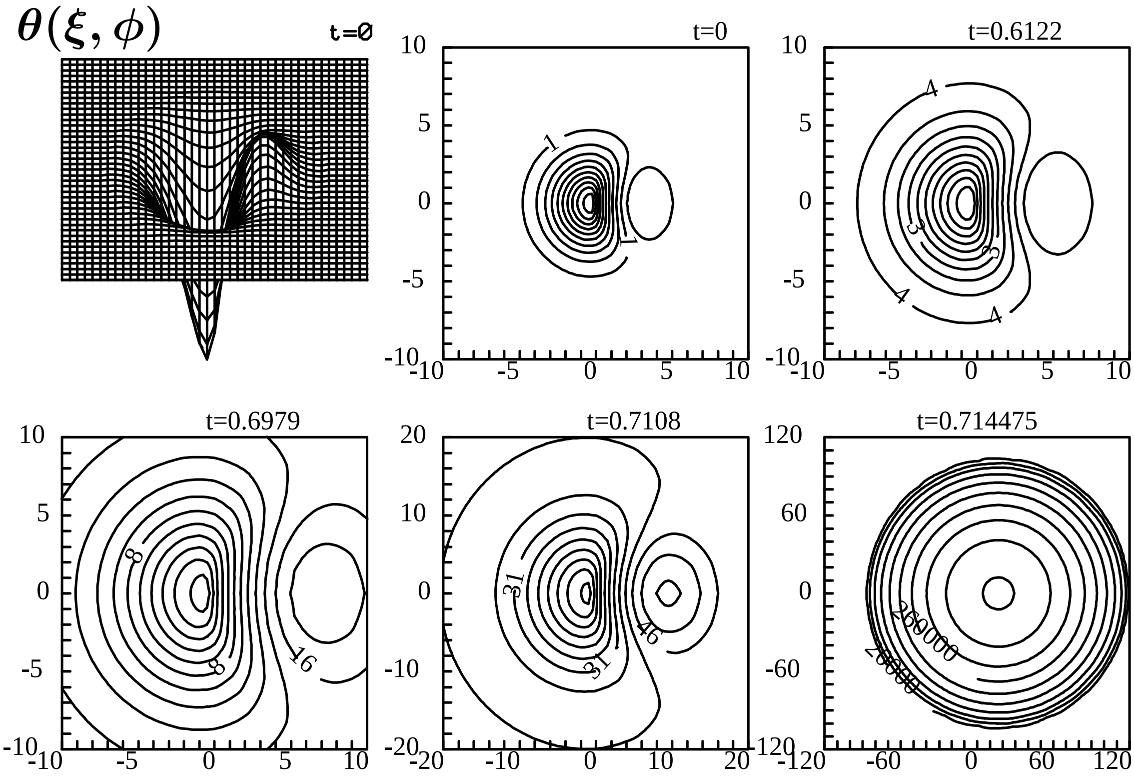


Fig. 14: Evolution of a complex wave in $\boldsymbol{HS}$-regime:
$\boldsymbol{\sigma = 2,\ \beta = 2.4,\ C_0 = 0,\ k = 1,\ T_0 = 0.(714285)}$.

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$\theta(\xi, \phi)$

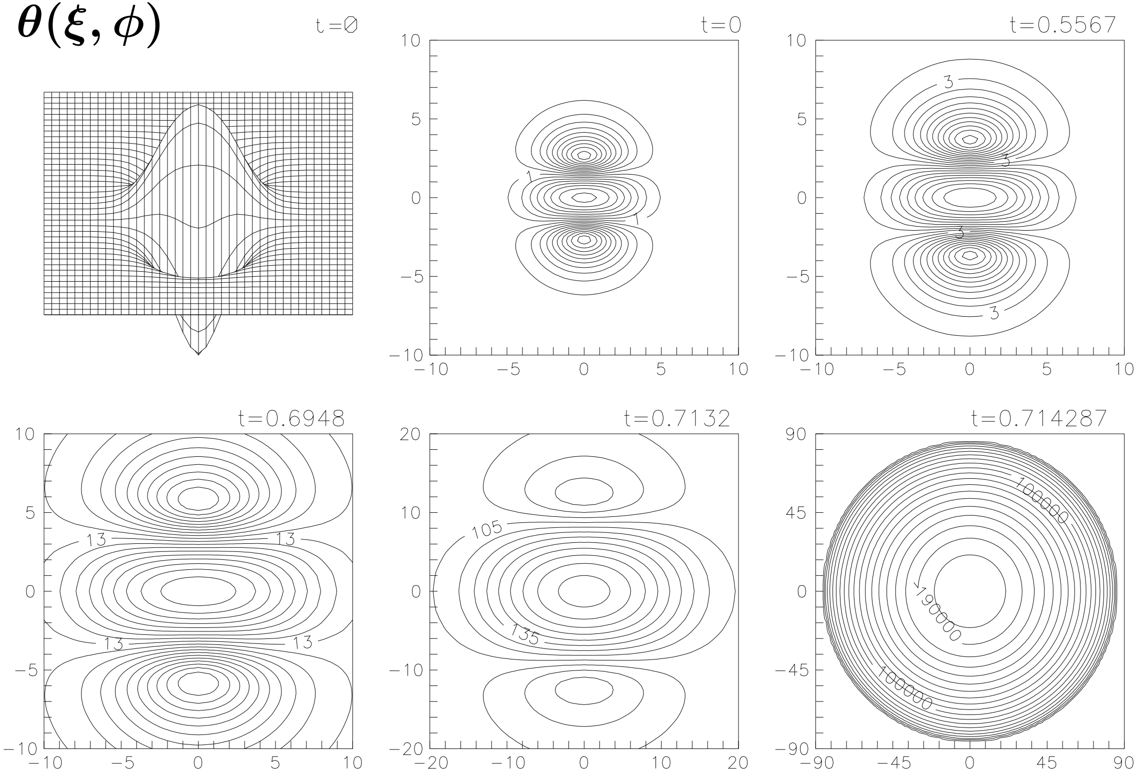

Fig. 15: Evolution of a complex wave in $\boldsymbol{HS}$-regime::
$\boldsymbol{\sigma = 2,\ \beta = 2.4,\ C_0 = 0,\ k = 2,\ T_0 = 0.(714285)}$.

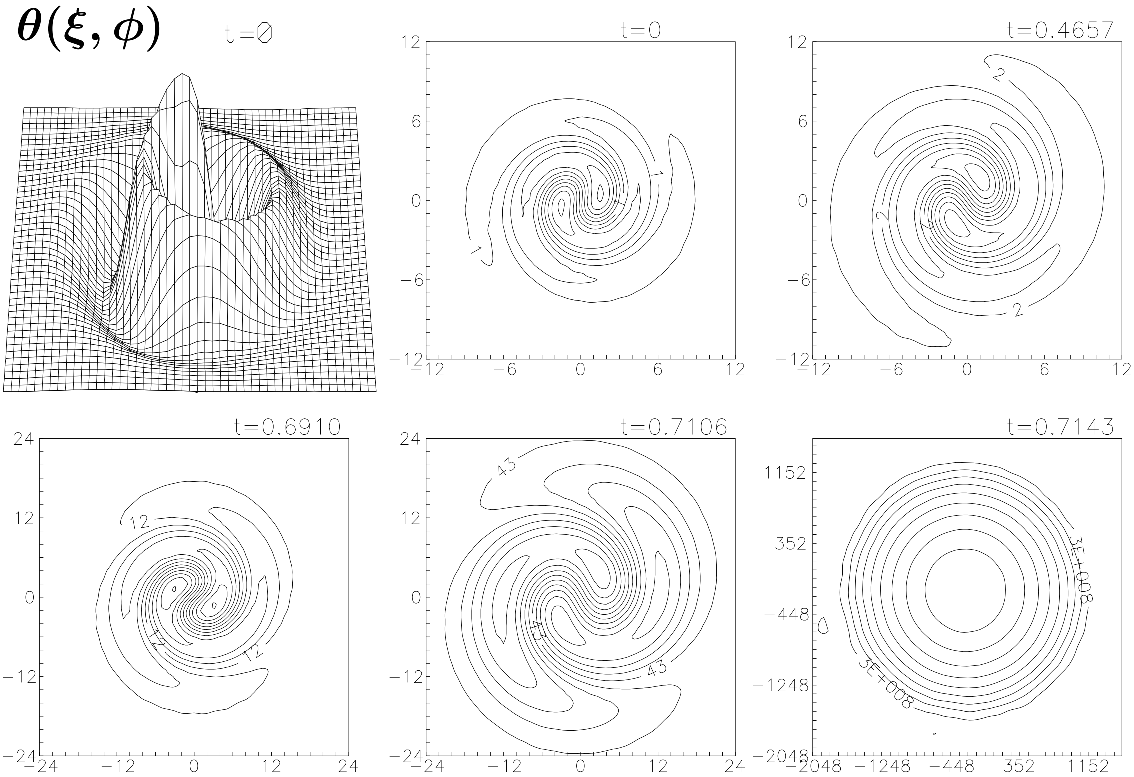

Fig. 16: Evolution of one-armed spiral wave in $\boldsymbol{HS}$-regime: $\boldsymbol{\sigma = 2,\ \beta = 2.4,\ C_0 = 1,\ k = 1,\ T_0 = 0.(714285)}$.

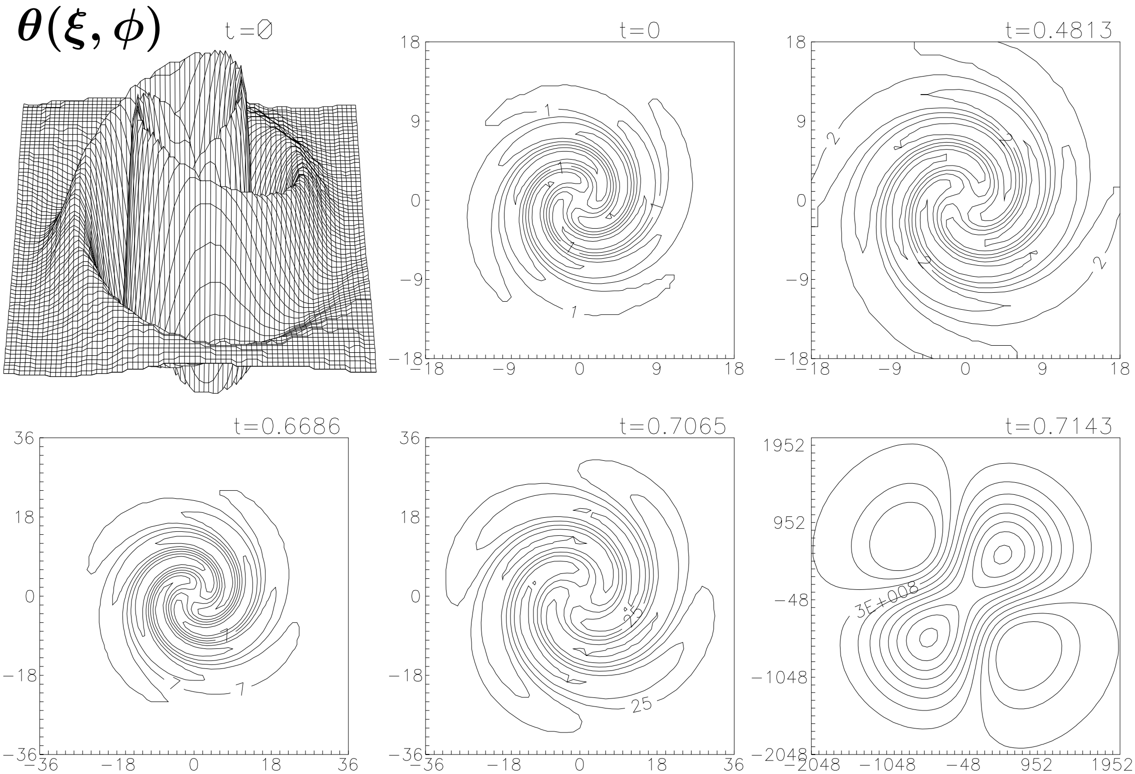

Fig. 17: Evolution of two-armed spiral wave in $\boldsymbol{HS}$-regime:
$\boldsymbol{\sigma = 2,\ \beta = 2.4,\ C_0 = 1,\ k = 2,\ T_0 = 0.(714285)}$.

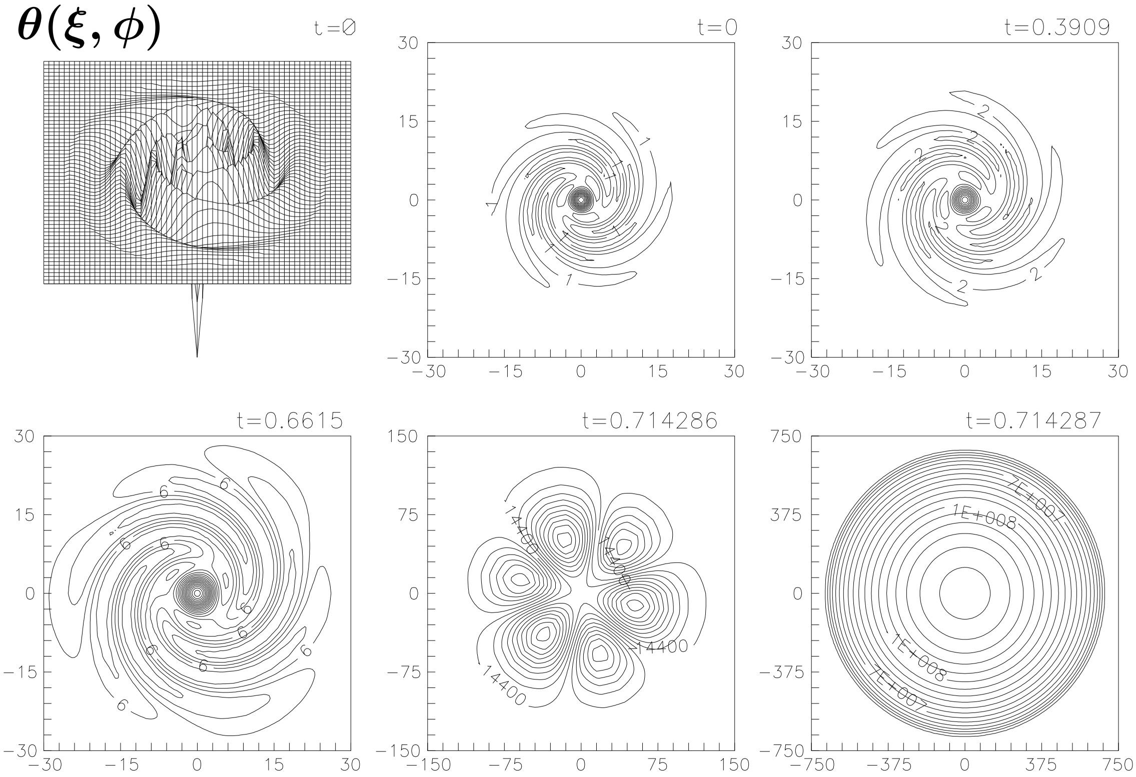

Fig. 18: Evolution of three-armed spiral wave in $\boldsymbol{HS}$-regime::
$\boldsymbol{\sigma = 2,\ \beta = 2.4,\ C_0 = 1,\ k = 3,\ T_0 = 0.(714285)}$.

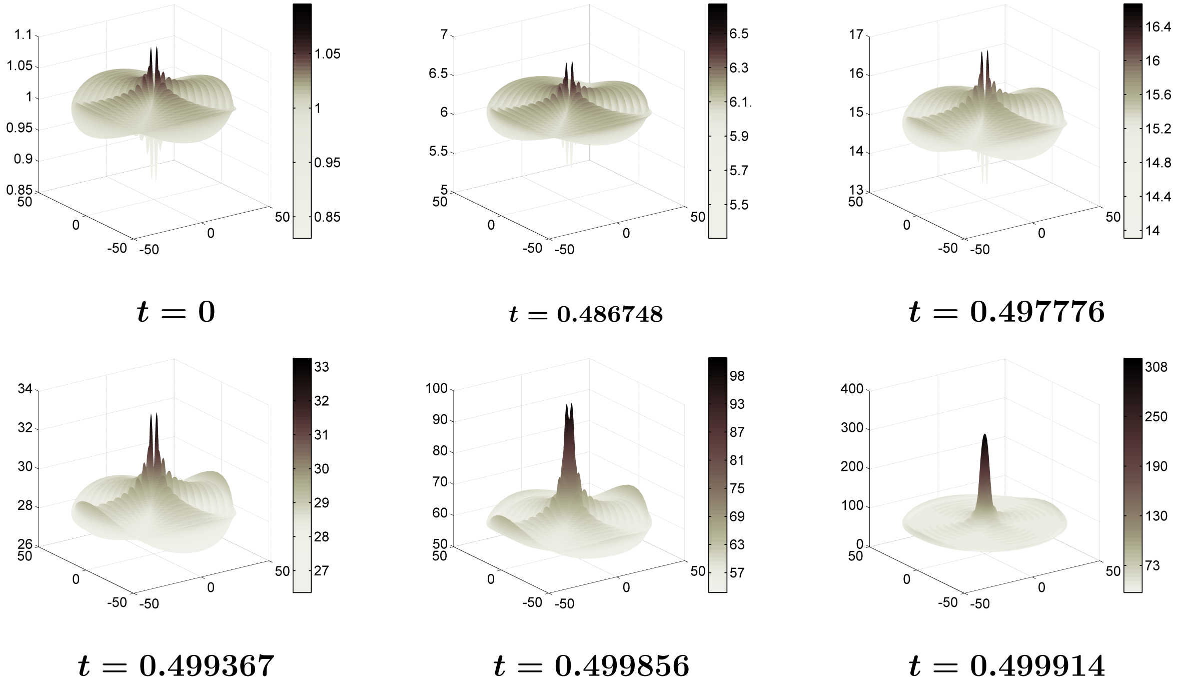

Fig. 19: Evolution of a complex wave in S-regime: $\boldsymbol{\sigma = 2}$, $\boldsymbol{\beta = 3}$, $\boldsymbol{c_0 = 0}$, $\boldsymbol{k = 2}$, $\boldsymbol{T_0 = 0.5}$.

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