# STRUCTURES AND WAVES IN A NONLINEAR HEAT-CONDUCTING MEDIUM

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$$u_t = \sum_{i=1}^N (oldsymbol{k}_i(oldsymbol{u})_{x_i} + oldsymbol{Q}(oldsymbol{u}), \ t > 0, \ x \in \mathbb{R}^N$$

 $k_i(u) \equiv \text{const}, Q(u) = \lambda e^u$  (Frank-Kamenetzki equation),

 $Q(u) = u^{\beta}, \beta > 1$ : H. Fujita (1966), J. Bebernes, A. Bressan, H. Brezis, D. Eberly, A. Friedman, V.A. Galaktionov, I.M. Gelfand, M.A. Herrero, R. Kohn, L.A. Lepin, S.A. Posashkov, A.A. Samarskii, J.L. Vazquez, L.J.L. Velazquez,...

 $k_i(u) \neq \text{const}$ : D.G. Aronson, A. Friedman, I.M. Gelfand, A.S. Kalashnikov, S. Kaplan, O.A. Ladyjenskaya, H.A. Levine, O.A. Oleinik, L.A. Pelitier, N.N. Uralceva...

 $k_i(u)=u^{\sigma_i},\;\sigma_i>0,\quad Q(u)=u^eta,\;eta>1$ 

A.A. Samarskii, M.I. Sobol(1963), S.P. Kurdyumov (1974), M.I. Bakirova,
V.A. Galaktionov, V.A. Dorodnicyn, G.G. Elenin, N.V. Zmitrenko,
E.S. Kurkina, A.P. Mihailov, Y.P. Popov, A.B. Potapov, M.N. LeRoux,
S. Svirshchevskii, H. Wilhelmsson,...

**>>** 

2D: Anisotropic medium. Directed heat diffusion

$$u_t = (u^{\sigma_1} u_{x_1})_{x_1} + (u^{\sigma_2} u_{x_2})_{x_2} + u^{eta}, \ (x_1, x_2) \in \mathbb{R}^2, \ \sigma_i > 0, \ eta > 1$$

**blow-up self-similar solution** (Dorodnicyn, Knyazeva):

$$u_s(t,x_1,x_2) = (1-t/T_0)^{-rac{1}{eta-1}} heta(\xi), \; \xi = (\xi_1,\xi_2) \in \mathbb{R}^2,$$

$$\xi_i = x_i (1 - t/T_0)^{-m_i/(eta - 1)} \quad m_i = (eta - \sigma_i - 1)/2, \; i = 1, 2.$$

$$egin{aligned} L( heta) &\equiv \sum_{i=1}^2 \left( -rac{\partial}{\partial \xi_i} \left( heta^{\sigma_i} rac{\partial heta}{\partial \xi_i} 
ight) + rac{eta - \sigma_i - 1}{2} \xi_i rac{\partial heta}{\partial \xi_i} 
ight) + heta - heta^eta &= 0. \end{aligned}$$
 $HS(1 < eta < \sigma_i + 1); \ S(1 < eta = \sigma_i + 1); \ LS(\sigma_i + 1 < eta)$ 



Fig. 1: S evolution:  $\sigma_1 = 2, \ \sigma_2 = 2, \ \beta = 3.$ 



## Fig. 2: HS - S evolution: $\sigma_1 = 3, \ \sigma_2 = 2, \ \beta = 3$ .



## Fig. 3: HS - LS evolution: $\sigma_1 = 3, \ \sigma_2 = 1, \ \beta = 3.$

Bakirova, M.I., Dimova, S.N., Dorodnicyn, V.A., Kurdyumov, S.P., Samarskii, A.A., Svirshchevskii, S.: Invariant solutions of heat-transfer equation, describing directed heat diffusion and spiral waves in nonlinear medium, Soviet Phys. Dokl. **33**(3), 187–189 (1988)

Image: A state of the state of the

$$egin{aligned} &u_t = rac{1}{x^{N-1}} (x^{N-1} u^\sigma u_x)_x + u^eta, \; x \in \mathbb{R}_+, \; t > 0, \ &u_x(t,0) = 0, \quad u(t,\infty) = 0, \quad u^\sigma u_x = 0 ext{ if } u = 0. \ &u_s(t,x) = arphi(t) heta(\xi) = (1-t/T_0)^{-rac{1}{eta-1}} heta(\xi), \ &\xi = x/\psi(t) = x(1-t/T_0)^{-rac{m}{eta-1}}, \;\; m = rac{eta - \sigma - 1}{2}, \ &u_s(0,x) = heta(x). \end{aligned}$$

$$egin{aligned} &-rac{1}{\xi^{N-1}}(\xi^{N-1} heta^{\sigma} heta')'+rac{eta-\sigma-1}{2(eta-1)T_0}\xi heta'+rac{1}{(eta-1)T_0} heta- heta^{eta}=0,\ & heta(0)=0,\ \ heta(\infty)=0,\ \ heta^{\sigma} heta'=0,\ heta=0.\ & heta(\xi)\equiv 0,\ \ heta(\xi)\equiv heta_H=1,\ \ T_0=1/(eta-1). \end{aligned}$$

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 $\sqrt{1} < \beta \leq \sigma + 1 \exists$  a finite support solution  $\theta(\xi) \geq 0$ , N > 1 - the solution is unique!!!

Zmitrenko, Kurdyumov, Samarskii, 1976

Samarskii, A.A., Galaktionov, V.A., Kurdyumov, S.P., Mikhailov, A.P.: Blowup in Problems for Quasilinear Parabolic Equations. Nauka, Moscow (1987); Valter de Gruyter&Co., Berlin (1995)

 ${\checkmarketa} > {\sigma}+1, N \geq 1$  no finite support solutions,  $heta(\xi) > 0$  in  $\mathbb{R}_+$ 

$$heta(\xi)=C_a\xi^{-2/(eta-\sigma-1)}[1+\omega(\xi)], \omega(\xi) o 0,\; \xi o\infty$$

Adyutov, Klokov Michailov, 1983; Kurkina (2004)

 $\checkmark N=1, \ \ eta > \sigma+1:$ 

 $\exists K = [a]: a \notin \mathbb{N}; \quad K = a - 1: a \in \mathbb{N}$ different solutions  $\theta_i(\xi), \ i = 1, 2, \dots K,$ 

 $a=rac{eta-1}{eta-\sigma-1}>1, \qquad a o\infty, \ \ eta o\sigma+1+0.$ 

 $\checkmark N>1, \;\; eta>\sigma+1:$  not known

Numerical Methods for the self-similar problems

$$egin{aligned} L( heta) &\equiv -rac{1}{\xi^{N-1}}(\xi^{N-1} oldsymbol{ heta}^{\sigma} oldsymbol{ heta}')' + m \xi oldsymbol{ heta}' + oldsymbol{ heta} - oldsymbol{ heta}^{eta} &= 0. \ &oldsymbol{ heta}'(0) &= 0, \ oldsymbol{ heta}(l) &= 0, \ oldsymbol{ heta} &\gg \xi_0, \ oldsymbol{ heta} &\leq \sigma + 1, \ &oldsymbol{ heta}'(0) &= 0, \ oldsymbol{ heta}'(l) + p rac{oldsymbol{ heta}(l)}{l} &= 0, \ p &= rac{1}{m}, \ oldsymbol{eta} > \sigma + 1. \end{aligned}$$

Continuous analog of the Newton's method (Gavurin, 1958)

$$egin{aligned} 0 &\leq t < \infty, & heta &= heta(\xi,t): \ L'( heta) &rac{\partial heta}{\partial t} &= -L( heta), & heta(\xi,0) &= heta_0(\xi) \ & rac{\partial heta}{\partial t} &= v(\xi,t) \end{aligned}$$

Dimova, S.N., Kaschiev, M.S., Kurdyumov, S.P.: Numerical analysis of the eigenfunctions of combustion of a nonlinear medium in the radial-symmetric case, USSR Comp. Math. Math. Phys. **29**(6), 61–73 (1989)

$$L'( heta_k)v_k = -L( heta_k)$$

$$egin{aligned} & heta_{k+1} = heta_k + au_k v_k, \quad 0 < au_k \leq 1, \quad k = 0, 1, \dots, \ & heta_k = heta_k(m{\xi}) = heta(m{\xi}, t_k), \quad v_k = v_k(m{\xi}) = v(m{\xi}, t_k), \ & heta_0(m{\xi}) - ext{initial approximation}, \ & au_k = \min\left(1, au_{k-1}rac{\delta_{k-1}}{\delta_k}
ight), ext{if } \delta_k < \delta_{k-1}, \quad \delta_k = \max_{\eta \in m{\omega}_h} |B( heta_k)ar{\Theta}_k|, \ & au_k = \max\left( au_0, au_{k-1}rac{\delta_{k-1}}{\delta_k}
ight), ext{if } \delta_k \geq \delta_{k-1}, \quad ext{KFKI-74-34, 1974} \end{aligned}$$

Galerkin FEM Lagrangian (linear and quadratic) finite elements

 $A( heta)ar{V} = -B( heta)ar{\Theta}$ 

$$V=\{v_k(\xi_j)\}_{j=1}^n, \hspace{1em} A=LU$$

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Initial approximations for the iteration process

$$eta > oldsymbol{\sigma} + 1: \quad heta(\xi) = 1 + arepsilon y(\xi), \mid arepsilon y(\xi) \mid \ll 1,$$

$$-rac{1}{\xi^{N-1}}(\xi^{N-1}y')'+rac{eta-\sigma-1}{2}\,\xi y'+(eta-1)y=0,$$

$$y(\xi) =_1 F_1(-(eta-1)/(eta-\sigma-1), \ N/2; \ (eta-\sigma-1)\xi^2/4)$$

$$ilde{ heta}_{0,i}(m{\xi}) = \left\{egin{array}{c} 1+oldsymbol{lpha}_i y(m{\xi}), \ 0\leq m{\xi}$$

$$\begin{split} & ilde{ heta}_{0,i}(\xi)\in C^2[0,l]\ (C^2- ext{approximations}), \ & ilde{ heta}_{0,i}(\xi)\in C^1[0,l]\ (C^1- ext{approximations}),\ y(\xi_i)=0. \ & ext{$O(h^2)$}\ ( ext{linear FE}),\ & ext{$O(h^4)$}\ ( ext{quadratic FE}) \end{split}$$

#### Nonsymmetric GFEM

$\tilde{L}(\theta)$	) =	$-(\xi^{\gamma} heta$	$\sigma \theta'$	$)'-\gamma(I)$	<b>V</b> -	$2)  heta^{\sigma}  heta'$ -	+m	$\xi^{1+\gamma}  heta' +$	$-\boldsymbol{\xi}^{\gamma}$ (	$ heta(1{-} heta^eta)$	$^{-1}) =$
		$\gamma =$	<b>0</b> f	or $oldsymbol{N}=$	<b>: 1</b> ,	$\gamma$ =	= 1	for $N$ $>$	> 1		
				$\beta = \sigma$	+1:	= 3 N =	: 3	$\theta_{k}(\xi)$			
	h	$\xi = 0.0$	$\alpha$	$\frac{\beta}{\xi = 0.8}$	$\frac{1}{\alpha}$	$\frac{\xi}{\xi} = 1.6$	$\frac{\alpha}{\alpha}$	$\frac{\xi = 2.4}{\xi = 2.4}$	$\alpha$	$\xi=2.8$	$\alpha$
I sch.	$0.40 \\ 0.20 \\ 0.10 \\ 0.05$	1.747684 1.726220 1.719496 1.717476	1.68 1.74	1.605931 1.599599 1.597994 1.597584	1.98 1.97	1.261843 1.260792 1.260536 1.260468	2.04 1.91	0.7624175 0.7634072 0.7636043 0.7636569	2.33 1.91	0.4792797 0.4808580 0.4810750 0.4811441	> 2.86 1.65
II sch	$0.40 \\ 0.20 \\ \cdot 0.10 \\ 0.05$	$\left.\begin{array}{c} 1.721136 \\ 1.717725 \\ 1.716929 \\ 1.716725 \end{array}\right\}$	$2.10 \\ 1.96$	$\begin{array}{c} 1.602726 \\ 1.598714 \\ 1.597768 \\ 1.597527 \end{array}$	2.08 1.97	$\begin{array}{c} 1.267529 \\ 1.262206 \\ 1.260889 \\ 1.260556 \end{array}$	$2.02 \\ 1.98$	0.7725607 0.7659756 0.7642477 0.7638178	1.93 2.01	0.4902303 0.4836629 0.4817792 0.4813203	> 1.80 2.04

Eriksson K., V. Thomee. Galerkin methods for singular boundary value problems in one space dimension. *Math. Comp.*, **42**,1984, pp. 345-367.

Eriksson K., Y. Y. Nie. Convergence analysis for a nonsymmetric Galerkin method for a class of singular boundary value problems in one space dimension. *Math. Comp.*, **49**, 1987, pp.167-186.

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Fig. 4: S-evolution,  $\sigma = 2, \beta = 3;$  HS-evolution,  $\sigma = 2, \beta = 2.4;$  N = 1, 2, 3

1.6 1.2

Φ<sub>0.8</sub>

0.4

0





8

٤

12 16

0

0

4



Fig. 6: LS-evolution:  $\sigma = 2, \beta = 3.6; N = 3; \sigma = 2, \beta = 3.18; N = 1, 2, 3$ 



 $LS \rightarrow S: N = 3, \ \sigma = 2, \ \beta = \{3.6(1); \ 3.38(2); \ 3.2(3); \ 3.08(4); \ 3.03(5)\}$ 



 $LS \rightarrow S: N = 3, \ \sigma = 2, \ \beta = \{3.2(1); \ 3.1(2); \ 3.08(3); \ 3.06(4); \ 3.03(5)\}$ 

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## Other critical exponents



Fig. 8:  $\sigma = 2$ , N = 15,  $\beta = 7 > \beta_p$ ;  $T_0 - \tilde{T}_0 = 0.0031$ 

Dimova, S.N., Chernogorova, T.P.: Nonsymmetric Galerkin finite element method with dynamic mesh refinement for singular nonlinear problems. J. Comp. Appl. Math. (Kiev University) **92**, 3–16 (2005)

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Numerical methods for the parabolic problems

$$egin{aligned} & u_t = rac{1}{r^{N-1}} (r^{N-1} oldsymbol{u}^\sigma u_r)_r + oldsymbol{u}^eta, & 0 < r < R, & 0 < t < T_0, \ & u_r(t,0) = 0, & 0 < t < T_0, & u(t,R) = 0, & 0 < t < T_0, \ & u(0,r) = u_0(r) \ge 0, & 0 \le r \le R, \end{aligned}$$

$$p(u) = \int_0^u w^\sigma dw = rac{u^{\sigma+1}}{\sigma+1}, \qquad q(u) = u^eta, \qquad ext{FEM}$$

$$u_h(t,r)=\sum_{i=1}^n u_i(t)arphi_i(r), \qquad u_0(r)=\sum_{i=1}^n u_0(r_i)arphi_i(r),$$

m

$$p(u) \sim p_I = \sum_{i=1}^n p(u_i) arphi_i(r), \qquad q(u) \sim q_I = \sum_{i=1}^n q(u_i) arphi_i(r).$$

 $\dot{\mathbf{u}} = -\tilde{\mathbf{M}}^{-1}\mathbf{K}\mathbf{p}(\mathbf{u}) + \mathbf{q}(\mathbf{u}), \qquad \mathbf{u}(0) = \mathbf{u}_0.$ 

Adaptive meshes, based on the invariant properties of the continuous model

$$u_t=Lu, \hspace{1em} u_a(t,r)=arphi(t) heta(\xi), \hspace{1em} \xi=r/\psi(t),$$

$$egin{aligned} \xi &= r \Gamma(t)^m, \quad \Delta \xi &= \Delta r \Gamma(t)^m, \quad \Gamma(t) = rac{\max_r u(t,r)}{\max_r u_0(r)} \ &m = rac{eta - \sigma - 1}{2}. \end{aligned}$$

#### Strategy for adaptation:

LS-evolution, m > 0:  $\Delta \xi^{(k)} = \Delta r^{(k)} \Gamma(t)^m \leq \lambda h_0$ . HS-evolution, m < 0:  $h_0 / \lambda \leq \Delta \xi^{(k)} = \Delta r^{(k)} \Gamma(t)^m$ .

Dimova, S.N., Vasileva, D.P.: Lumped-mass finite element method with interpolation of the nonlinear coefficients for a quasilinear heat transfer equation. *Numerical Heat Transfer, Part B* **28**, 199–215 (1995)



Fig. 9: S-evolution,  $u_0(x) = heta(x), \ T_0 = 0.5$ 

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### 2D Isotropic medium

$$u_t = rac{1}{r}(ru^{\sigma}u_r)_r + rac{1}{r^2}(u^{\sigma}u_arphi)_arphi + u^{eta}, \ 0 < r < \infty, \ 0 \leq arphi < 2\pi.$$

Свирщевский, 1985: 
$$u_a(t,r,arphi) = (1-rac{\iota}{T_0})^{-rac{1}{eta-1}} heta_a(\xi,\phi),$$

$$\xi = r(1-rac{t}{T_0})^{-rac{m}{eta-1}}, \ \phi = arphi + rac{C_0}{eta-1} \ln(1-rac{t}{T_0}), \ m = rac{eta-\sigma-1}{2}$$

 $oldsymbol{C}_0$  - parameter of the family of solutions.

$$-rac{1\partial}{\xi\partial\xi}(\xi heta^{\sigma}rac{\partial heta}{\partial\xi})-rac{1}{\xi^2}rac{\partial}{\partial\phi}( heta^{\sigma}rac{\partial heta}{\partial\phi})+m\xirac{\partial heta}{\partial\xi}-C_0rac{\partial heta}{\partial\phi}+ heta- heta^{eta}=0.$$

C<sub>0</sub> = 0, LS-evolution: Kurdyumov, Kurkina, Potapov, Samarskii, 1984, 1986
 Koleva, M.G., Dimova, S.N., Kaschiev M.S.: Analisys of the eigen functions of combustion of a nonlinear medium in polar coordinates. Math. Modeling 3, 76–83 (1992)

 E1M3
 E2M2
 E2M3

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E1M2



E1M4



E2M5







E2M4



Fig. 12:  $\sigma = 2, \ \beta = 3.25: E1Mm, \ E2Mm, \ m = 2, \ldots, 7$ 

2D Isotropic medium,  $C_0 \neq 0$ 

$$\xi e^{s\phi}=re^{sarphi}= ext{const}, \ \ s=rac{eta-\sigma-1}{2C_0}$$

$$\lim_{\xi o\infty} heta(\xi,\phi)= heta_{H}^{0}\equiv 0:$$

only radially symmetric functions in HS- and S- regimes.

$$\lim_{\xi \to \infty} \theta(\xi, \phi) = \theta_H \equiv 1:$$

two new classes of solutions in HS-regime:

- complex symmetry solutions for  $C_0=0,$
- spiral wave-solutions for  $C_0 \neq 0$ .

$$rac{1}{\xi}(\xi y_{\xi})_{\xi}+rac{1}{\xi^2}y_{\phi\phi}-rac{eta-\sigma-1}{2}\xi y_{\xi}+m{C_0}m{y_{\phi}}+(eta-1)y=0$$

 $Y_k(\xi,\phi)=R_k(\xi)e^{ik\phi}, \ k
eq 0, \ k-$ intiger (for periodicity)

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$$R_k'' + \left(rac{1}{\xi} - rac{eta - \sigma - 1}{2}\xi
ight)R_k' + \left(-rac{k^2}{\xi^2} + oldsymbol{C}_0ki + eta - 1
ight)R_k = 0$$

$$eta = oldsymbol{\sigma} + 1: \; R_k(\xi) = J_k(z), \; \; z = (\sigma + C_0 k i)^{1/2} \xi,$$

$$eta 
eq \sigma + 1: \; R_k(\xi) = \xi^k \; _{_1}F_1(a,b;z), \; \; z = rac{eta - \sigma - 1}{4} \xi^2$$

$$a=-rac{eta-1+m{C_0ki}}{eta-\sigma-1}+rac{k}{2}, \;\;b=1+k$$
 $meta<\sigma+1:\;|R_k(\xi)| o 0,$  when  $\xi o\infty.$ 

$$y_k(\xi,arphi) = |R_k(r)| \cos(rg(R_k(\xi) + karphi))$$

Dimova, S.N., Vasileva, D.P.: Numerical realization of blow-up spiral wave solutions of a nonlinear heat-transfer equation. Int. J. Num. Meth. Heat Fluid Flow 4(6), 497–511 (1994)



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Asymptotics of the linearized approximations:

$$y_k(\xi,\phi)\sim c\xi^{(eta-1)/m}\cos k\left(\phi+rac{1}{s}\ln\xi
ight), \;\; \xi
ightarrow\infty$$

Asymptotics of the s.-s. functions  $heta_k(\xi,\phi)$ ,  $k=1,2,\ldots$  :

$$heta_k(\xi,\phi) \sim 1 + \gamma \xi^{(eta-1)/m} \cos k \left( \phi + rac{1}{s} \ln \xi 
ight), \,\, \xi o \infty, \,\, \gamma = lpha c$$

and approximations

$$heta_k(\xi,\phi)=1+lpha y_k(\xi,\phi), \ lpha= ext{const}$$
  
Third kind boundary conditions at  $m{\xi}=m{l}\gg 1$ :

$$egin{aligned} rac{\partial heta_k}{\partial \xi} &= rac{ heta_k - 1}{m^* \xi} - rac{\gamma k}{s} \xi^{(m^* - 1)/m^*} \sin k \left( \phi + rac{1}{s} \ln \xi 
ight), \ m^* &= rac{m}{eta - 1}, \ rac{\partial heta_k}{\partial \xi} &= rac{ heta_k - 1}{m^* \xi}, \ m^* &= rac{m}{eta - 1} \ ext{for } oldsymbol{C}_0 &= oldsymbol{0}. \end{aligned}$$



Fig. 14: Evolution of a complex wave in HS-regime:  $\sigma = 2, \ \beta = 2.4, \ C_0 = 0, \ k = 1, \ T_0 = 0.(714285).$  26/31



Fig. 15: Evolution of a complex wave in HS-regime::  $\sigma = 2, \ \beta = 2.4, \ C_0 = 0, \ k = 2, \ T_0 = 0.(714285).$  27/31



Fig. 16: Evolution of one-armed spiral wave in *HS*-regime:  $\sigma = 2, \ \beta = 2.4, \ C_0 = 1, \ k = 1, \ T_0 = 0.(714285).$  28/31



Fig. 17: Evolution of two-armed spiral wave in *HS*-regime:  $\sigma = 2, \ \beta = 2.4, \ C_0 = 1, \ k = 2, \ T_0 = 0.(714285).$  29/31



Fig. 18: Evolution of three-armed spiral wave in HS-regime::  $\sigma = 2, \ \beta = 2.4, \ C_0 = 1, \ k = 3, \ T_0 = 0.(714285).$ 

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Fig. 19: Evolution of a complex wave in S-regime:  $\sigma = 2$ ,  $\beta = 3$ ,  $c_0 = 0$ , k = 2,  $T_0 = 0.5$ .

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