Comparison of Two Numerical Approaches to Boussinesq Paradigm Equation

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Dedicated to the memory of Professor Christo I. Christov, who initiated our collaborative research for Boussinesq Paradigm Equation.

Motivation

- Boussinesq equation is the first model for surface waves in shallow fluid layer that accounts for both nonlinearity and dispersion. The balance between the steepening effect of the nonlinearity and the flattening effect of the dispersion maintains the shape of the waves;

  J. V. Boussinesq, Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond, Journal de Mathématiques Pures et Appliquées 17 (1872) 55–108.

- In the 60s it was discovered that these permanent waves can behave in many instances as particles and they were called *solitons* by Zabusky and Kruskal;

  N. J. Zabusky, M. D. Kruskal, Interaction of 'solitons' in collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. 15 (1965) 240–243.

- A plethora of deep mathematical results have been obtained for solitons in the 1D case, but it is of crucial importance to investigate also the 2D case, because of the different phenomenology and the practical importance;

- The accurate derivation of the Boussinesq system combined with an approximation, that reduces the full model to a single equation, leads to the Boussinesq Paradigm Equation (BPE)

$$u_{tt} = \Delta \left[u - F(u) + \beta_1 u_{tt} - \beta_2 \Delta u \right],$$

$F(u) := \alpha u^2$ or $F(u) := \alpha(u^3 - \sigma u^5)$,

u is the surface elevation, $\beta_1 > 0,\ \beta_2 > 0$ - dispersion coefficients,

$\alpha > 0$ - amplitude parameter, $\beta_2 = \alpha = 1$ without loosing of generality.

C. I. Christov, An energy-consistent dispersive shallow-water model, Wave Motion 34 (2001) 161–174.

- 2D BPE admits stationary translating soliton solutions, which can be constructed using either finite differences, perturbation technique, or Galerkin spectral method;

  J. Choudhury, C.I. Christov, 2D solitary waves of Boussinesq equation. APS Conference Proceedings 755 (2005), 85–90.

  C. I. Christov, Numerical implementation of the asymptotic boundary conditions for steadily propagating 2D solitons of Boussinesq type equations, Math. Comp. Simulat. 82 (2012) , 1079–1092.

  C. I. Christov, J. Choudhury, Perturbation solution for the 2D shallow-water waves, Mech. Res. Commun. 38 (2011) 274–281.

  C.I. Christov, M.T. Todorov, M.A. Christou, Perturbation solution for the 2D shallow-water waves. AIP Conference Proceedings 1404 (2011), 49–56.

  M.A. Christou, C.I. Christov, Fourier-Galerkin method for 2D solitons of Boussinesq equation, Math. Comput. Simul. 74 (2007) 82–92.

- It is of utmost importance to investigate the properties of these solutions when they are allowed to evolve in time and to answer the question about their structural stability, i.e., what is their behaviour when used as initial conditions for time-dependent computations of the Boussinesq equation;

- To obtain reliable knowlegde about the time evolution of the stationary soliton solutions, it is imperative to develop different techniques for solving the unsteady 2D BPE;

- Some preliminary results for quadratic nonlinearity in

  A. Chertock, C. I. Christov, A. Kurganov, Central-upwind schemes for the Boussinesq paradigm equation. Computational Science and High Performance Computing IV, NNFM, 113, 267–281 (2011).

  C.I. Christov, N. Kolkovska, D. Vasileva, On the Numerical Simulation of Unsteady Solutions for the 2D Boussinesq Paradigm Lecture Notes Computer Science 6046 (2011), 386–394.

  C.I. Christov, N. Kolkovska, D. Vasileva, Numerical Investigation of Unsteady Solutions for the 2D Boussinesq Paradigm Equation, 5th Annual Meeting of the Bulgarian Section of SIAM, BGSIAM'10 Proceedings (2011), 11–16.

Numerical approaches for solving BPE

Approach A1.

$$\begin{aligned} v(x,y,t) &:= u - \beta_1 \Delta u, \\ v_{tt} &= \frac{\beta_2}{\beta_1}\Delta v + \frac{\beta_1 - \beta_2}{\beta_1^2}(u - v) - \Delta F(u). \end{aligned}$$

The following implicit time stepping is designed

$$\begin{aligned} \frac{v_{ij}^{n+1} - 2v_{ij}^{n} + v_{ij}^{n-1}}{\tau^2} &= \frac{\beta_2}{2\beta_1}\Lambda\left[v_{ij}^{n+1} + v_{ij}^{n-1}\right] - \Lambda G(u_{ij}^{n+1}, u_{ij}^{n}, u_{ij}^{n-1}) \\ &+ \frac{\beta_1 - \beta_2}{2\beta_1^2}[u_{ij}^{n+1} - v_{ij}^{n+1} + u_{ij}^{n-1} - v_{ij}^{n-1}], \\ u_{ij}^{n+1} - \beta_1 \Lambda u_{ij}^{n+1} &= v_{ij}^{n+1}, \quad i = 1, \ldots, N_x, \;\; j = 1, \ldots, N_y, \end{aligned}$$

τ is the time increment, $\Lambda = \Lambda^{xx} + \Lambda^{yy}$ is the difference approximation of the Laplace operator Δ on a uniform or non-uniform grid,

$G(u_{ij}^{n+1}, u_{ij}^{n}, u_{ij}^{n-1})$ is an approximation to the nonlinear term $F(u)$:

$$G(u_{ij}^{n+1}, u_{ij}^{n}, u_{ij}^{n-1}) = \frac{2[g((u_{ij}^{n+1} + u_{ij}^{n})/2) - g((u_{ij}^{n} + u_{ij}^{n-1})/2)]}{u_{ij}^{n+1} - u_{ij}^{n-1}},$$

where $g(u) = \int_0^u F(s)\,ds$. The nonlinear term G is linearized using Picard method, i.e., we perform successive iterations for u and v on the $(n+1)$-st stage, starting with initial condition from the already computed n-th stage.

The unconditional stability of the scheme, the convergence and the conservation of the energy are shown in

N. Kolkovska, Two Families of Finite Difference Schemes for Multidimensional Boussinesq Equation. AIP Conference Series, 1301 (2010), 395–403.

N. Kolkovska, Convergence of Finite Difference Schemes for a Multidimensional Boussinesq Equation, LNCS 6046 (2011), 469–476.

N. Kolkovska, M. Dimova, A new conservative finite difference scheme for Boussinesq paradigm equation, Cent. Eur. J. Math. 10(3) (2012) 1159–1171.

Uniform and non-uniform grids on the computational domain $\Omega_{\mathrm{h}} = [-L_1, L_1] \times [-L_2, L_2]$ are used

$$x_i = -L_1 + ih_x,\ i = 0, \ldots, N_x + 1, \quad h_x = 2L_1/(N_x + 1),$$
$$y_j = -L_2 + jh_y,\ j = 0 \ldots, N_y + 1, \quad h_y = 2L_2/(N_y + 1),$$

$$x_i = \sinh[\hat{h}_x(i - n_x)],\ x_{N_x+1-i} = -x_i,$$
$$i = n_x + 1, \ldots, N_x + 1,\ x_{n_x} = 0,$$
$$y_j = \sinh[\hat{h}_y(j - n_y)],\ y_{N_y+1-j} = -y_j,$$
$$j = n_y + 1, \ldots, N_y + 1,\ y_{n_y} = 0,$$

where N_x, N_y are odd numbers, $n_x = (N_x + 1)/2$, $n_y = (N_y + 1)/2$ and $\hat{h}_x = 2D_x/(N_x + 1)$, $\hat{h}_y = 2D_y(/N_y + 1)$. The numbers D_x and D_y are selected in a manner to have $L_1 = \sinh(D_x)$ and $L_2 = \sinh(D_y)$.

The boundary conditions can be set equal to zero, because of the localization of the wave profile. The second set of b.c.'s used here are the asymptotic boundary conditions

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \approx -2u, \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} \approx -2v, \qquad \sqrt{x^2+y^2} \gg 1.$$

The first of these asymptotic boundary conditions is approximated as

$$\begin{aligned}
u^{n+1}_{i,N_y+1} &= u^{n+1}_{i,N_y-1} + \frac{h^y_{N_y} + h^y_{N_y-1}}{y_{N_y}}\Big[-2u^{n+1}_{i,N_y} - \frac{x_i}{h^x_i + h^x_{i-1}}(u^{n+1}_{i+1,N_y} - u^{n+1}_{i-1,N_y})\Big], \\
u^{n+1}_{N_x+1,j} &= u^{n+1}_{N_x-1,j} + \frac{h^x_{N_x} + h^x_{N_x-1}}{x_{N_x}}\Big[-2u^{n+1}_{N_x,j} - \frac{y_j}{h^y_j + h^y_{j-1}}(u^{n+1}_{N_x,j+1} - u^{n+1}_{N_x,j-1})\Big],
\end{aligned}$$

$i = 0, \ldots, N_x$, $j = 0, \ldots, N_y$. The approximation of the second asymptotic boundary condition is done in the same way.

The first initial condition

$$u(x,y,0) = u_0(x,y), \quad v(x,y,0) = u_0(x,y) - \beta_1 \Delta u_0(x,y)$$

is approximated as

$$u_{ij}^0 = u_0(x_i,y_j), \quad v_{ij}^0 = u_{ij}^0 - \beta_1 \Delta u_0(x_i,y_j).$$

For the second unitial condition

$$u_t(x,y,0) = u_1(x,y), \quad v_t(x,y,0) = u_1(x,y) - \beta_1 \Delta u_1(x,y)$$

the approximations

$$(u_{ij}^1 - u_{ij}^{-1})/(2\tau) = u_1(x_i,y_j), \quad (v_{ij}^1 - v_{ij}^{-1})/(2\tau) = u_1(x_i,y_j) - \beta_1 \Delta u_1(x_i,y_j)$$

are used and the corresponding finite difference equation is modified for $t = \tau$.

The coupled system of equations is solved by the Bi-Conjugate Gradient Stabilized Method with ILU preconditioner.

Approach A2.

$$B\left(\frac{u_{ij}^{n+1}-2u_{ij}^{n}+u_{ij}^{n-1}}{\tau^2}\right)=\Lambda u_{ij}^{n}-\beta_2\Lambda^2 u_{ij}^{n}+\Lambda G(u_{ij}^{n+1},u_{ij}^{n},u_{ij}^{n-1}),$$

$$B=I-(\beta_1+\theta\tau^2)\Lambda+\theta\tau^2\beta_2\Lambda^2.$$

Here I is the identity operator, $\Lambda^2=(\Lambda^{xxxx}+2\Lambda^{xxyy}+\Lambda^{yyyy})$ is the discrete biLaplacian. In approximations to Λu and $\Lambda^2 u$ θ-weighted approximation to u_{ij}^{n} is used: $u_{ij}^{\theta,n}=\theta u_{ij}^{n+1}+(1-2\theta)u_{ij}^{n}+\theta u_{ij}^{n-1}$.

An $O(|h|^2+\tau^2)$ approximation to the second initial condition is given by

$$u_{i,j}^{1}=u_0(x_i,y_j)+\tau u_1(x_i,y_j)+\frac{\tau^2}{2(I-\beta_1\Lambda)}\big(\Lambda u_0-\beta_2\Lambda^2 u_0+\Lambda F(u_0)\big)(x_i,y_j).$$

The boundary conditions $u_{ij}^{n+1}=0,\quad \Lambda u_{ij}^{n+1}=0$ for $i=0,N_x+1$ or $j=0,N_y+1$ are used here.

The operator B is replaced by the the factorized operator $\tilde{B} = B_1 B_2 B_3$, where

$$B_1 = (I - \theta\tau^2\Lambda^{xx} + \theta\tau^2\beta_2\Lambda^{xxxx}), B_2 = (I - \theta\tau^2\Lambda^{yy} + \theta\tau^2\beta_2\Lambda^{yyyy}), B_3 = (I - \beta_1\Lambda).$$

The factorized scheme

$$B_1 B_2 B_3 \left(\frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{\tau^2} \right) = \Lambda u_{ij}^n - \beta_2 \Lambda^2 u_{ij}^n + \Lambda G(u_{ij}^{n+1}, u_{ij}^n, u_{ij}^{n-1})$$

is obtained by the regularization method, is second order convergent in space and time, preserves the discrete energy and is unconditionally stable for $\theta \geq 1/2$:

N. Kolkovska, M. Dimova, A new conservative finite difference scheme for Boussinesq paradigm equation, Cent. Eur. J. Math. 10(3) (2012) 1159–1171.

The theoretical results are confirmed numerically in the 1D case in

M. Dimova, N. Kolkovska, Comparison of some finite difference schemes for Boussinesq paradigm equation, LNCS 7125 (2012), 215–220.

The factorized scheme can be split to a sequence of three simpler schemes. Since the scheme is nonlinear we apply Picard method for the linearization.

Step 1: Solve the problem for the unknown $w_{ij}^{(1)}$:

$$\begin{array}{ll} B_1 w_{ij}^{(1)} = \Lambda u_{ij}^n - \beta_2 \Lambda^2 u_{ij}^n + \alpha \Lambda G(u_{ij}^{n+1}, u_{ij}^n, u_{ij}^{n-1}), & i \neq 0, N_x+1, \\ w_{ij}^{(1)} = 0, \qquad \Lambda^{xx} w_{ij}^{(1)} = 0, & i = 0, N_x+1. \end{array}$$

Step 2: Define the unknown $w_{ij}^{(2)}$ as a solution of the following problem:

$$\begin{array}{ll} B_2 w_{ij}^{(2)} = w_{ij}^{(1)}, & j \neq 0, N_y+1, \\ w_{ij}^{(2)} = 0, \Lambda^{yy} w_{ij}^{(2)} = 0, & j = 0, N_y+1. \end{array}$$

Step 3: Compute $w_{i,j}^{(3)}$ by solving

$$\begin{array}{ll} B_3 w_{ij}^{(3)} = w_{ij}^{(2)}, & i \neq 0, N_x+1, \quad j \neq 0, N_y+1, \\ w_{ij}^{(3)} = 0, & i = 0, N_x+1 \ \text{ or } \ j = 0, N_y+1. \end{array}$$

Step 4: Finally, compute the solution:

$$u_{ij}^{n+1} = 2u_{ij}^{n} - u_{ij}^{n-1} + \tau^2 w_{ij}^{(3)}.$$

The discrete operators B_1 and B_2 are one-dimensional, i.e., the solution of the first problem is reduced to a sequence of 1D problems on the rows of the domain Ω_{h}, while for the second problem we have a sequence of 1D problems on the columns of Ω_{h}. For both problems the resulting systems of linear algebraic equations are five-diagonal with constant matrix coefficients and we apply a nonmonotonic Gaussian elimination with pivoting:

Christov, C.I.: Gaussian elimination with pivoting for multidiagonal systems. Internal Report, University of Reading, 4 (1994)

The third problem is solved by a Conjugate Gradient type Method specially designed for the discrete Laplacian equation:

Samarskii, A.A., Vabishchevich, P.N.: Numerical Methods for Solving Inverse Problems of Mathematical Physics. Walter de Gruyter, 2007.

Numerical experiments

Let $u^s(x,y;c)$ be the best-fit approximation to the stationary translating with velocity c solution of BPE, obtained in

C. I. Christov, J. Choudhury, Perturbation solution for the 2D shallow-water waves, Mech. Res. Commun. 38 (2011) 274–281.

C.I. Christov, M.T. Todorov, M.A. Christou, Perturbation solution for the 2D shallow-water waves. AIP Conference Proceedings 1404 (2011), 49–56.

$$u^s(x,y;c) = f^s(x,y) + c^2 g^s(x,y;\beta_1) + c^2 h^s(x,y;\beta_1)\cos\left[2\arctan(y/x)\right].$$

The parameters β_2 and σ are set to $\beta_2 = 1$ and $\sigma = 3/16$ or $\sigma = 0.95$.

The initial conditions

$$u_0(x,y) := u^s(x,y;c), \qquad u_1(x,y) := -c u^s_y(x,y;c),$$

correspond to a solution moving along the y-axis with the velocity c.

In the numerical experiments $\beta_1 = 3,\ \beta_2 = 1,\ \alpha = 1$;

Two different uniform grids in the domain $x, y \in [-25, 25]^2$ with 500^2, and 1000^2 grid points respectively;

On the coarse grid $\tau = 0.1$, on the fine grid $\tau = 0.05$;

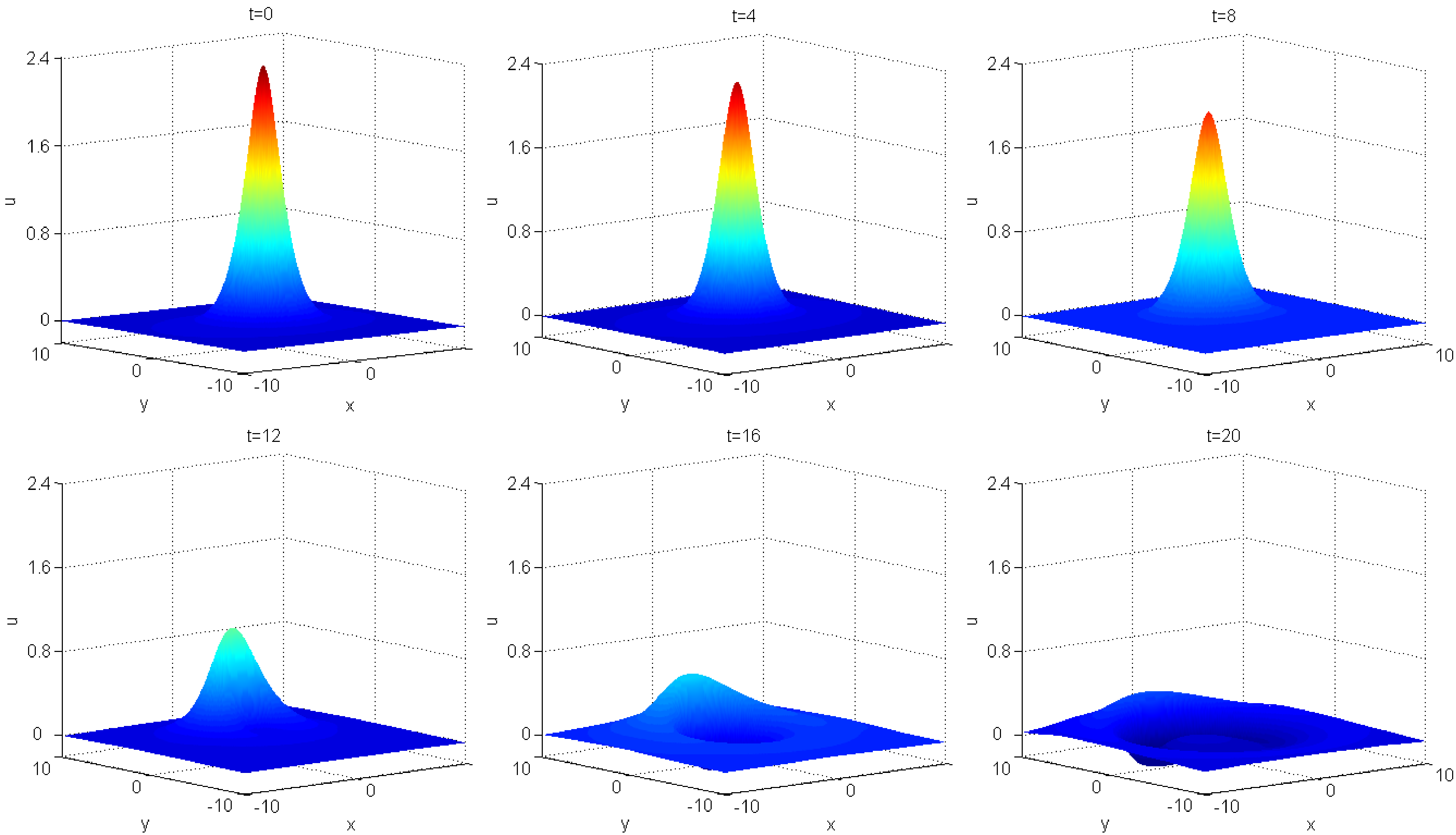
A nonuniform grid in the region $[-250, 250]^2$ with 500^2 grid points and $\tau = 0.1$;

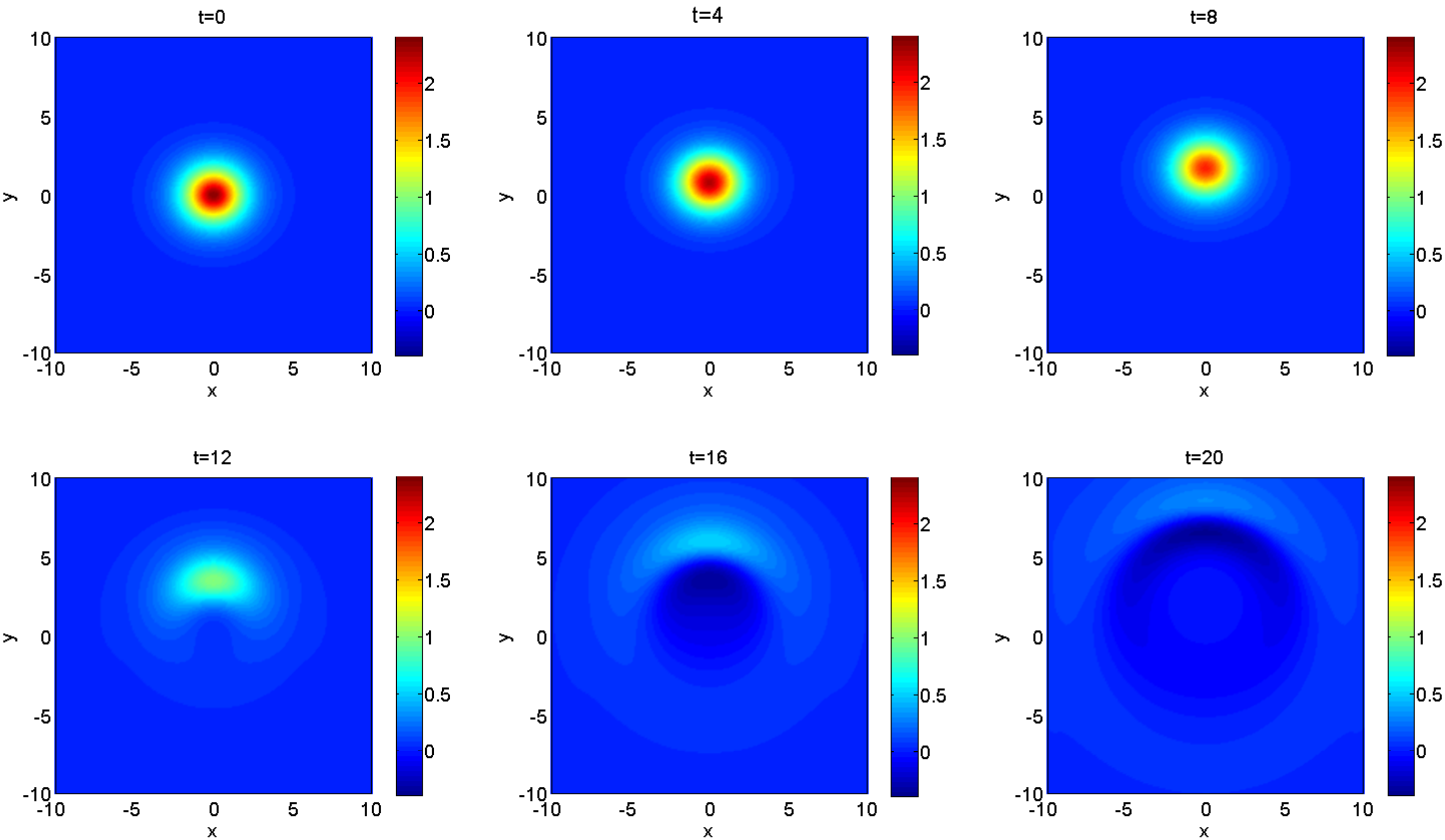
A uniform grid in $[-25, 25]^2$ with 500^2 grid points and $\tau = 0.1$, using the asymptotic boundary conditions.

(i) The case of quadratic nonlinearity, $F(u) = \alpha u^2$

The previous numerical results show that the behaviour of the solution significantly changes when the velocity $c \in [0.2, 0.3]$. That is why we are focusing on these values of c.

Example 1. $c = 0.2$




t=0
t=4
t=8
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y
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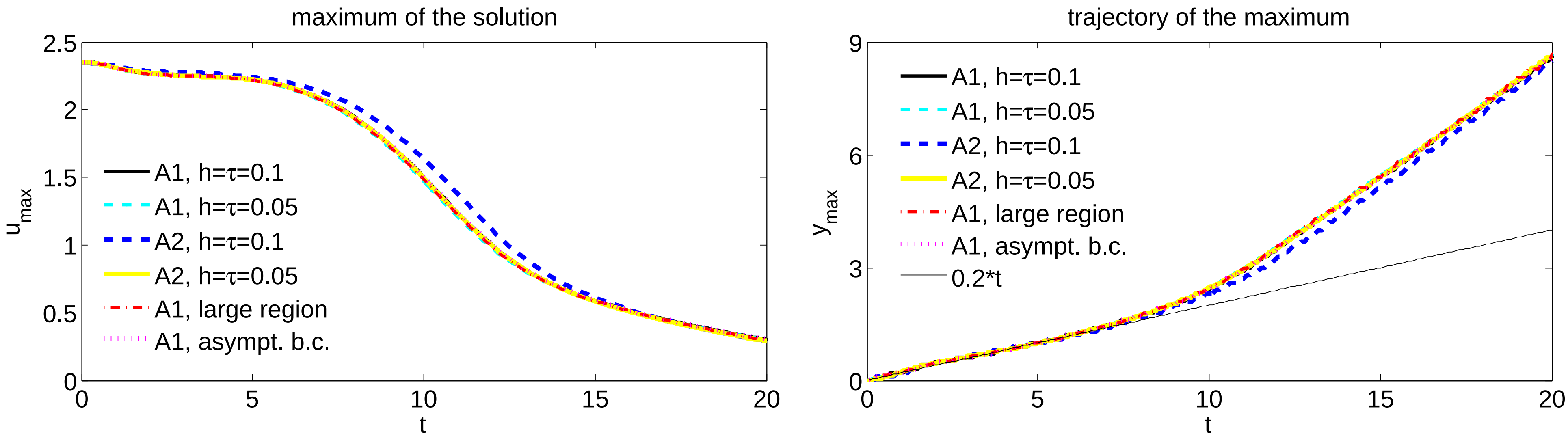
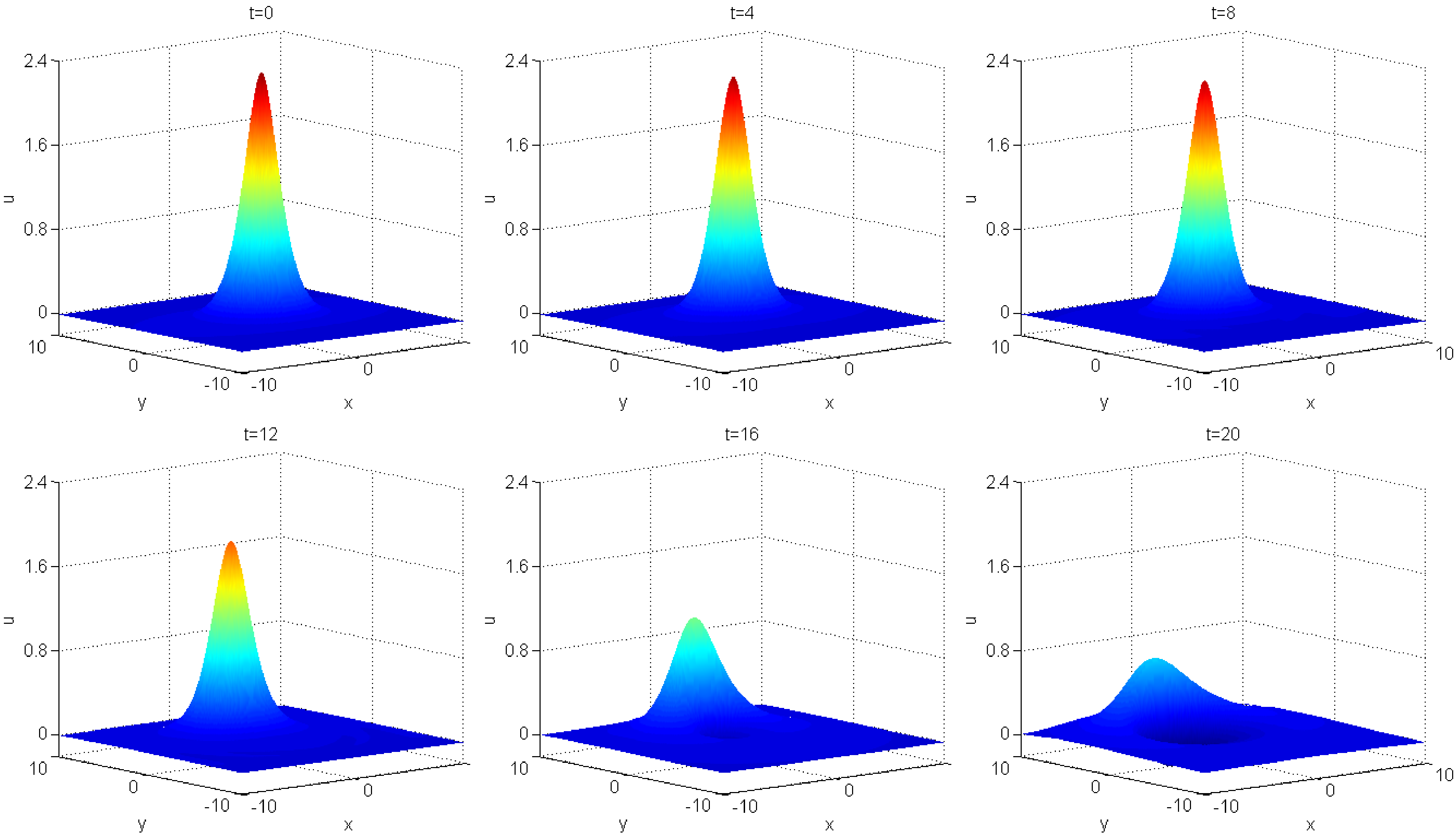
2
1.5
1
0.5
0

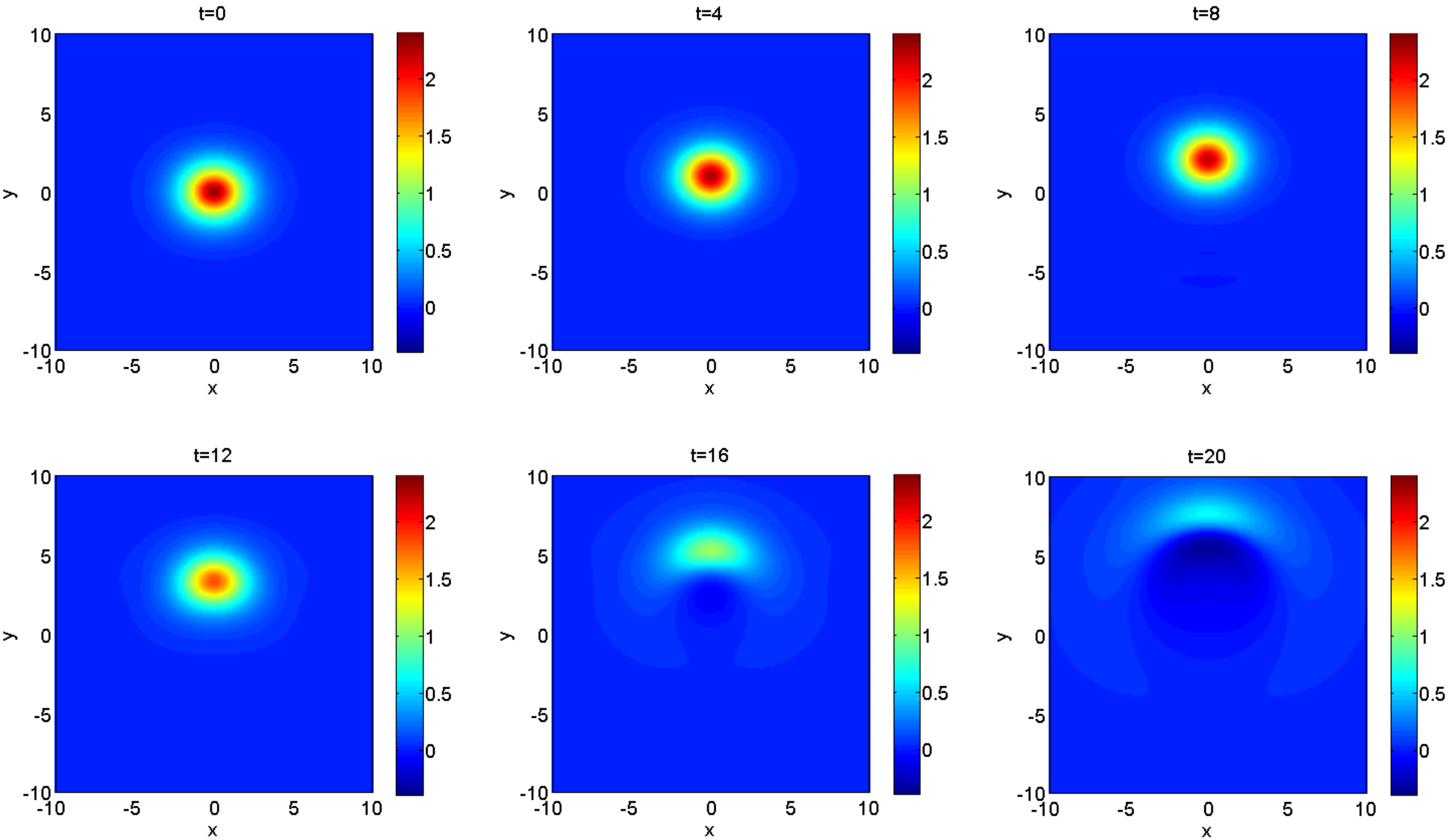


As it is seen, for $t > 8$ the solution cannot keep the form, and transforms into a propagating wave. For $t < 8$, the solution not only moves with a velocity, close to $c = 0.2$, but also behaves like a soliton, i.e., preserves its shape, albeit its maximum decreases slightly. For larger times, the solution transforms into a diverging propagating wave with a front deformed in the direction of propagation.

The behaviour of the solution is the same on all grids and for all times steps, and does not depend on the type of the boundary conditions used (the trivial one or asymptotic). The approach A2 produces slightly different results for the maximum of the solution and its position on the coarse grid, but on the fine grid the results are very close to those obtained by A1.

Example 2. $c = 0.26$

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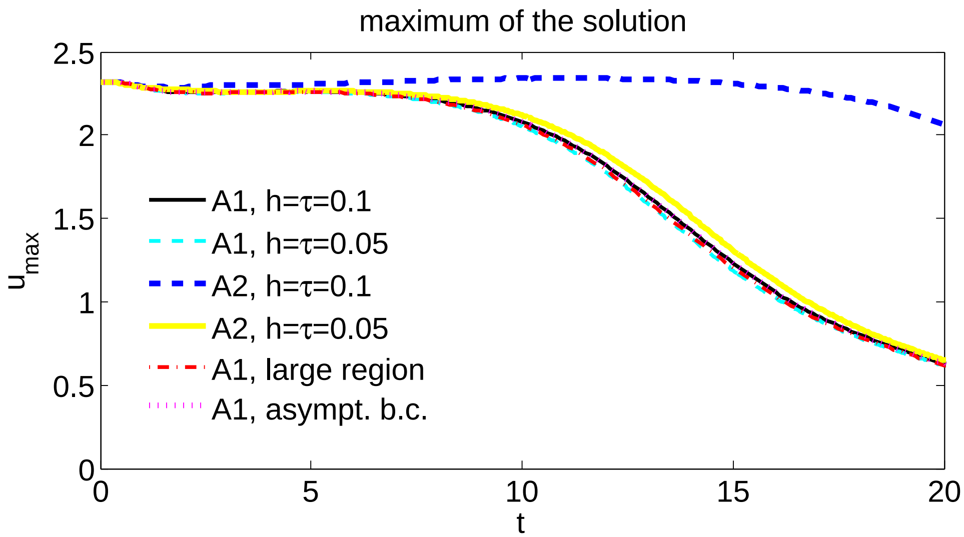
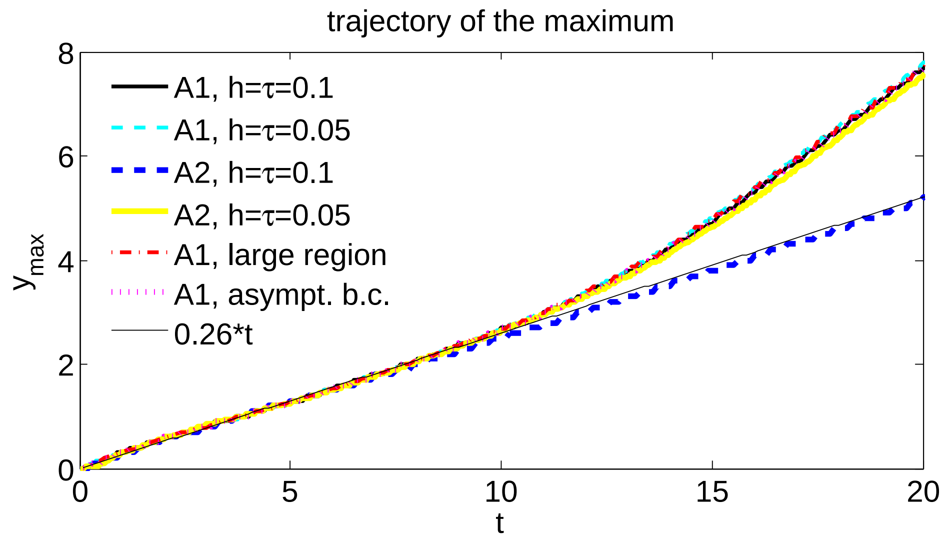
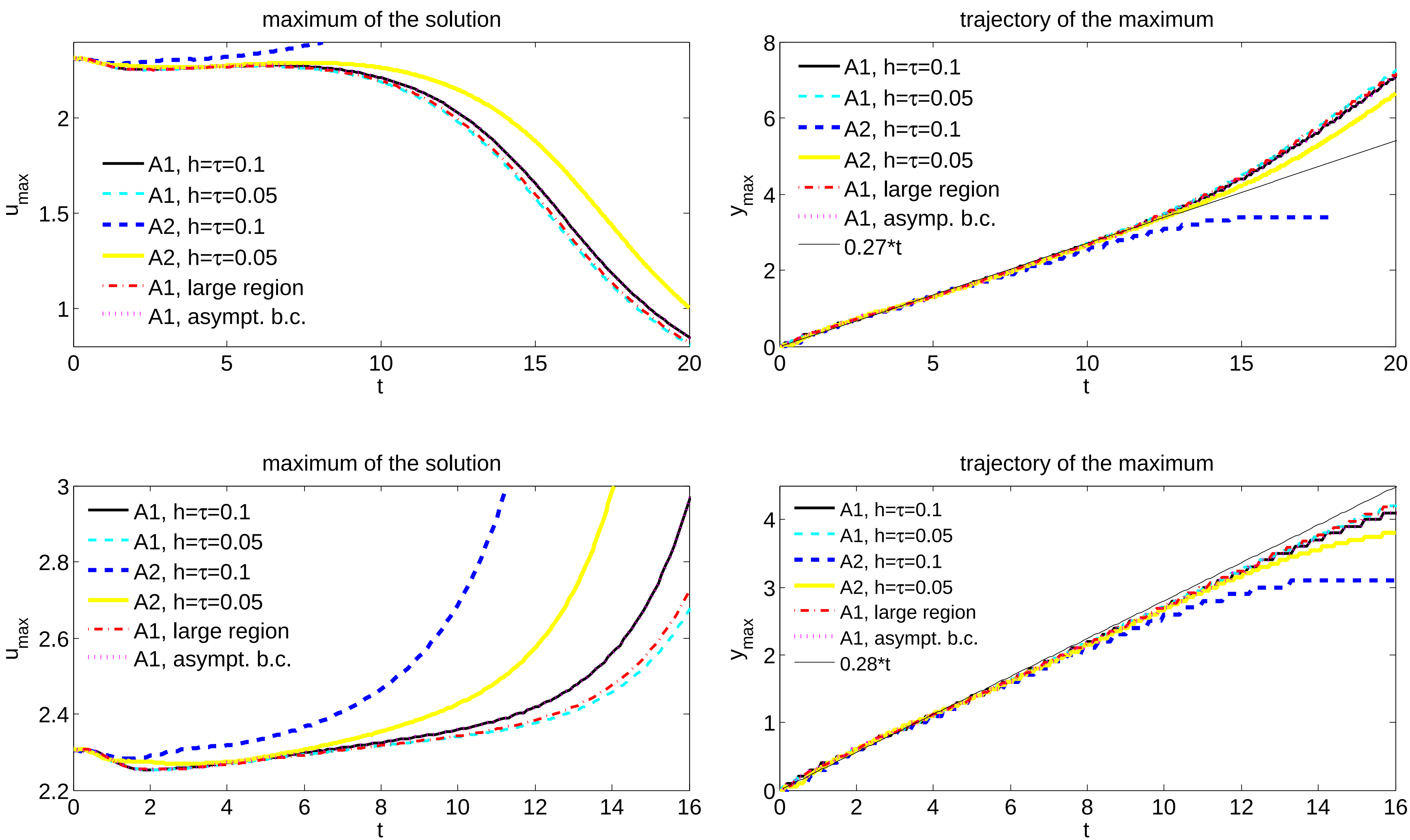
t=4
t=8
t=12
t=16
t=20
x
y





For $t < 10$ the solution moves with a velocity, very close to $c = 0.26$, and behaves like a soliton. For larger times the solution transforms into a diverging propagating wave, except in the case of A2 on the coarser grid, where the soliton keeps its form till $t < 20$. But on the finer grid A2 leads to a solution, very close to those, produced by A1 on all grids and with both boundary conditions.

Example 3. $c = 0.27$ and $c = 0.28$



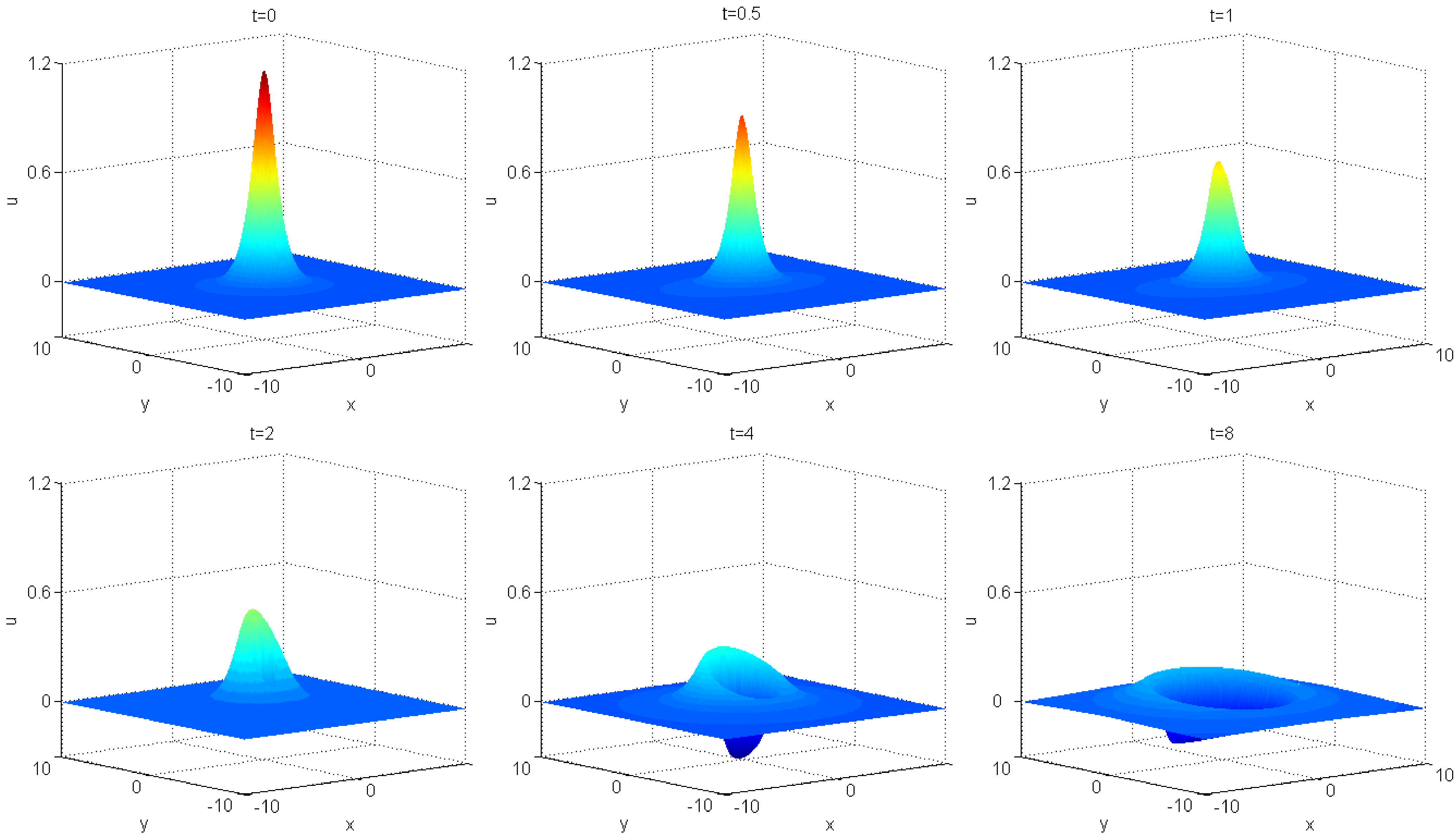
The results computed for $c = 0.27$ and $c = 0.28$ show that the behaviour of all A1 solutions and the A2 solution on the finer grid is similar – the solution keeps its form and moves with the prescribed velocity till $t \approx 10$. After that it transforms into a diverging wave for $c = 0.27$ or blows-up for $c = 0.28$. On the coarser grid the A2 solution blows-up for $c = 0.27$.

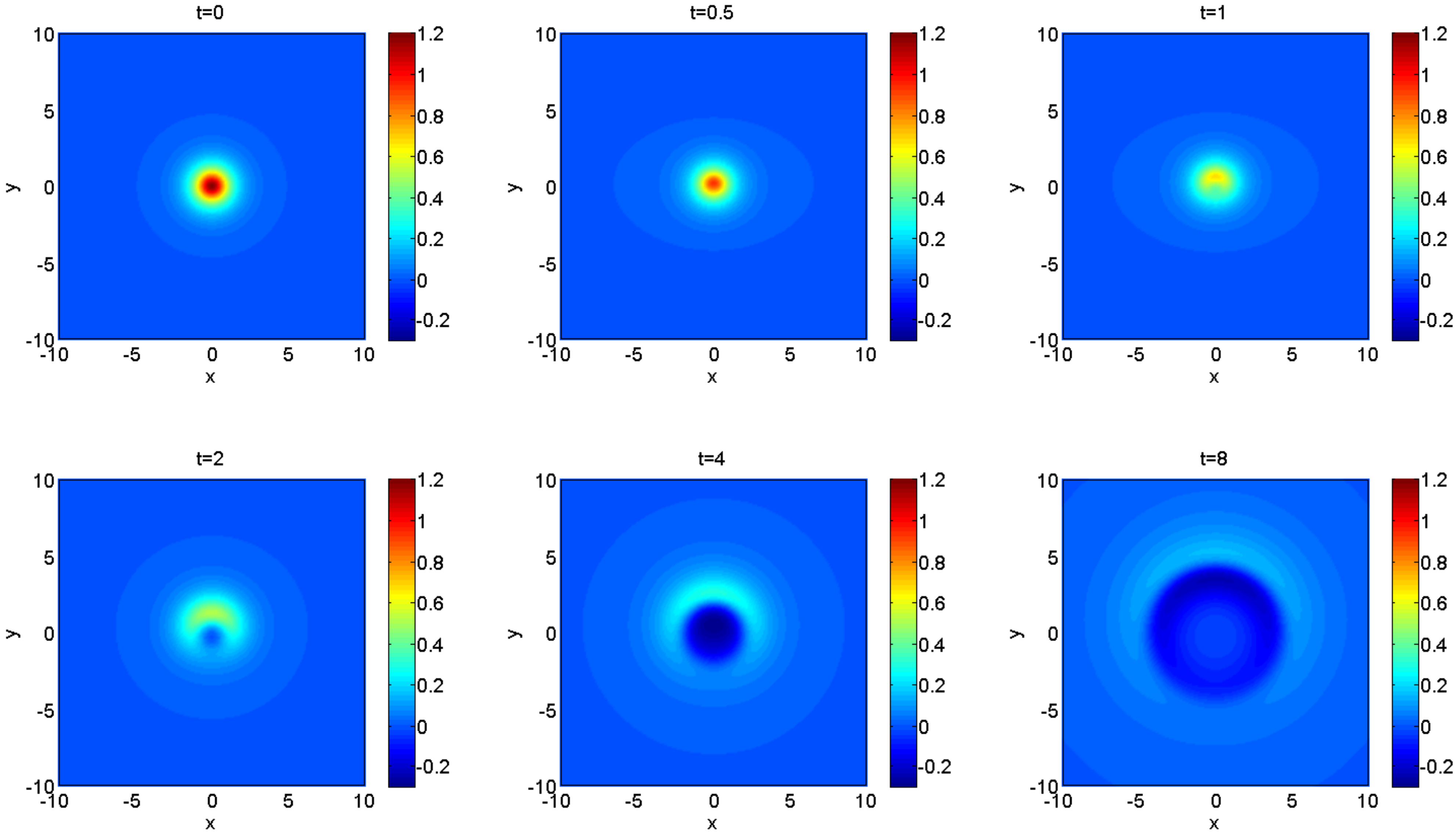
The results from these experiments confirm once again that a mechanism for having a balance between the nonlinearity and dispersion is present, but the solution is not robust (even when it is stable as a time stepping process) and takes the path to the attractor presented by the propagating wave for $c \leq 0.27$ or blows-up for $c \geq 0.28$.

These results were a motivation for investigating BPE with a different nonlinear term.

(ii) The case of qubic-quintic nonlinearity, $F(u) = \alpha(u^3 - \sigma u^5)$

Example 4. $\sigma = 3/16, \;\; c = 0.3$




t=0
t=0.5
t=1
t=2
t=4
t=8
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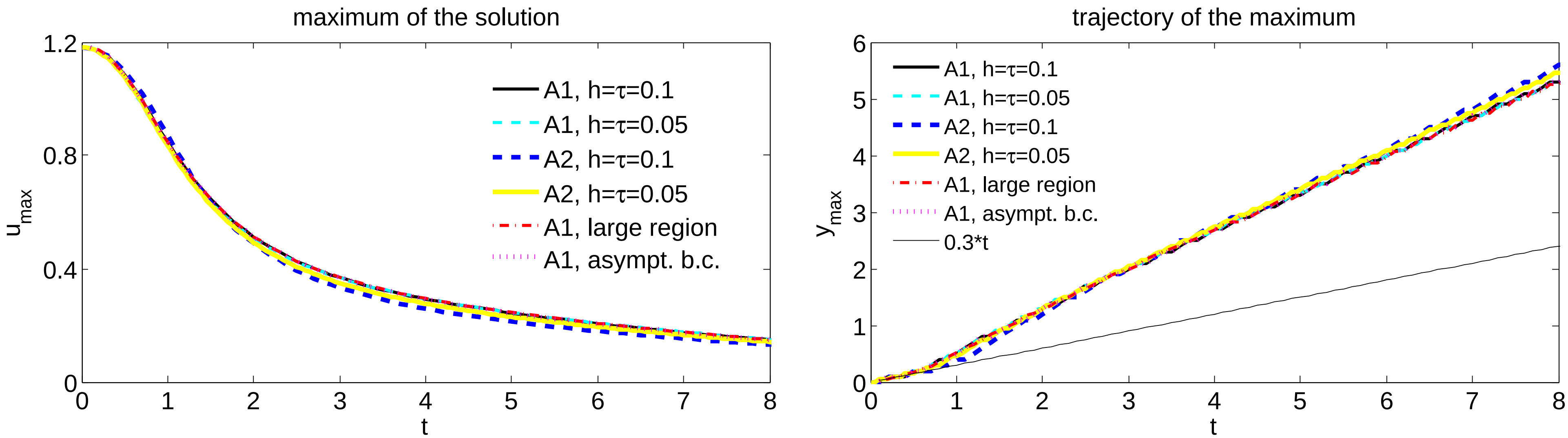
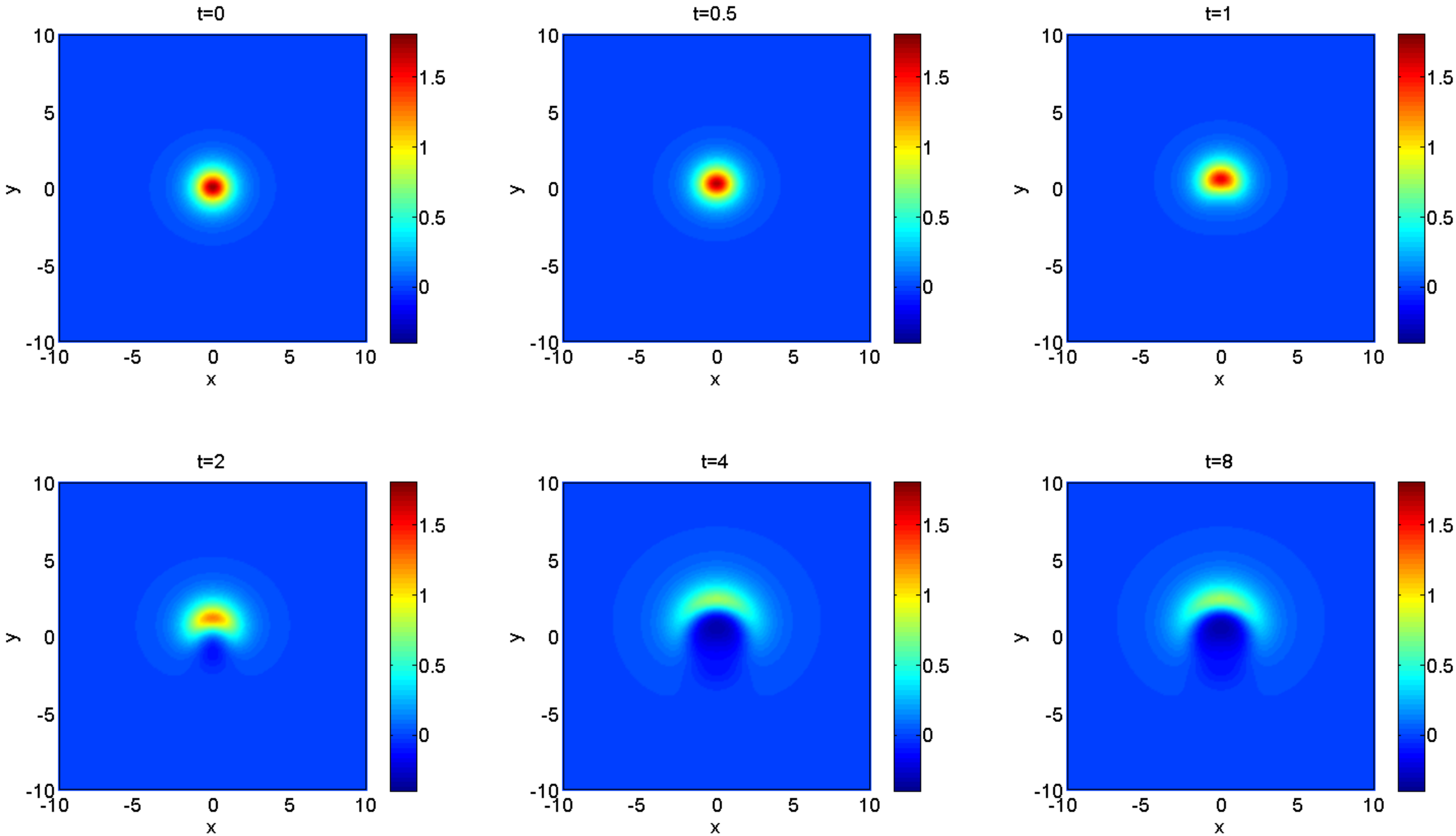
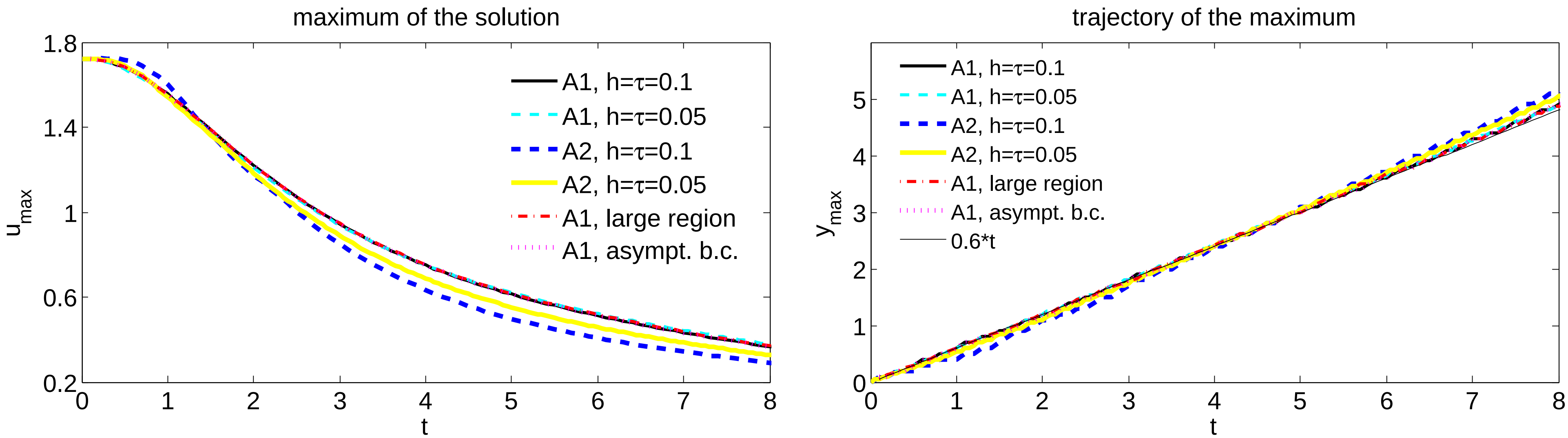


The solution cannot keep its form even for small times, and transforms into a propagating wave, which is almost concentric for $t > 8$. The maximum of the solution moves with a velocity, much faster than $c = 0.3$. The behaviour of the solution is the same on all grids and for all times steps, and does not depend on the type of the boundary conditions used.

Example 5. $\sigma = 3/16$, $c = 0.6$



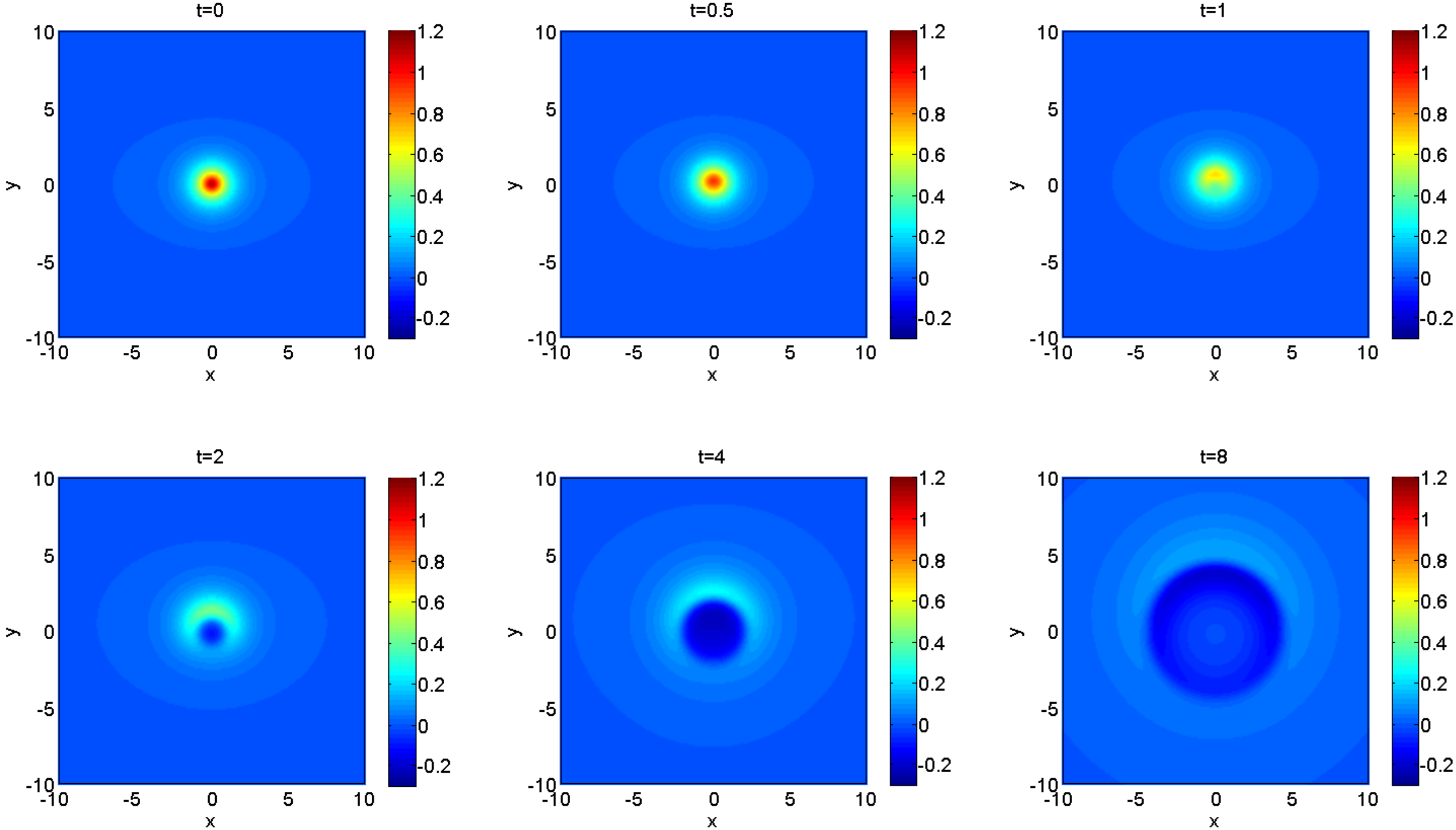


The results are very similar to those in the previous example, i.e., the solution cannot keep its form and transform into a diverging wave. Slightly different results are obtained for the maximum of the A2 solution on the coarse grid, but on the fine grid the results are closer to those for A1.

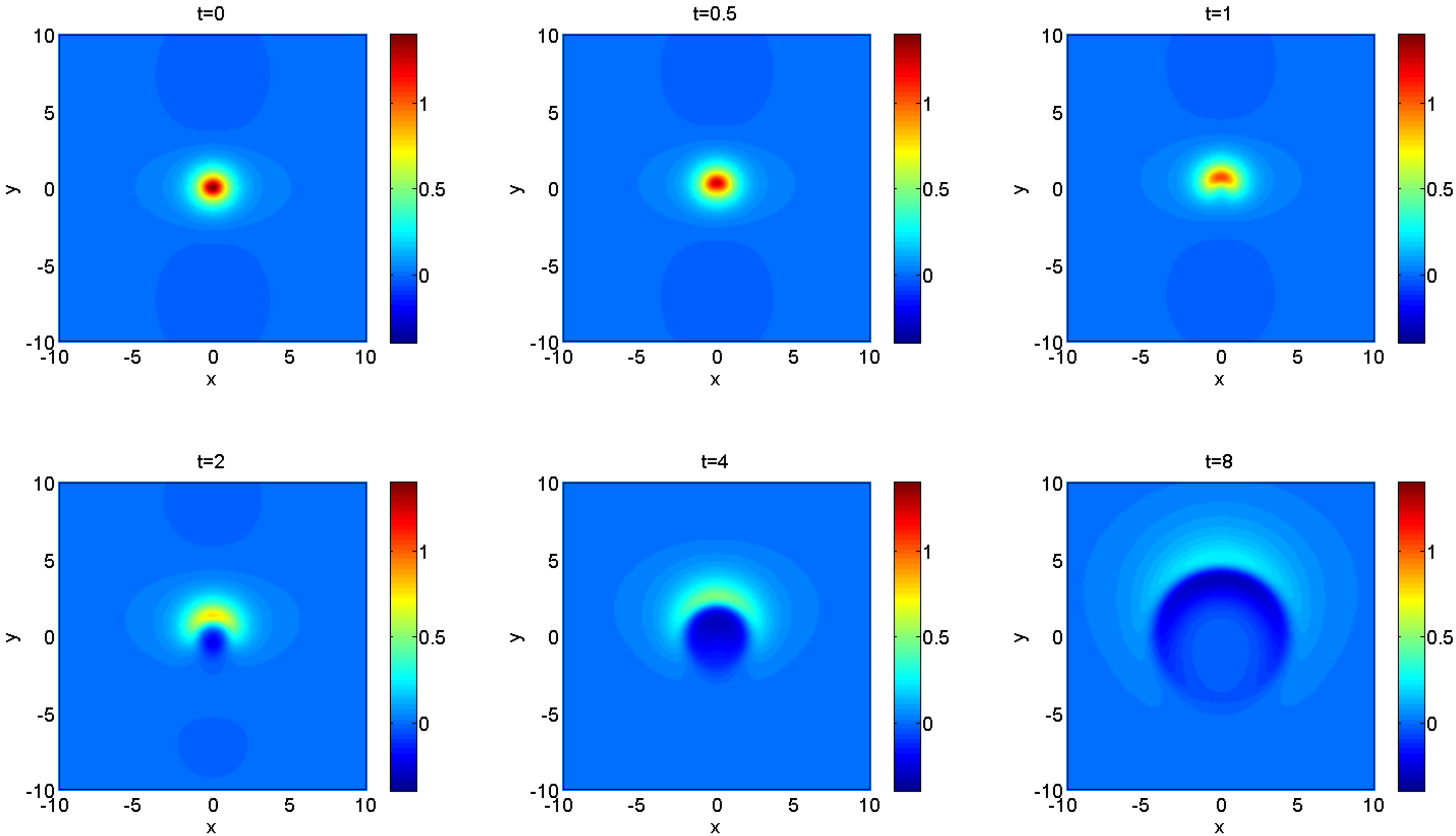
Example 6. $\boldsymbol{\sigma = 0.95}$, $\boldsymbol{c = 0.3}$ or $\boldsymbol{c = 0.6}$

The results for $\sigma = 0.95$ are very similar to the already presented results for $\sigma = 3/16$. Let us note that the investigated here 2D solutions of BPE with qubic-quintic nonlinearity do not blow-up even for larger values of c, but unfortunately they seem to be less structurally stable in comparison with the 2D solutions of BPE with quadratic nonlinearity.

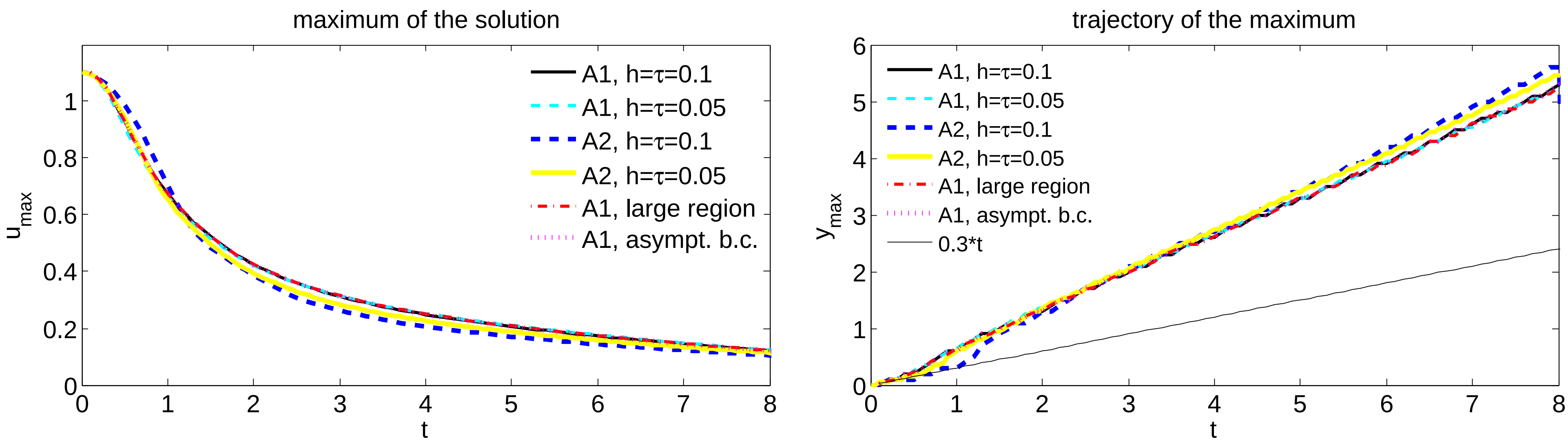
$\sigma = 0.95, \;\; c = 0.3$



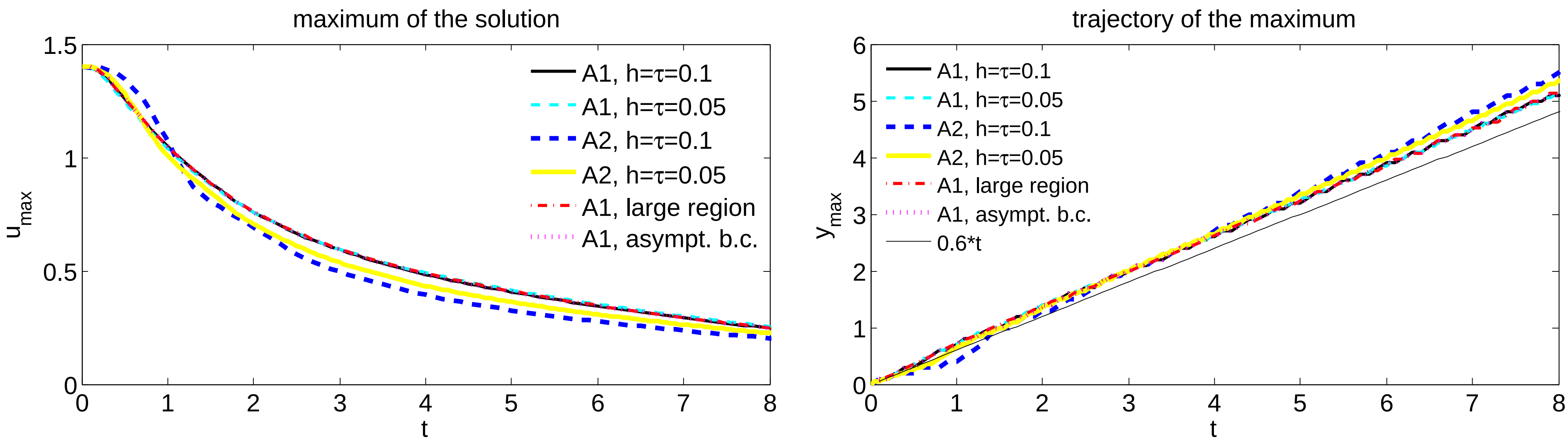
$\sigma = 0.95, \;\; c = 0.6$



$c = 0.3$


maximum of the solution
u_max
t
A1, h=τ=0.1
A1, h=τ=0.05
A2, h=τ=0.1
A2, h=τ=0.05
A1, large region
A1, asympt. b.c.
trajectory of the maximum
y_max
0.3*t

$c = 0.6$


maximum of the solution
u_max
t
A1, h=τ=0.1
A1, h=τ=0.05
A2, h=τ=0.1
A2, h=τ=0.05
A1, large region
A1, asympt. b.c.
trajectory of the maximum
y_max
0.6*t

Conclusion

We compared the results obtained by the aproach A1 with these obtained by A2 for quadratic and qubic-quintic nonlinearity and showed that they are in good agreement with each other. In the case of quadratic nonlinearity we confirmed the results obtained in previous works – the solution preserves its shape for small times, but for larger times it either disperses in the form of decaying ring wave or blows-up. The threshold for the appearance of blow-up seems to be near $c \approx 0.28$. For qubic-quintic nonlinearity the solution does not blow-up even for relatively large values of c, but is much less stable and transforms into a diverging propagating wave.

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Some papers and presentations about BPE may be found at

http://www.math.bas.bg/~nummeth/boussinesq/

Thanks for your attention!