Numerical Realization of Unsteady Solutions for the 2D Boussinesq Paradigm Equation

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Motivation

- Boussinesq equation is the first model for surface waves in shallow fluid layer that accounts for both nonlinearity and dispersion. The balance between the steepening effect of the nonlinearity and the flattening effect of the dispersion maintains the shape of the waves;

  J. V. Boussinesq, Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond, Journal de Mathématiques Pures et Appliquées 17 (1872) 55–108.

- In the 60s it was discovered that these permanent waves can behave in many instances as particles and they were called *solitons* by Zabusky and Kruskal;

  N. J. Zabusky, M. D. Kruskal, Interaction of 'solitons' in collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. 15 (1965) 240–243.

- A plethora of deep mathematical results have been obtained for solitons in the 1D case, but it is of crucial importance to investigate also the 2D case, because of the different phenomenology and the practical importance;

- The accurate derivation of the Boussinesq system combined with an approximation, that reduces the full model to a single equation, leads to the Boussinesq Paradigm Equation (BPE)

$$u_{tt} = \Delta \left[u - F(u) + \beta_1 u_{tt} - \beta_2 \Delta u \right], \quad F(u) := \alpha u^2,$$

  u is the surface elevation, $\beta_1 > 0$, $\beta_2 > 0$ - dispersion coefficients, $\alpha > 0$ - amplitude parameter, $\beta_2 = \alpha = 1$ without loosing of generality;

  C. I. Christov, An energy-consistent Galilean-invariant dispersive shallow-water model, Wave Motion 34 (2001) 161–174.

- 2D BPE admits stationary translating soliton solutions, which can be constructed using either finite differences, perturbation technique, or Galerkin spectral method;

  M. A. Christou, C. I. Christov, Fourier-Galerkin method for 2D solitons of Boussinesq equation, Math. Comput. Simul. 74 (2007) 82–92.

  J. Choudhury, C. I. Christov, 2D solitary waves of Boussinesq equation, in: "ISIS International Symposium on Interdisciplinary Science", Natchitoches, October 6-8, 2004, APS Conference Proceedings 755, Washington D.C., 2005, pp. 85–90.

  C. I. Christov, J. Choudhury, Perturbation solution for the 2D shallow-water waves, Mech. Res. Commun. Submitted.

  C. I. Christov, Numerical implementation of the asymptotic boundary conditions for steadily propagating 2d solitons of Boussinesq type equations, Math. Comp. Simulat. Accepted.

- Virtually nothing is known about the properties of these solutions when they are allowed to evolve in time and it is of utmost importance to answer the questions about their structural stability, i.e., what is their behaviour when used as initial conditions for time-dependent computations of the Boussinesq equation;

- To obtain reliable knowlegde about the time evolution of the stationary soliton solutions, it is imperative to develop different techniques for solving of the unsteady 2D BPE;

- Some preliminary results in

  A. Chertock, C. I. Christov, A. Kurganov, Central-upwind schemes for the Boussinesq paradigm equation, to appear in Proc. 4th Russian-German Advanced Research Workshop on Computational Science and High Performance Computing, 2010.

  C.I. Christov, N. Kolkovska, D. Vasileva, On the Numerical Simulation of Unsteady Solutions for the 2D Boussinesq Paradigm Equation, to appear in Proc. NMA'10 Conference, Lecture Notes Computer Science, 6046.

Numerical method

We introduce the following new dependent function

$$v(x,y,t) := u - \beta_1 \Delta u.$$

and get the following equation for v

$$v_{tt} = \frac{\beta_2}{\beta_1}\Delta v + \frac{\beta_1-\beta_2}{\beta_1^2}(u-v) - \alpha\Delta F(u).$$

The following implicit time stepping schemes can be designed

$$\frac{v_{ij}^{n+1}-2v_{ij}^{n}+v_{ij}^{n-1}}{\tau^2} = \frac{\beta_2}{2\beta_1}\Lambda\left[v_{ij}^{n+1}+v_{ij}^{n-1}\right] + \frac{\beta_1-\beta_2}{2\beta_1^2}[u_{ij}^{n+1}-v_{ij}^{n+1}+u_{ij}^{n-1}-v_{ij}^{n-1}] - \Lambda F(u_{ij}^{n}), \quad \text{or} \qquad (1)$$

$$\frac{v_{ij}^{n+1}-2v_{ij}^{n}+v_{ij}^{n-1}}{\tau^2} = \frac{\beta_2}{2\beta_1}\Lambda\left[v_{ij}^{n+1}+v_{ij}^{n-1}\right] + \frac{\beta_1-\beta_2}{2\beta_1^2}[u_{ij}^{n+1}-v_{ij}^{n+1}+u_{ij}^{n-1}-v_{ij}^{n-1}] - \Lambda G(u_{ij}^{n+1},u_{ij}^{n-1}); \qquad (2)$$

$$u_{ij}^{n+1} - \beta_1\Lambda u_{ij}^{n+1} = v_{ij}^{n+1}, \quad i=0,\dots,N_x+1, j=0,\dots,N_y+1.$$

τ is the time increment, $G(u_{ij}^{n+1},u_{ij}^{n-1}) = \left[(u_{ij}^{n+1})^2 + u_{ij}^{n+1}u_{ij}^{n-1} + (u_{ij}^{n-1})^2\right]/3$, $\Lambda = \Lambda^{xx}+\Lambda^{yy}$ is a difference approximation of the Laplace operator Δ on a non-uniform grid

$$\Lambda^{xx}\phi_{ij} = \frac{2\phi_{i-1j}}{h_{i-1}^x(h_i^x+h_{i-1}^x)} - \frac{2\phi_{ij}}{h_i^x h_{i-1}^x} + \frac{2\phi_{i+1j}}{h_i^x(h_i^x+h_{i-1}^x)} = \frac{\partial^2\phi}{\partial x^2}\Big|_{ij} + O(|h_i^x - h_{i-1}^x|),$$

$$\Lambda^{yy}\phi_{ij} = \frac{2\phi_{ij-1}}{h_{j-1}^y(h_j^y+h_{j-1}^y)} - \frac{2\phi_{ij}}{h_i^y h_{i-1}^y} + \frac{2\phi_{ij+1}}{h_j^y(h_j^y+h_{j-1}^y)} = \frac{\partial^2\phi}{\partial y^2}\Big|_{ij} + O(|h_i^y - h_{i-1}^y|).$$

Thus, we have two *coupled* equations for the two unknown grid functions $u_{ij}^{n+1}, v_{ij}^{n+1}$.
The second scheme conserves the energy, but is nonlinear and the nonlinear term G may be linearized using Picard method, i.e., using successive iterations.

We use the following non-uniform grid in the x- and y-directions:

$$\begin{aligned} x_i &= \sinh[\hat{h}_1 i],\ x_{N_x+1-i} = -x_i,\ i = \frac{N_x+1}{2}+1,\dots,N_x+1,\ x_{\frac{N_x+1}{2}} = 0, \\ y_j &= \sinh[\hat{h}_2 j],\ y_{N_y+1-j} = -y_j,\ j = \frac{N_y+1}{2}+1,\dots,N_y+1,\ y_{\frac{N_y+1}{2}} = 0, \end{aligned}$$

where $\hat{h}_1 = D_1/N_x$, $\hat{h}_2 = D_2/N_y$ and D_1, D_2 are selected in a manner to have large enough computational region.

For a smooth distribution of the nonuniform grid (as the one considered here) one has

$$O(|h_i^x - h_{i-1}^x|) \approx \frac{\partial h^x}{\partial x} O(|h_{i-1}|^2) = O(|h_{i-1}|^2).$$

The unconditional stability and the convergence of the schemes are shown in

N. Kolkovska, Two Families of Finite Difference Schemes for Multidimensional Boussinesq Equation. Accepted for publication in AIP.
N. Kolkovska, Convergence of Finite Difference Schemes for a Multidimensional Boussinesq Equation. Accepted for publication in LNCS.

The numerical experiments in the 1D case with the analogue of the presented scheme confirm the findings in

C. I. Christov and M. G. Velarde. Inelastic interaction of Boussinesq solitons. J. Bifurcation & Chaos, 4 (1994), 1095–1112.
about the structural stability of 1D soliton solutions of BPE - they preserve their shape for all times and even after interaction.

The boundary conditions can be set equal to zero, because of the localization of the wave profile.

For smaller computational box - asymptotic boundary conditions can be formulated as

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \approx -2u, \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} \approx -2v, \qquad \sqrt{x^2+y^2} \gg 1.$$

$$\begin{aligned} u^{n+1}_{iN_y+1} &= u^{n+1}_{iN_y-1} + \frac{h^y_{N_y} + h^y_{N_y-1}}{y_{N_y}}\Big[-2u^{n+1}_{iN_y} - \frac{x_i}{h^x_i + h^x_{i-1}}(u^{n+1}_{i+1N_y} - u^{n+1}_{i-1N_y})\Big], \quad i = 0, \dots, N_x, \\ u^{n+1}_{N_x+1j} &= u^{n+1}_{N_x-1j} + \frac{h^x_{N_x} + h^x_{N_x-1}}{x_{N_x}}\Big[-2u^{n+1}_{N_xj} - \frac{y_j}{h^y_j + h^y_{j-1}}(u^{n+1}_{N_x,j+1} - u^{n+1}_{N_x,j-1})\Big], \quad j = 0, \dots, N_y. \end{aligned}$$

The coupled system of equations is solved by the Bi-Conjugate Gradient Stabilized Method with ILU preconditioner.

Numerical experiments

We use the following best fit approximation for the shape of the stationary propagating soliton with velocity c

C. I. Christov, J. Choudhury, Perturbation solution for the 2D shallow-water waves, Submitted to Mech. Res. Commun.

$$u^s(x,y;c) = f(x,y) + c^2\left[(1-\beta_1)g_a(x,y) + \beta_1 g_b(x,y)\right] + c^2\left[(1-\beta_1)h_1(x,y) + \beta_1 h_2(x,y)\right]\cos(2\arctan(y/x).$$

$$f(x,y) = \frac{2.4(1+0.24r^2)}{\cosh(r)(1+0.095r^2)^{1.5}}, \quad g_a(r) = -\frac{1.2(1-0.177r^{2.4})}{\cosh(r)(1+0.11r^{2.1})}, \quad g_b(r) = -\frac{1.2(1+0.22r^2)}{\cosh(r)(1+0.11r^{2.4})},$$

$$h_i(x,y) = \frac{a_i r^2 + b_i r^3 + c_i r^4 + v_i r^6}{1 + d_i r + e_i r^2 + f_i r^3 + g_i r^4 + h_i r^5 + q_i r^6 + w_i r^8}, \quad r(x,y) = \sqrt{x^2+y^2}, \quad \theta(x,y) = \arctan(y/x),$$

$a_1 = 1.03993,\ a_2 = 31.2172,\ b_1 = 6.80344,\ b_2 = -10.0834,\ c_1 = -0.22992,\ c_2 = 3.97869,\ d_1 = 12.6069,\ d_2 = 77.9734,$

$e_1 = 13.5074,\ e_2 = -76.9199,\ f_1 = 2.46495,\ f_2 = 55.4646,\ g_1 = 2.45953,\ g_2 = -12.9335,\ h_1 = 1.03734, h_2 = 1.0351,$

$q_1 = -0.0246084,\ q_2 = 0.628801,\ v_1 = 0.0201666,\ v_2 = -0.0290619,\ w_1 = 0.00408432,\ w_2 = -0.00573272.$

In the examples below $u^s(x,y;c)$ for $\beta_1 = 3$ is taken as initial data for $t = 0$ and the second initial condition may be chosen as

$$\partial u/\partial t = -c\,\partial u^s/\partial y, \quad t = 0, \quad \text{or} \quad u(x,y,-\tau) = u^s(x,y+c\tau;c).$$

The solutions are computed on 3 different grids in the region $x,y \in [-50,50]$ (with 161×161, 321×321 and 641×641 grid points), with at least 3 different time steps ($\tau = 0.2$, 0.1 and 0.05), with or without using some symmetry conditions at $x = 0$ (and $y = 0$ for $c = 0$), and using zero or asymptotic boundary conditions.

Example 1. $c = 0$ – the profile of the initial condition is a standing soliton. The nonlinearity is not strong enough and the solution transforms into a propagating cylindrical wave, similar to the one generated on a water surface when an object is dropped into it. The behaviour of the solution is one and the same on all grids, for all times steps, and does not depend on the scheme and the type of the boundary conditions used.

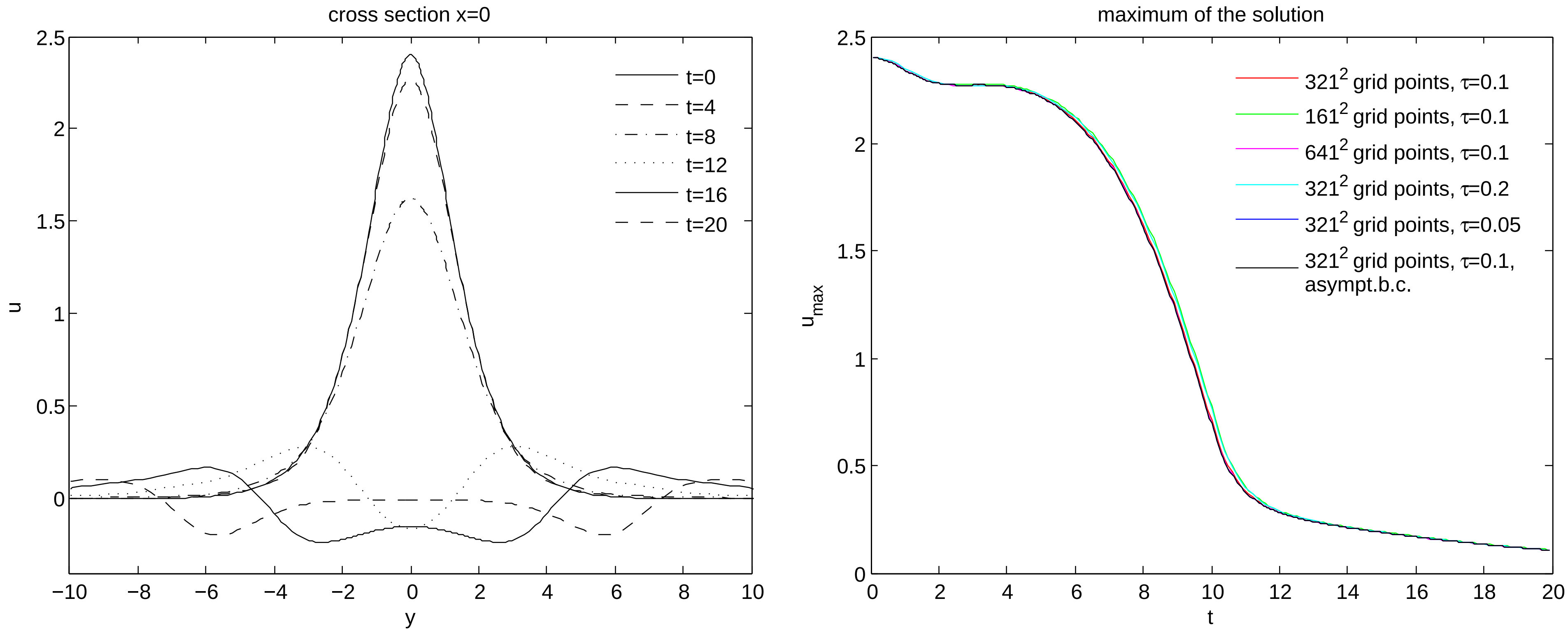

Figure 1: Evolution of the solution for $c = 0$ – its shape for $x = 0$ and the values of the maximum

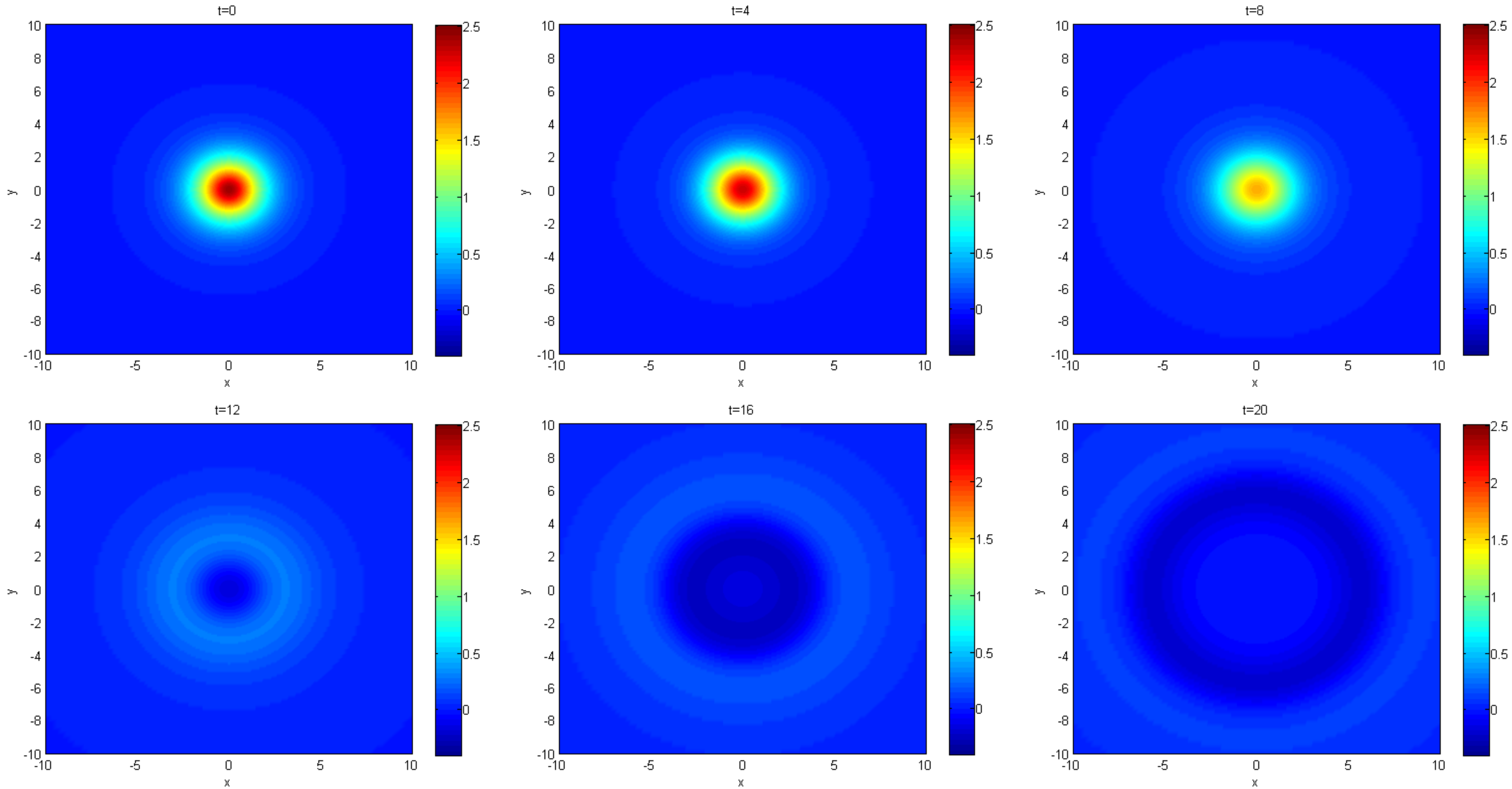

Figure 2: Evolution of the solution for $c = 0$

$$\alpha = \log_2 \left(|u_{\max}^{\text{prev}} - u_{\max}^{\text{prev,prev}}| / |u_{\max} - u_{\max}^{\text{prev}}| \right)$$

Table 1: Values of the maximum of the solution, convergence in space and time, $c = 0$

		$t=4$			$t=8$			$t=12$		
τ	N_x+1	$u_{\max}$	$u_{\max}^{\text{prev}} - u_{\max}$	α	$u_{\max}$	$u_{\max}^{\text{prev}} - u_{\max}$	α	$u_{\max}$	$u_{\max}^{\text{prev}} - u_{\max}$	α
		first scheme with second IC according to $\partial u/\partial t = -c\,\partial u^s/\partial y$								
0.1	160	2.271219			1.647044			2.87575e-1		
0.1	320	2.264753	6.47e-3		1.605527	4.15e-2		2.80298e-1	7.28e-3	
0.1	640	2.263138	1.62e-3	2.0	1.595311	1.02e-2	2.0	2.78648e-1	1.65e-3	2.1
0.2	320	2.265164			1.617918			2.82138e-1		
0.1	320	2.264753	4.11e-4		1.605527	1.24e-2		2.80298e-1	1.84e-3	
0.05	320	2.264636	1.17e-4	1.8	1.602381	3.14e-3	2.0	2.79847e-1	4.51e-4	2.0
0.025	320	2.264606	3.00e-5	2.0	1.601592	7.89e-4	2.0	2.79736e-1	1.11e-4	2.0
		first scheme with second IC according to $u(x, y, -\tau) = u^s(x, y + c\tau; c)$								
0.1	160	2.270163			1.629898			2.84523e-1		
0.1	320	2.263502	6.66e-3		1.587706	4.22e-2		2.77527e-1	6.996e-3	
0.1	640	2.261839	1.66e-3	2.0	1.577328	1.04e-2	2.0	2.75936e-1	1.591e-3	2.1
0.025	320	2.264285	-2.87e-4	0.8	1.597096	-3.68e-3	0.6	2.79032e-1	-5.84e-4	0.7
0.0125	320	2.264438	-1.53e-4	0.9	1.599145	-2.05e-3	0.8	2.79355e-1	-3.23e-4	0.9
0.00625	320	2.264516	-7.80e-5	1.0	1.600220	-1.08e-3	0.9	2.79524e-1	-1.69e-4	0.9
0.003125	320	2.264556	-4.00e-5	1.0	1.600769	-5.49e-4	1.0	2.79611e-1	-8.67e-5	1.0
		second scheme with second IC according to $\partial u/\partial t = -c\,\partial u^s/\partial y$								
0.1	160	2.270904			1.641639			2.866134e-1		
0.1	320	2.264401	6.50e-3		1.599935	4.16e-2		2.794467e-1	7.17e-3	
0.1	640	2.262777	1.62e-3	1.9	1.589674	1.03e-2	2.0	2.778184e-1	1.16e-3	2.1
0.2	320	2.263859			1.596044			2.787405e-1		
0.1	320	2.264401	-5.42e-4		1.599935	-3.95e-3		2.794467e-1	-7.06e-4	
0.05	320	2.264546	-1.45e-4	1.9	1.600970	-9.77e-4	2.0	2.796343e-1	-1.87e-4	1.9
0.025	320	2.264583	-3.70e-5	2.0	1.601238	-2.68e-4	1.9	2.796827e-1	-4.84e-5	2.0

Example 2. The next case is for a phase speed $c = 0.25$. For $t < 8$ the soliton not only moves with a speed, close to $c = 0.25$, but also behaves like a soliton, i.e., preserves its shape, although its maximum slightly decreases. For larger times the solution transforms into a diverging propagating wave, but without a cylindrical symmetry because of the propagation of the wave.

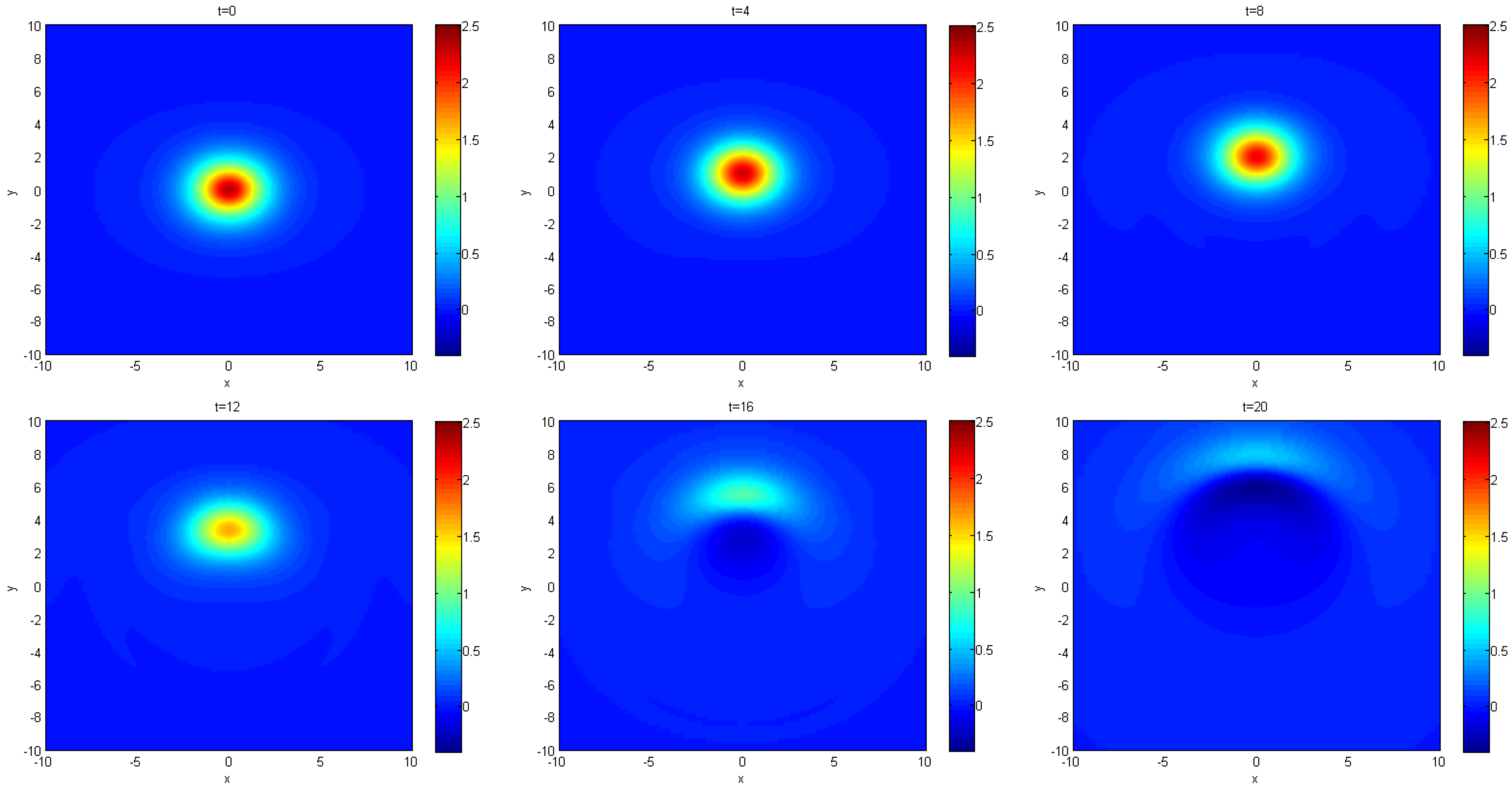

Figure 3: Evolution of the solution for $c = 0.25$

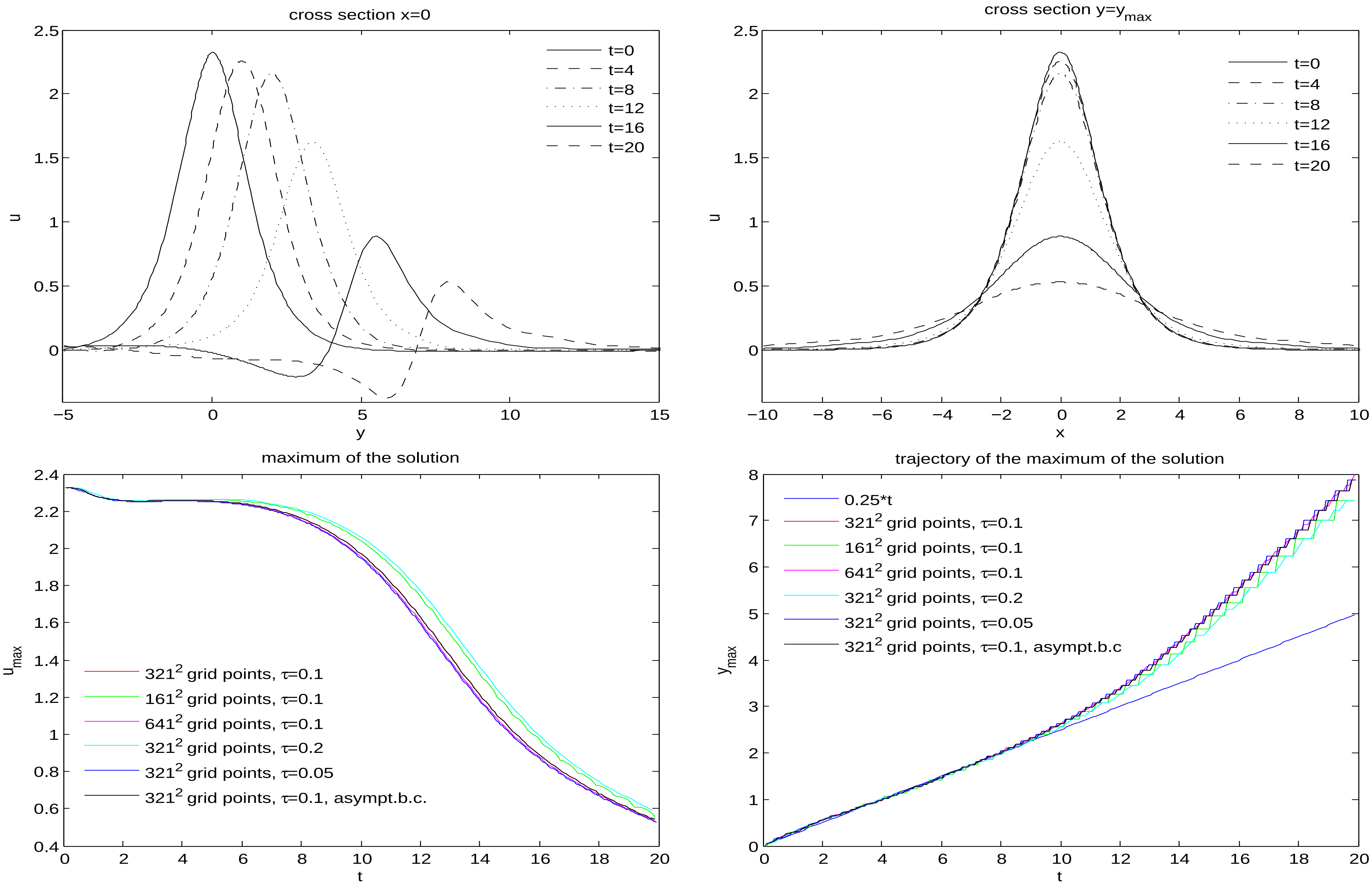

Figure 4: Evolution of the solution for $c = 0.25$ – its shape for $x = 0$, for $y = y_{\max}$, the values and the trajectory of the maximum

Table 2: Values of the maximum of the solution, convergence in space and time, $c = 0.25$

		$t=4$			$t=8$			$t=12$		
τ	N_x+1	$u_{\max}$	$u_{\max}^{\text{prev}}-u_{\max}$	α	$u_{\max}$	$u_{\max}^{\text{prev}}-u_{\max}$	α	$u_{\max}$	$u_{\max}^{\text{prev}}-u_{\max}$	α
		first scheme with second IC according to $\partial u/\partial t = -c\,\partial u^s/\partial y$								
0.1	160	2.261156			2.191684			1.725273		
0.1	320	2.257642	3.51e-3		2.165738	2.59e-2		1.639348	8.59e-2	
0.1	640	2.256689	9.53e-4	1.9	2.158619	7.12e-3	1.9	1.619535	1.98e-2	2.1
0.2	320	2.268606			2.226354			1.848499		
0.1	320	2.257642	1.10e-2		2.165738	6.06e-2		1.639348	2.09e-1	
0.05	320	2.254871	2.77e-3	2.0	2.148196	1.75e-2	1.8	1.588800	5.05e-2	2.0
		first scheme with second IC according to $u(x, y, -\tau) = u^s(x, y + c\tau; c)$								
0.1	160	2.261550			2.189987			1.718885		
0.1	320	2.256804	4.75e-3		2.156155	3.38e-2		1.609205	1.10e-1	
0.1	640	2.255469	1.34e-3	1.8	2.147008	9.15e-3	1.9	1.584249	2.50e-2	2.1
0.2	320	2.264348			2.195763			1.734455		
0.1	320	2.256804	7.54e-3		2.156155	3.96e-2		1.609205	1.25e-1	
0.05	320	2.254958	1.85e-3	2.0	2.146491	9.66e-3	2.0	1.583778	2.54e-2	2.3
		second scheme with second IC according to $u(x, y, -\tau) = u^s(x, y + c\tau; c)$								
0.1	160	2.258272			2.170509			1.656640		
0.1	320	2.253499	4.77e-3		2.137724	3,28e-2		1.556919	9.97e-2	
0.1	640	2.252157	1.34e-3	1.8	2.128562	9.16e-3	1.8	1.532585	2.43e-2	2.0
0.2	320	2.250649			2.120342			1.510212		
0.1	320	2.253499	-2.85e-3		2.137724	-1.74e-2		1.556919	-4.67e-2	
0.05	320	2.254140	-6.41e-4	2.2	2.141967	-4.24e-3	2.0	1.570970	-1.41e-2	1.7

Example 3. Results for a phase speed $c = 0.3$ are presented. For $t < 4$ the behaviour of the solution is similar to that in the previous example, but for larger times it turns to grow and blows-up for $t \approx 16$.

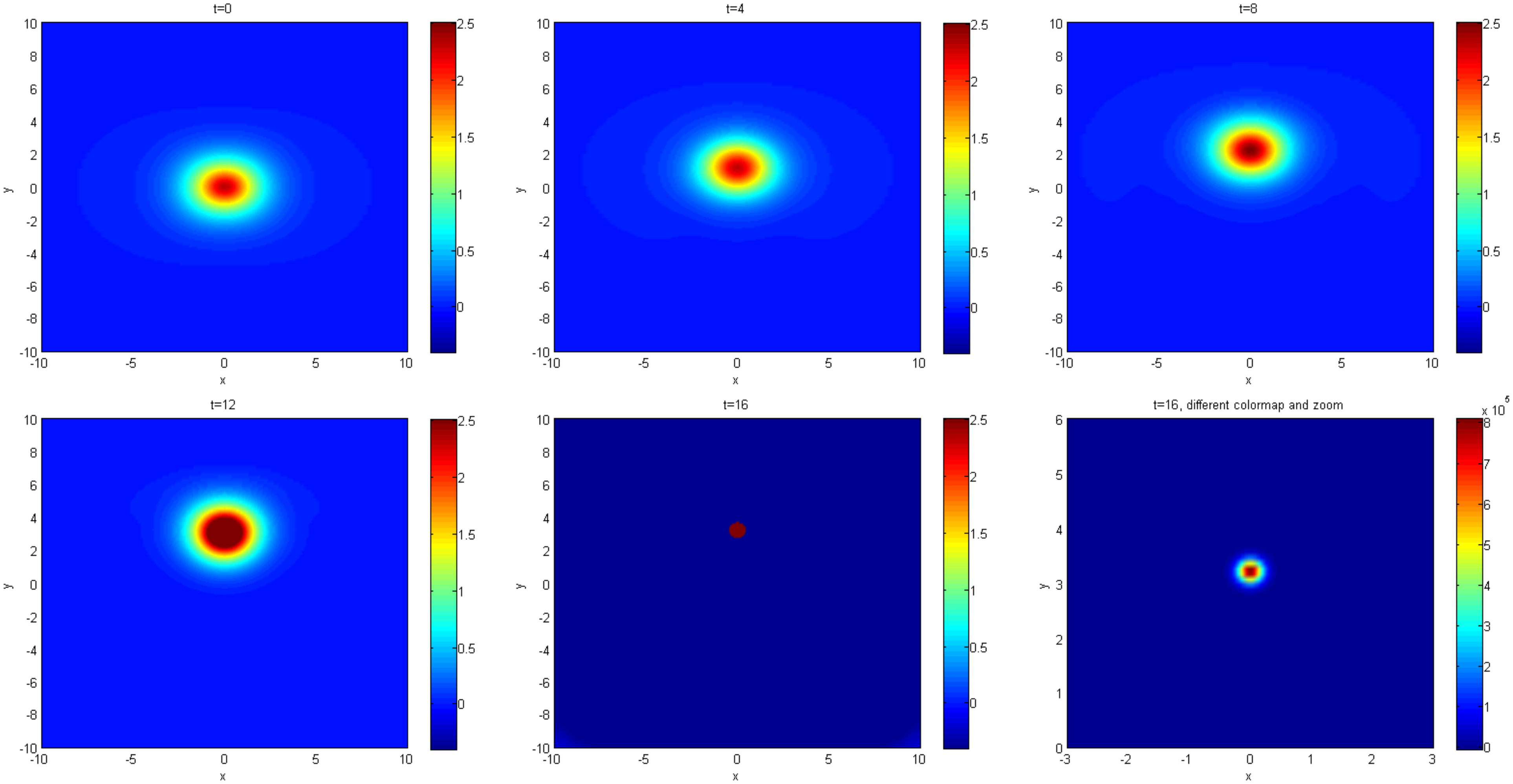

Figure 5: Evolution of the solution for $c = 0.3$

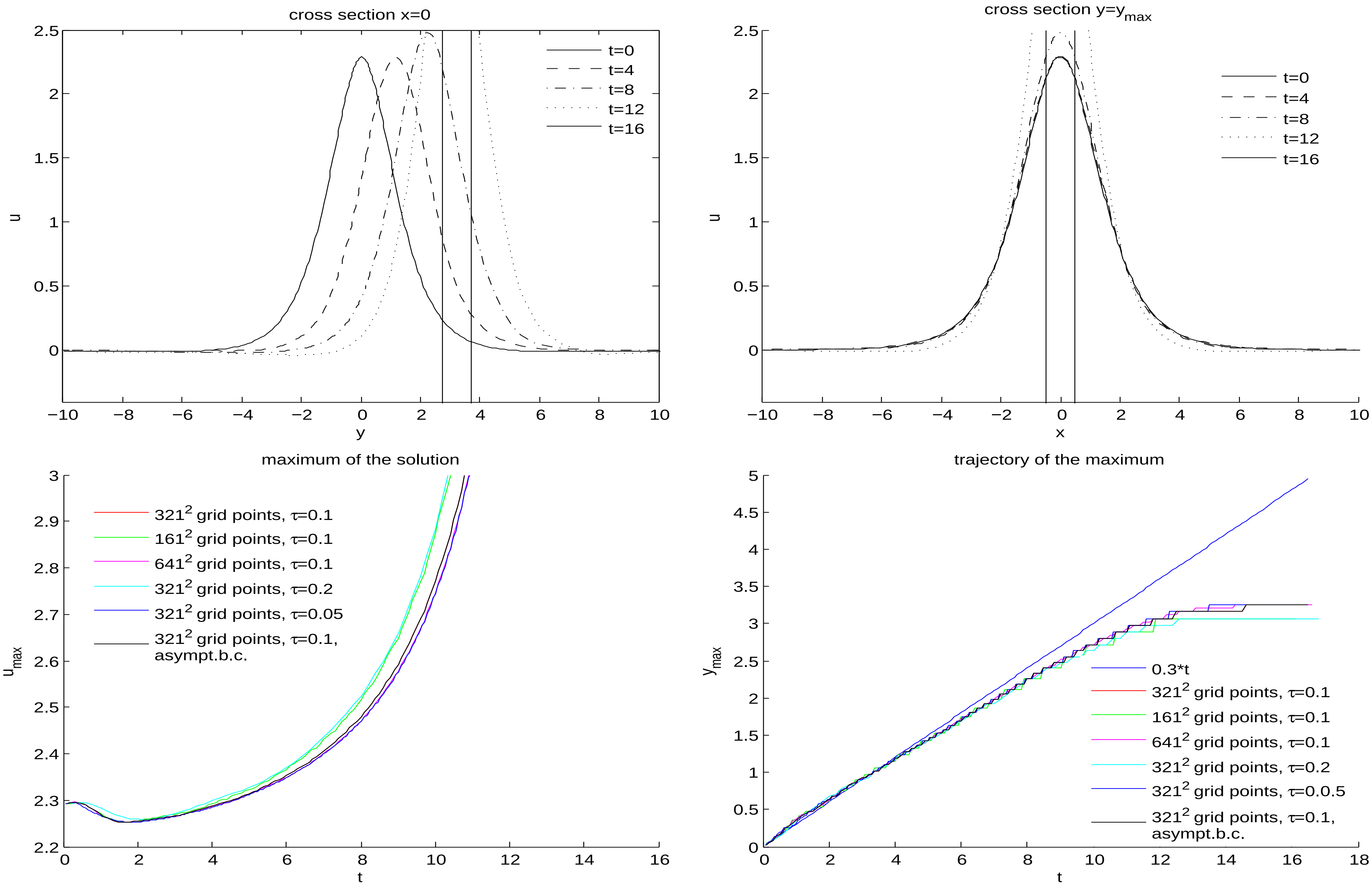

Figure 6: Evolution of the solution for $c = 0.3$ – its shape for $x = 0$, for $y = y_{\max}$, the values and the trajectory of the maximum

Table 3: Values of the maximum of the solution, convergence in space and time, $c = 0.3$

		$t=4$			$t=8$			$t=12$		
τ	$N_x{+}1$	$u_{\max}$	$u_{\max}^{\text{prev}}-u_{\max}$	α	$u_{\max}$	$u_{\max}^{\text{prev}}-u_{\max}$	α	$u_{\max}$	$u_{\max}^{\text{prev}}-u_{\max}$	α
		first scheme with second IC according to $\partial u/\partial t = -c\,\partial u^s/\partial y$								
0.1	160	2.292808			2.523852			4.122404		
0.1	320	2.290190	2.62e-3		2.495695	2.82e-2		3.819252	3.03e-1	
0.1	640	2.289419	7.71e-4	1.8	2.488447	7.25e-3	2.0	3.747515	7.17e-2	2.1
0.2	320	2.308133			2.577895			4.733447		
0.1	320	2.290190	1.79e-2		2.495695	8.22e-2		3.819252	9.14e-1	
0.05	320	2.285470	4.72e-3	1.9	2.473195	2.25e-2	1.9	3.597565	2.22e-1	2.0
		first scheme with second IC according to $u(x,y,-\tau) = u^s(x, y+c\tau; c)$								
0.1	160	2.294405			2.528720			4.175907		
0.1	320	2.289804	4.60e-3		2.489582	3.91e-2		3.753627	4.22e-1	
0.1	640	2.288430	1.37e-3	1.7	2.479694	9.89e-3	2.0	3.655287	9.83e-2	2.1
0.2	320	2.304409			2.547474			4.340873		
0.1	320	2.289804	1.46e-2		2.489582	5.79e-2		3.753627	5.87e-1	
0.05	320	2.286092	3.71e-3	2.0	2.474824	1.48e-2	2.0	3.612671	1.41e-1	2.1
		second scheme with second IC according to $u(x,y,-\tau) = u^s(x, y+c\tau; c)$								
0.1	160	2.290448			2.505289			3.932897		
0.1	320	2.285503	4.95e-3		2.466035	3.93e-2		3.533640	3.99e-1	
0.1	640	2.284210	1.29e-3	1.9	2.456697	9.34e-3	2.1	3.445910	8.77e-2	2.2
0.2	320	2.287450			2.454126			3.398348		
0.1	320	2.285503	1.95e-3		2.466035	-1.19e-2		3.533640	-1.35e-1	
0.05	320	2.285062	4.41e-4	2.1	2.469062	-3.03e-3	2.0	3.558793	-2.52e-2	2.4

Example 4. Interaction of two solitonic-like structures is investigated for different values of c. In most cases, when both structures clash, the solution blows up (for $c = 0.25, c = 0.2$ and various initial distances between the structures). Another behaviour appears for $c = 0.15$, when the initial distance between the centres of the structures is not very small, on the figure - the initial distance is 15. The dispersion had some time to begin acting, and when the structures hit each other, a clear interference pattern onsets.

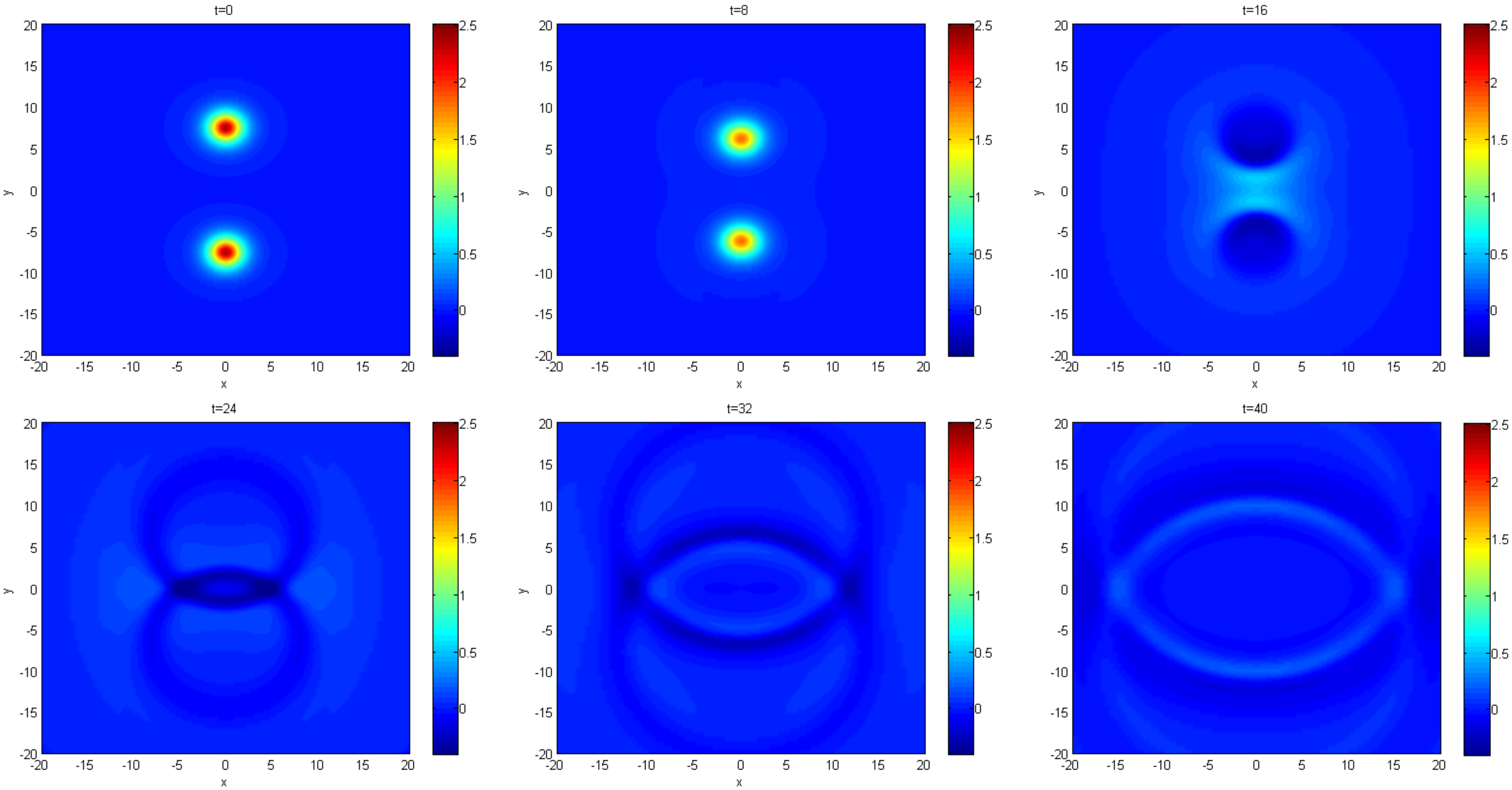

Figure 7: Evolution of two interacting structures for $c = 0.15$

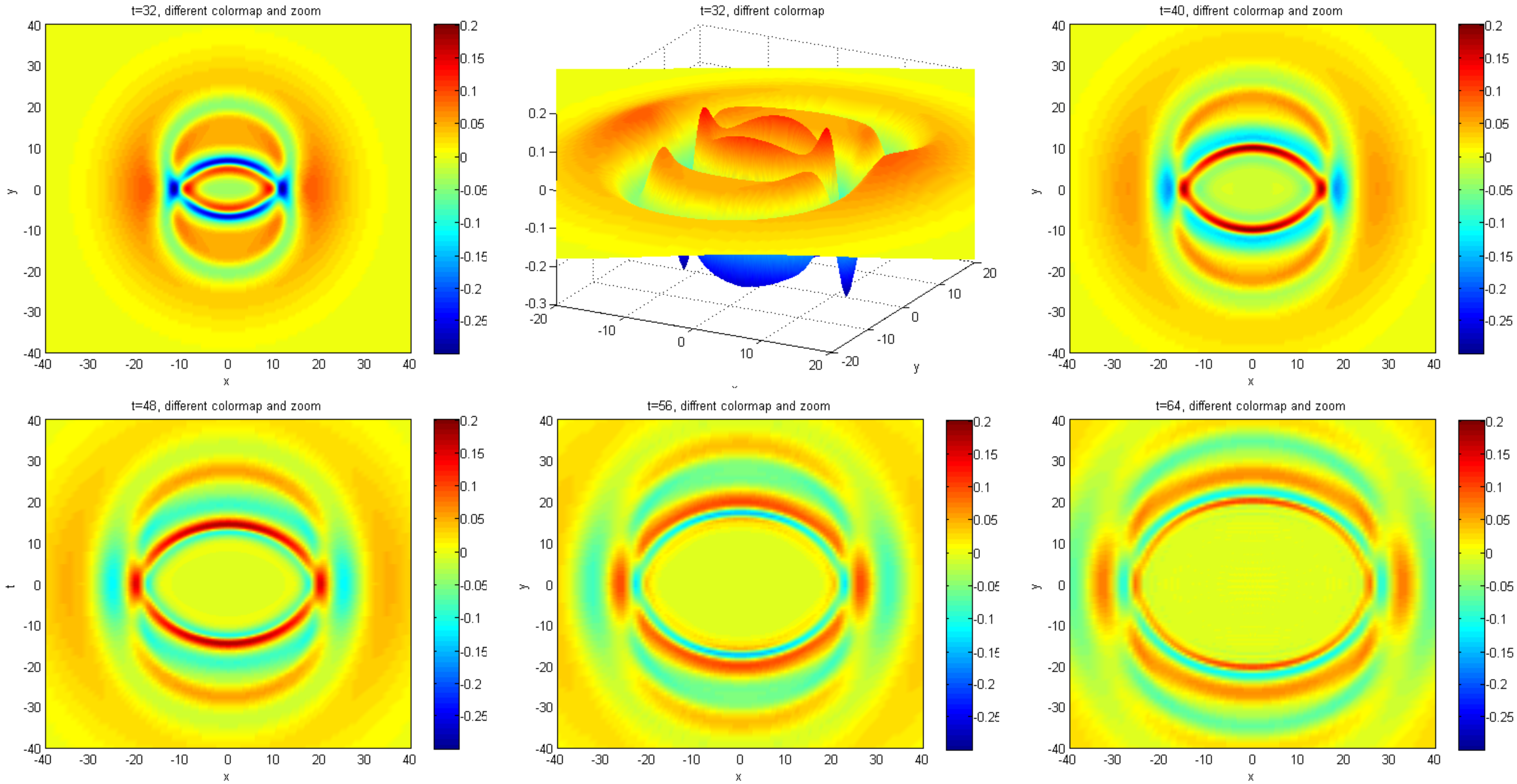

Figure 8: Evolution of two interacting structures for $c = 0.15$

Conclusion

- Our results are very close to the results of Chertock, Christov and Kurganov from

  A. Chertock, C. I. Christov, A. Kurganov, Central-upwind schemes for the Boussinesq paradigm equation, to appear in Proc. 4th Russian-German Advanced Research Workshop on Computational Science and High Performance Computing, 2010.

  and confirm the solitonic-like behaviour of the solutions for relatively small times.

- Unfortunately, the investigated solutions are not structurally stable and transform either in diverging propagating waves or blow-up.

- For $c \approx 0.3$, an time interval exists in which the solution is virtually preserving its shape whils steadily translating means that 2D solitons could be found for equations from the class of the BPE. This means that the nonlinearity is strong enough to balance the dispersion which is now much stronger than in the 1D case.

- Probably, the quadratic nonlinearity in BPE is not enough for modeling permanent soliton-like waves. Our future plans include experiments with different types of nonlinearities in the source term and in the coefficients of the equation, as well as development of faster solvers for the linear systems, arising after the discretization.