Non-overlapping domain decomposition methods of optimal computational complexity

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Schwartz's original method was proposed in 1869, which considered the Poisson equation with Dirichlet boundary conditions in a domain consisting of a circle and an overlapping square. Nowadays this approach provides powerful tools for efficient parallel solution of large-scale systems of algebraic equations arising from the discretization of partial differential equations. Two kinds of preconditioners are constructed in this way: overlapping and non-overlapping domain decomposition (DD).

We consider the second order elliptic equation $-\mathcal{A}u = f$ in $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, equipped with appropriate boundary conditions on $\Gamma = \partial\Omega$. Assume that the finite element method is applied for numerical solution of the problem using linear elements on a quasi-uniform triangulation $\mathcal{T}_h$, thus obtaining the linear system $A\mathbf{u} = \mathbf{f}$. We now assume that $\{\Omega_i\}_{i=1}^m$ is a non-overlapping partitioning of Ω with interface $\gamma \subset \mathbb{R}^{d-1}$. The stiffness matrix A is written in the form

$$A = \begin{pmatrix} A_D & A_{D\gamma} \\ A_{\gamma D} & A_\gamma \end{pmatrix} = \begin{pmatrix} A_D & \\ A_{\gamma D} & S \end{pmatrix} \begin{pmatrix} I & A_D^{-1} A_{D\gamma} \\ & I \end{pmatrix},$$

$A_D = Diag\,(A_1, A_2, \cdots, A_m)$, the blocks A_i correspond to the subdomains Ω_i, $i = 1, 2, \cdots, m$, A_γ - to the interface, and S is the Schur complement. Then following new non-overlapping DD preconditioner $C_{DD,k}^{BURA}$ is analyzed

$$A = \begin{pmatrix} A_D & \\ A_{\gamma D} & \sigma C_{1/2,k}^{BURA}(\Lambda) \end{pmatrix} \begin{pmatrix} I & A_D^{-1} A_{D\gamma} \\ & I \end{pmatrix}.$$

Here $C_{1/2,k}^{BURA}(\Lambda)$ is the best uniform rational approximation of degree k of $\Lambda^{1/2}$, Λ is the discrete Laplacian corresponding to $\mathcal{T}_h \cap \gamma$, $\sigma > 0$ is a scalling parameter. The BURA based non-overlapping DD preconditioner has optimal computational complexity. Key to the theory is the spectral equivalence between the energy norm associated with the Steklov-Poincaré operator on γ and the corresponding Sobolev norm of index $1/2$. Estimates are independent of the geometry of γ. The theoretical results are illustrated by numerical experiments.