Nonexistence of Global Solutions to Generalized Boussinesq Equation with Arbitrary Energy

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Outline

1. Introduction
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3. Finite time blow up for arbitrary high positive energy
    - Modification of the concavity method of Levine
    - Construction of the initial data
    - Numerical experiments

Boussinesq equation with linear restoring force ($\mathbf{BE_{lrf}}$)

$$\beta_2 u_{tt} - u_{xx} - \beta_1 u_{ttxx} + u_{xxxx} + mu + f(u)_{xx} = 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}$$
$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x),$$

$\beta_1 \geq 0, \quad \beta_2 > 0, \quad m \geq 0; \quad m = 0$ – generalized double dispersion equation

$$u_0 \in \mathrm{H}^1(\mathbb{R}), \ (-\Delta)^{-1/2} u_0 \in \mathrm{L}^2(\mathbb{R}) \quad u_1 \in \mathrm{H}^1(\mathbb{R}), \ (-\Delta)^{-1/2} u_1 \in \mathrm{L}^2(\mathbb{R})$$

Nonlinearities

$$f(u) = \sum_{k=1}^{l} a_k |u|^{p_k - 1} u - \sum_{j=1}^{s} b_j |u|^{q_j - 1} u, \qquad a_1 > 0, \ a_k \geq 0, \ b_j \geq 0$$

$$1 < q_s < q_{s-1} < \cdots < q_1 < p_1 < p_2 < \cdots < p_l$$

$f(u) = au^5 \pm bu^3, \ a > 0, \ b > 0$ – in the theory of atomic chains and shape memory alloys

Modeling

- A.M. Samsonov, Nonlinear waves in elastic waveguides, Springer, 1994
- A.D. Mishkis, P.M. Belotserkovskiy, *ZAMM Z. Angew. Math. Mech.* 79 (1999) 645–647 – transverse deflections of an elastic rod on an elastic foundation
- A.V. Porubov, Amplification of nonlinear strain waives in solids, World Scientific, 2003

Numerical Study

- C.I. Christov, T.T. Marinov, R.S. Marinova, *Math. Comp. Simulation* 80 (2009) 56–65
- M.A. Cristou, *AIP Conf. Proc.* 1404 (2011) 41–48

Theoretical Investigations

- N. Kutev, N. Kolkovska, M. Dimova, *AIP Conf. Proc.* 1629 (2014) 172–185
- N. Kutev, N. Kolkovska, M. Dimova, *Pliska Studia Mathematica Bulgarica* 23 (2014) 81–94

Notations: $\langle u,v\rangle = \beta_2\left((-\Delta)^{-1/2}u,(-\Delta)^{-1/2}v\right) + \beta_1(u,v)$

$u,v\in \mathrm{H}^1,\ (-\Delta)^{-1/2}u,(-\Delta)^{-1/2}v\in \mathrm{L}^2$

Conservation of full energy

$$E(t)=E(0) \qquad \forall\, t\in[0,T_m),$$

$$E(t)=\frac{1}{2}\left(\langle u_t,u_t\rangle + \|u\|^2_{\mathrm{H}^1} + m\left\|(-\Delta)^{-1/2}u\right\|^2_{\mathrm{L}^2}\right) - \int_{\mathbb{R}}\int_0^u f(y)\,dy\,dx$$

$E(0)\le 0$

$0<E(0)\le d$

$E(0)>d$

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- Non-positive energy →

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- Non-positive energy –
- Subcritical and critical energy – Potential Well Method –

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$E(0) \leq 0$ | $0 < E(0) \leq d$ | $E(0) > d$

- Non-positive energy → $E(0) \leq 0$
- Subcritical and critical energy – Potential Well Method → $0 < E(0) \leq d$
- Supercritical energy → $E(0) > d$

Potential well method ($0 < E(0) < d$)

- Payne LE, Sattinger DH, *Israel Journal of Mathematics*, 1975 – wave equation
- Wang S, Chen G, *Nonlinear Anal* 2006 – $m = 0$,
  $uf(u) \geq (2+\varepsilon)\int_0^u f(s)\,ds, \ \forall u \in R, \quad \varepsilon > 0$
- Liu Y, Xu R, *J Math Anal Appl* 2008 – $m = 0$, $f(u) = \pm|u|^p, \pm|u|^{p-1}u$
- Xu R, *Math Meth Appl Sci* 2011 – $m = 0$, $\beta_1 = 0$, combined power-type nonlinearity
- Polat N, Ertas A, *Math Anal Appl* 2009 – $m = 0$, damped Boussinesq equation

Potential energy functional $J(u)$:

$$J(u) = \frac{1}{2}\|u\|^2_{\mathrm{H}^1} + \frac{m}{2}\left\|(-\Delta)^{-1/2}u\right\|^2_{\mathrm{L}^2} - \int_{\mathbb{R}}\int_0^u f(y)\,dydx$$

Nehari functional $I(u)$:

$$I(u) = J'(u)u = I(u) = \|u\|^2_{\mathrm{H}^1} + m\left\|(-\Delta)^{-1/2}u\right\|^2_{\mathrm{L}^2} - \int_{\mathbb{R}}\int_0^u f(y)\,dydx$$

Depth of the potential well (critical energy constant) d:

$$d = \inf_{u\in\mathcal{N}} J(u), \qquad \mathcal{N} = \{u \in \mathrm{H}^1 : \ I(u) = 0, \ \|u\|_{\mathrm{H}^1} \neq 0\}$$

Potential well method ($0 < E(0) < d$)

Theorem

Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$, and $(-\Delta)^{-1/2}u_1 \in \mathrm{L}^2$. Assume that $0 < E(0) < d$.

- If $I(0) > 0$ or $\|u_0\|_{\mathrm{H}^1} = 0$ then the weak solution of $\mathbf{BE_{lrf}}$ is globally defined for $t \in [0, \infty)$;
- If $I(0) < 0$ then the weak solution of $\mathbf{BE_{lrf}}$ blows up for a finite time.

Lemma (Sign preserving property of $I(u(t))$)

If $0 < E(0) < d$ then the sign of the Nehari functional $I(u(t))$ is invariant under the flow of $\mathbf{BE_{lrf}}$.

Kalantarov–Ladyzhenskaya 1978 (Levine 1974, $\delta = 0$, $\mu = 0$)

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq -2\delta\Psi(t)\Psi'(t) - \mu\Psi^2(t), \qquad \gamma > 1,\ \delta \geq 0,\ \mu \geq 0$$

$$\Psi(t) = \langle u(t), u(t) \rangle$$

Arbitrary high positive energy

Theorem (Finite time blow up)

Suppose $u_0 \in \mathrm{H}^1$, $(-\Delta)^{-1/2}u_0 \in \mathrm{L}^2$, $u_1 \in \mathrm{L}^2$ and $(-\Delta)^{-1/2}u_1 \in \mathrm{L}^2$. If $m_0 = \min\left(\frac{m}{\beta_2}, \frac{1}{\beta_1}\right)^{1/2}$ and

$$\langle u_0, u_0\rangle \neq 0, \quad \langle u_0, u_1\rangle > 0, \quad \frac{m_0^2}{2}\frac{(p_1-1)}{(p_1+1)}\langle u_0, u_0\rangle + \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} > E(0) > 0$$

then the weak solution of $\mathbf{BE_{lrf}}$ blows up for a finite time $t_* < \infty$.

More precisely:

If $\frac{m_0^2}{2}\frac{(p_1-1)}{(p_1+1)}\langle u_0, u_0\rangle \geq E(0)$, then $t_* \leq \boldsymbol{t_*^1} = \frac{2}{(p_1-1)}\frac{\langle u_0, u_0\rangle}{\langle u_0, u_1\rangle} < \infty$;

If $\frac{m_0^2}{2}\frac{(p_1-1)}{(p_1+1)}\langle u_0, u_0\rangle \leq E(0)$, then

$$t_* \leq \boldsymbol{t_*^2} = \frac{\sqrt{2}}{(p_1-1)}\frac{\langle u_0, u_0\rangle}{\sqrt{\frac{m_0^2}{2}\frac{(p_1-1)}{(p_1+1)}\langle u_0, u_0\rangle + \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} - E(0)}}.$$

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$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq \alpha\Psi^2(t) - \beta\Psi(t), \qquad \gamma > 1,\ \alpha \geq 0,\ \beta > 0$$

Straughan 1975, Korpusov 2012

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq -\beta\Psi(t), \qquad \gamma > 1,\ \beta \geq 0$$

Lemma (Kutev, Kolkovska, Dimova 2015)

Suppose that a twice-differentiable function $\Psi(t)$ satisfies on $t \geq 0$ the following inequality and initial conditions:

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq \alpha\Psi^2(t) - \beta\Psi(t), \qquad \gamma > 1, \quad \alpha \geq 0, \quad \beta > 0,$$

$$\Psi(0) > 0, \qquad \Psi'(0) > 0, \qquad \beta < \frac{(2\gamma - 1)}{2}\frac{\Psi'^2(0)}{\Psi(0)} + \alpha\Psi(0).$$

Then $\Psi(t) \to \infty$ for $t \to t_* < \infty$.

- Cai Donghong, Ye Jianjun, Blow-up of solution to Cauchy problem for the generalized damped Boussinesq equation, *WSEAS Transactions on Mathematics*, 13 (2014), 122–131, $m > 0$

$$\frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} > E(0) > 0$$

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq -\beta\Psi(t), \qquad \gamma > 1, \ \beta \geq 0$$

- Kutev, Kolkovska, Dimova 2015

$$\frac{m_0^2}{2}\frac{(p-1)}{(p+1)}\langle u_0, u_0\rangle + \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} > E(0) > 0$$

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq \alpha\Psi^2(t) - \beta\Psi(t), \qquad \gamma > 1, \ \alpha \geq 0, \ \beta > 0$$

Construction of initial data

Theorem

For every positive constant K there exist infinitely many initial data u_0^K, u_1^K such that $E(u_0^K, u_1^K) = K$ and the existing time for the corresponding solution u^K is finite.

Choice of initial data

$$u_0^K(x) = qw'(x), \quad u_1^K(x) = qw'(x) + sv'(x),$$

$$w(x), v(x) \in \mathrm{H}^2(\mathbb{R}) : \|w\|_{\mathrm{H}^2} \neq 0, \quad \|v\|_{\mathrm{H}^2} \neq 0, \quad \langle w, v\rangle = 0, \quad \langle w', v'\rangle = 0$$

q, s – positive constants

- $\langle u_0^K, u_0^K\rangle \neq 0, \quad \langle u_0^K, u_1^K\rangle > 0, \quad \dfrac{m_0^2(p_1-1)}{2(p_1+1)}\langle u_0^K, u_0^K\rangle + \dfrac{\langle u_0^K, u_1^K\rangle^2}{2\langle u_0^K, u_0^K\rangle} > E(0) > 0$
- $E(u_0^K, u_1^K) = E(0) > K,$ $\quad$ K is an arbitrary positive constant

Family of finite difference schemes, $f(u) = a|u|^{p-1}u, a > 0$

$$B\left(\frac{v_i^{n+1} - 2v_i^n + v_i^{n-1}}{\tau^2}\right) - \Lambda^{xx} v_i^n + \beta_2 \Lambda^{xxxx} v_i^n - m v_i^n$$
$$- \frac{a}{(p+1)}\Lambda^{xx}\left(\frac{|v_i^{n+1}|^{(p+1)} - |v_i^{n-1}|^{(p+1)}}{v_i^{n+1} - v_i^{n-1}}\right) = 0,$$
$$B = (\beta_2 + \frac{m}{2}\tau^2)I - (\beta_1 + \sigma\tau^2)\Lambda^{xx} + \sigma\tau^2\beta_2\Lambda^{xxxx}$$

- v_i^n – a discrete approximation to u at (x_i, t_n), τ is a time-step
- $\Lambda^{xx} v_i = \frac{v_{i+1} - 2vu_i + v_{i-1}}{h^2}$, $\quad \Lambda^{xxxx} v_i = \dfrac{v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}}{h^4}$, $\quad \sigma$ - parameter

Properties:

- **Convergence:** The schemes have second order of convergence in space and time $O(h^2 + \tau^2)$.
- **Stability:** The schemes are unconditionally stable for $\sigma > 1/4$.
- **Conservativeness:** The discrete energy is conserved in time, i.e. $E_h(v^{(n)}) = E_h(v^{(0)}), \quad n = 1, 2, \ldots$

Example 1: $\beta_1 = \beta_2 = 1$, $m = 1$, $m_0 = 1$, $a = 2$, $p = 3$ $(f(u) = 2u^3)$, $K = 10 > d$

$$u_0(x) = qw'(x), \quad u_1(x) = qw'(x) + sv'(x)$$

$$w(x) = -\frac{2}{e^{2x}+1} + \frac{1}{e^{(0.4x-2)}+1} + \frac{1}{e^{(0.4x+2)}+1}, \qquad v(x) = w'(x)$$

$$q = 1.2, \quad s = 5.78, \quad E_h(0) \approx 10.50249147 > d$$

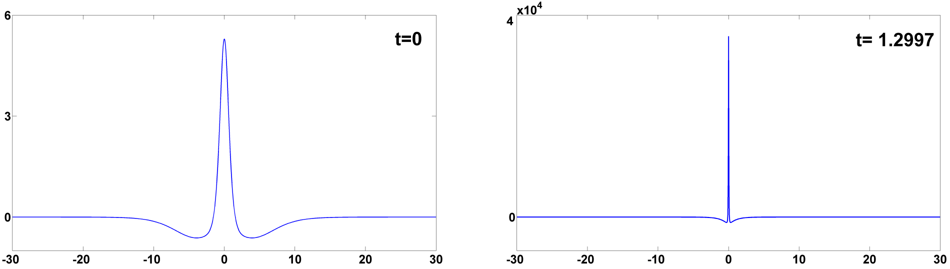


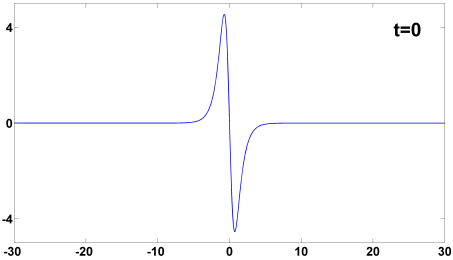
Figure: Profiles of the numerical solution $u(x,t)$ at evolution times $t = 0$ and t=1.2997; $\widetilde{t}_* = 1.2998$ – blow up time ($\tau = 0.0001$).

Example 2: $\beta_1 = \beta_2 = 1$, $m = 1$, $m_0 = 1$, $a = 2$, $p = 3$ $(f(u) = 2u^3)$, $K = 20 > d$

$$u_0(x) = qw'(x), \quad u_1(x) = qw'(x) + sv'(x)$$

$$w(x) = (\cosh(x))^{-1}, \quad v(x) = -\sinh(x)\tanh(x)$$

$$q = 0.85, \quad s = 3.64, \quad E_h(0) \approx 20.09279508 > d$$



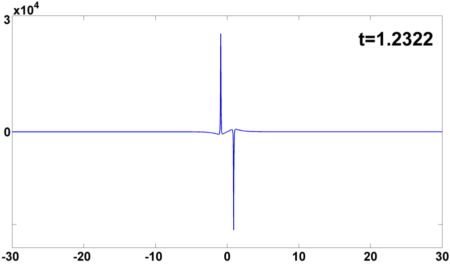


Figure: Profiles of the numerical solution $u(x,t)$ at evolution times $t = 0$ and t=1.2322; $\widetilde{t}_* = 1.2323$ – blow up time ($\tau = 0.0001$)

Thank you
for your attention!