On N-soliton Interactions of Gross-Pitaevsky Equation in Two Space-time Dimensions

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Outline

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- Variational Approach and Perturbed CTC
- Non-perturbed and Perturbed CNSE. Manakov System.
- Choice of Initial Conditions and Potential Perturbations
- Effects of Polarization Vectors
- Conservative Numerical Method vs Variational Approach
- Main Results and Discussion

Gross-Pitaevsky Equation. Adiabatic Approximation

We consider the N-soliton interactions of the two-component Gross-Pitaevsky eq. in two space-time dimensions:

$$i\vec{u}_t + \frac{1}{2}\vec{u}_{xx} + (\vec{u}^{\,\dagger}, \vec{u})\vec{u} = V(x)\vec{u}(x,t) + c_1\sigma_1\vec{u}(x,t), \;\; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

in the presence of inter-channel interaction, i.e., $c_1 \neq 0$. This equation describes quasi-one-dimensional Bose-Einstein condensates. In the adiabatic approximation the propagation of the soliton trains of eq. (1) is described by a perturbed complex Toda chain (PCTC).

We demonstrate that PCTC correctly models the effects of several types of potentials:
a) harmonic $V(x) = v_2 x^2 + v_1 x + v_0$;
b) periodic $V(x) = A\cos(\Omega x + \Omega_0)$;
c) shallow potential wells $V(x) = c_0(\tanh(x - x_{\mathrm{f}}) - \tanh(x - x_{\mathrm{in}}))$, $c_0 \ll 1$ and $x_{\mathrm{in}} < x_{\mathrm{f}}$.
We demonstrate that the PCTC adequately models the soliton train dynamics for a wide region of the initial soliton parameters as well as for $c_1 \neq 0$.

Idea of Adiabatic Approximation

The idea of the adiabatic approximation to the soliton interactions (Karpman&Solov'ev (1981)) led to effective modeling of the N-soliton trains of the perturbed scalar NLS eq.:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u(x,t) = iR[u]. \qquad (2)$$

By N-soliton train we mean a solution of the NLSE (2) with initial condition

$$u(x, t=0) = \sum_{k=1}^{N} u_k(x, t=0), \qquad (3)$$

$$u_k(x,t) = 2\nu_k \mathrm{e}^{i\phi_k} \mathrm{sech} z_k, \quad z_k = 2\nu_k(x - \xi_k(t)), \quad \xi_k(t) = 2\mu_k t + \xi_{k,0},$$

$$\phi_k = \frac{\mu_k}{\nu_k} z_k + \delta_k(t), \quad \delta_k(t) = 2(\mu_k^2 + \nu_k^2)t + \delta_{k,0}.$$

Here μ_k are the amplitudes, ν_k – the velocities, δ_k – the phase shifts, ξ_k - the centers of solitons.

Idea of Adiabatic Approximation

The adiabatic approximation holds if the soliton parameters satisfy the restrictions

$$|\nu_k - \nu_0| \ll \nu_0, \; |\mu_k - \mu_0| \ll \mu_0, \; |\nu_k - \nu_0||\xi_{k+1,0} - \xi_{k,0}| \gg 1, \quad (4)$$

where ν_0 and μ_0 are the average amplitude and velocity respectively. In fact we have two different scales:

$$|\nu_k - \nu_0| \simeq \varepsilon_0^{1/2}, \quad |\mu_k - \mu_0| \simeq \varepsilon_0^{1/2}, \quad |\xi_{k+1,0} - \xi_{k,0}| \simeq \varepsilon_0^{-1}.$$

In this approximation the dynamics of the N-soliton train is described by a dynamical system for the $4N$ soliton parameters.

Perturbed Manakov System and Perturbed Vector Complex Toda Chain

In the present paper we generalize the above results to the perturbed vector NLS

$$i\vec{u}_t + \frac{1}{2}\vec{u}_{xx} + (\vec{u}^\dagger, \vec{u})\vec{u}(x,t) = iR[\vec{u}]. \qquad (5)$$

The corresponding vector N-soliton train is determined by the initial condition

$$\vec{u}(x,t=0) = \sum_{k=1}^{N} \vec{u}_k(x,t=0), \quad \vec{u}_k(x,t) = 2\nu_k e^{i\phi_k} \operatorname{sech} z_k \vec{n}_k, \quad (6)$$

and the amplitudes, the velocities, the phase shifts, and the centers of solitons are as in Eq.(3). The phenomenology, however, is enriched by introducing a constant polarization vectors $\vec{n}_k$ that are normalized by the conditions

$$\langle \vec{n}_k^\dagger, \vec{n}_k \rangle = 1, \qquad \sum_{s=1}^{n} \arg \vec{n}_{k;s} = 0.$$

More precisely after involving these vectors we derive a generalized version of the CTC (GCTC) model, which allows to have in mind the polarization effects in the N-soliton train of the vector NLS.

Manakov solitons and the CTC model

The dynamical system that describes the evolution of the Manakov soliton trains is

$$\frac{d(\mu_k + i\nu_k)}{dt} = 4\nu_0 \left[\langle \vec{n}_k, \vec{n}_{k-1} \rangle e^{q_k - q_{k-1}} - \langle \vec{n}_{k+1}, \vec{n}_k \rangle e^{q_{k+1} - q_k} \right],$$

$$\frac{dq_k}{dt} = -4\nu_0(\mu_k + i\nu_k), \tag{7}$$

where

$$q_k = -2\nu_0\xi_k + k \ln 4\nu_0^2 - i(\delta_k + \delta_0 + k\pi - 2\mu_0\xi_k),$$

$$\nu_0 = \frac{1}{N}\sum_{s=1}^{N} \nu_s, \qquad \mu_0 = \frac{1}{N}\sum_{s=1}^{N} \mu_s, \qquad \delta_0 = \frac{1}{N}\sum_{s=1}^{N} \delta_s. \tag{8}$$

Thus we get the CTC:

$$\frac{d^2 q_k}{dt^2} = 16\nu_0^2 \left[\langle \vec{n}_{k+1}, \vec{n}_k \rangle e^{q_{k+1} - q_k} - \langle \vec{n}_k, \vec{n}_{k-1} \rangle e^{q_k - q_{k-1}} \right], \tag{9}$$

All terms in the right hand sides of the evolution equations for $\vec{n}_k$ are of the order of ϵ, so we can neglect the evolution of $\vec{n}_k$ and to approximate them with their initial values. It is easy to see, that if all $\langle \vec{n}_{k+1}{}^{\dagger}, \vec{n}_k \rangle = \mathrm{const} \neq 0$ then the CTC (9) is a completely integrable dynamical system, just like the real Toda chain.

In order to have in mind the effects of the external potentials we consider the perturbed CTC system:

$$\frac{d\lambda_k}{dt} = -4\nu_0 \left(e^{q_{k+1}-q_k} \langle \vec{n}^\dagger_{k+1}, \vec{n}_k \rangle - e^{q_k - q_{k-1}} \langle \vec{n}^\dagger_k, \vec{n}_{k-1} \rangle \right) + M_k + iN_k,$$
$$\frac{dq_k}{dt} = -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \qquad \frac{d\vec{n}_k}{dt} = \mathcal{O}(\epsilon), \tag{10}$$

where $\lambda_k = \mu_k + i\nu_k$, $X_k = 2\mu_k \Xi_k + D_k$ and

Perturbed Vector Complex Toda Chain Model

$$N_k = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{dz_k}{\cosh z_k}\,\mathfrak{I}(z_k), \quad M_k = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz_k\ \sinh z_k}{\cosh^2 z_k}\,\mathfrak{R}(z_k),$$

$$\Xi_k = -\frac{1}{4\nu_k^2}\int_{-\infty}^{\infty} \frac{dz_k\ z_k}{\cosh z_k}\,\mathfrak{I}(z_k), \; D_k = \frac{1}{2\nu_k}\int_{-\infty}^{\infty} \frac{dz_k\,(1 - z_k\tanh z_k)}{\cosh z_k}\mathfrak{R}(z_k),$$

$$\mathfrak{I}(z_k) = \left(V(y_k)u_k \mathrm{e}^{-i\phi_k}\right), \quad \mathfrak{R}(z_k) = \left(V(y_k)u_k \mathrm{e}^{-i\phi_k}\right)$$

where $y_k = z_k/(2\nu_0) + \xi_k$.

The corresponding integrals take the form $N_k = 0$, $\Xi_k = 0$ and:

$$M_k = 2c_s\nu_k(P_0(z_k - y_\mathrm{f}) - P_0(z_k - y_\mathrm{i}),$$

$$P_0(w) = \frac{\sinh(w) - w\cosh(w)}{\sinh^2(w)},$$

$$D_k = \frac{2c_s}{\nu_k}(R_0(z_k - y_\mathrm{f}) - R_0(z_k - y_\mathrm{i}) - \frac{c_1}{4}\sin(2\theta_k)\cos(2\beta_k),$$

$$R_0(w) = \frac{e^{-w}\sinh^2(w) + w^2\cosh(w) - 2w\sinh(w)}{2\sinh^2(w)}.$$

Effects of the polarization vectors on the soliton interaction

We formulate a condition on $\vec{n}_s$ that is compatible with the adiabatic approximation. We also formulate the conditions on the initial vector soliton parameters responsible for the different asymptotic regimes.

The CTC is completely integrable model; it allows Lax representation $L_t = [A.L]$, where:

$$
\begin{aligned}
L &= \sum_{s=1}^{N} \left(b_s E_{ss} + a_s (E_{s,s+1} + E_{s+1,s}) \right), \\
A &= \sum_{s=1}^{N} \left(a_s (E_{s,s+1} - E_{s+1,s}) \right),
\end{aligned}
\qquad (11)
$$

where $a_s = \exp((Q_{s+1} - Q_s)/2)$, $b_s = \mu_{s,t} + i\nu_{s,t}$ and the matrices E_{ks} are determined by $(E_{ks})_{pj} = \delta_{kp}\delta_{sj}$. The eigenvalues of L are integrals of motion and determine the asymptotic velocities.

The GCTC is also a completely integrable model because it allows Lax representation $\tilde{L}_t = [\tilde{A}.\tilde{L}]$, where:

$$\begin{aligned} \tilde{L} &= \sum_{s=1}^{N} \left(\tilde{b}_s E_{ss} + \tilde{a}_s (E_{s,s+1} + E_{s+1,s}) \right), \\ \tilde{A} &= \sum_{s=1}^{N} \left(\tilde{a}_s (E_{s,s+1} - E_{s+1,s}) \right), \end{aligned} \tag{12}$$

where $\tilde{a}_s = m_{0k}^2 e^{2i\phi_{0k}} a_s$, $b_s = \mu_{s,t} + i\nu_{s,t}$. Like for the scalar case, the eigenvalues of $\tilde{L}$ are integrals of motion. If we denote by $\zeta_s = \kappa_s + i\eta_s$ (resp. $\tilde{\zeta}_s = \tilde{\kappa} + i\tilde{\eta}_s$) the set of eigenvalues of L (resp. $\tilde{L}$) then their real parts κ_s (resp. $\tilde{\kappa}_s$) determine the asymptotic velocities for the soliton train described by CTC (resp. GCTC).

While for the RTC the set of eigenvalues ζ_s of the Lax matrix are all real, for the CTC they generically take complex values, e.g., $\zeta_s = \kappa_s + \mathrm{i}\eta_s$.
Hence, the only possible asymptotic behavior in the RTC is an asymptotically separating, free motion of the solitons. In opposite, for the CTC the real parts $\kappa_s \equiv \mathfrak{Re}\zeta_s$ of eigenvalues of the Lax matrix ζ_s determines the asymptotic velocity of the sth soliton.

Effects of the polarization vectors on the soliton interaction

Thus, starting from the set of initial soliton parameters we can calculate $L|_{t=0}$ (resp. $\tilde{L}|_{t=0}$), evaluate the real parts of their eigenvalues and thus determine the asymptotic regime of the soliton train.

- **Regime (i)** $\kappa_k \neq \kappa_j$ $(\tilde{\kappa}_k \neq \tilde{\kappa}_j)$ for $k \neq j$ – asymptotically separating, free solitons;
- **Regime (ii)** $\kappa_1 = \kappa_2 = \cdots = \kappa_N = 0$ $(\tilde{\kappa}_1 = \tilde{\kappa}_2 = \cdots = \tilde{\kappa}_N = 0)$ – a "bound state;"
- **Regime (iii)** group of particles move with the same mean asymptotic velocity and the rest of the particles will have free asymptotic motion.

Varying only the polarization vectors one can change the asymptotic regime of the soliton train.

Effects of the external potentials on the GCTC. Numeric checks vs Variational approach

The predictions and validity of the CTC and GCTC are compared and verified with the numerical solutions of the corresponding CNSE using fully explicit difference scheme of Crank-Nicolson type, which conserves the energy, the mass, and the pseudomomentum. The scheme is implemented in a complex arithmetics. Such comparison is conducted for all dynamical regimes considered.

- First we study the soliton interaction of the pure Manakov model (without perturbations, $V(x) \equiv 0$) and with vanishing cross-modulation $\alpha_2 = 0$;
- 5-soliton configurations and transitions between different asymptotic regimes;
- 5-soliton configurations and transitions under the effect of external potential and evaluate its effect as a perturbation.

Criterium for perturbation

The effect of the potential on the soliton train can be viewed as an adiabatic perturbation, namely $|H_{\rm V}| \ll |H_0|$

$$H_{\rm V} = \int_{-\infty}^{\infty} dx\ V(x)(\vec{u}^\dagger, \vec{u})(x,t),$$
$$H_0 = \int_{-\infty}^{\infty} dx\ \left((\vec{u}_x^\dagger, \vec{u}_x) - \frac{1}{2}(\vec{u}^\dagger, \vec{u})^2 \right). \qquad (13)$$

If the ratio $|H_{\rm V}|/|H_0| \simeq \varepsilon$ then the PCTC matches very well the soliton trajectories of the perturbed Manakov model. If this ratio is of the order of 1 then the potential strongly prevails the soliton interactions and determines the soliton dynamics.

Remark: H_0 and $H_{\rm V}$ depend on the soliton parameters. The leading term of H_0, however, depends only on ν_k and μ_k, while $H_{\rm V}$ depends substantially also on the positions ξ_k of the solitons and their number.

Criterium for perturbation

$$
\begin{aligned}
H_0 &\simeq 8 \sum_{k=1}^{N} \left(\mu_k^2 \nu_k^2 - \frac{\nu_k^3}{3} \right), \\
H_{V_{\mathrm{anh}}} &= \sum_{k=1}^{N} \left(4\nu_k V_{\mathrm{anh}}(\xi_k) + \frac{\pi^2}{24\nu_k} V''_{\mathrm{anh}}(\xi_k) + \frac{7\pi^4}{960\nu_k^3} V_4 \right); \\
H_{V_{\mathrm{per}}} &= \sum_{k=1}^{N} \frac{\pi A \Omega \cos(\Omega \xi_k + \Omega_0)}{\sinh\left(\frac{\pi\Omega}{4\nu_k}\right)}; \\
H_{V\,\mathrm{ww}} &= \int_{-\infty}^{\infty} V(x) |u_{NS}|^2 = \int_{-\infty}^{\infty} V(x) \sum_k |u_k|^2 = -c_0 H_0.
\end{aligned}
\tag{14}
$$

Example 1

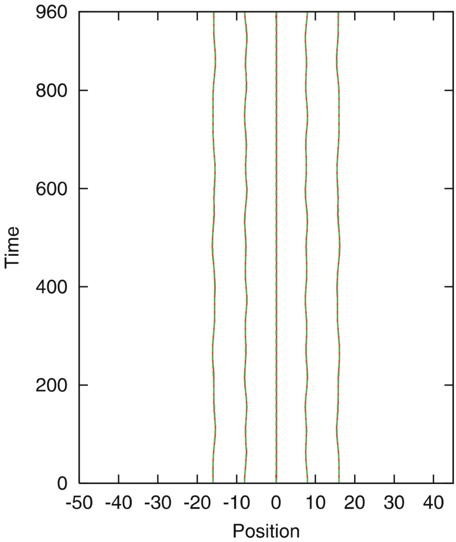

Figure: 1. $V_{H_\mathrm{h}} = 0.000036x^2$, $\frac{|H_{V_\mathrm{h}}|}{|H_0|} \approx 2.78\%$.

Example 2

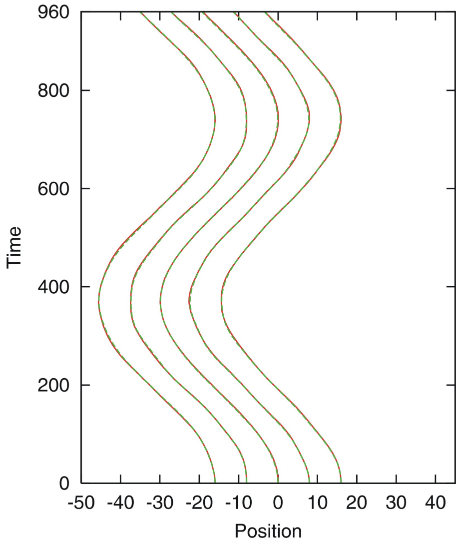

Figure: 2. $V_{H_\mathrm{h}} = 0.000036(x+15)^2$, $\frac{|H_{V_\mathrm{h}}|}{|H_0|} \approx 7.64\%$.

Example 3

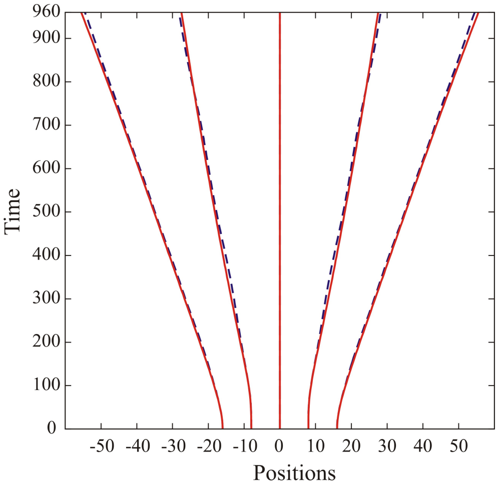

Figure: 3. $V_{H_{\rm per}} = A\cos\frac{\pi}{4}x$, $A = -0.0001$, $\frac{|H_{V_{\rm per}}|}{|H_0|} \approx 0.047\%$.

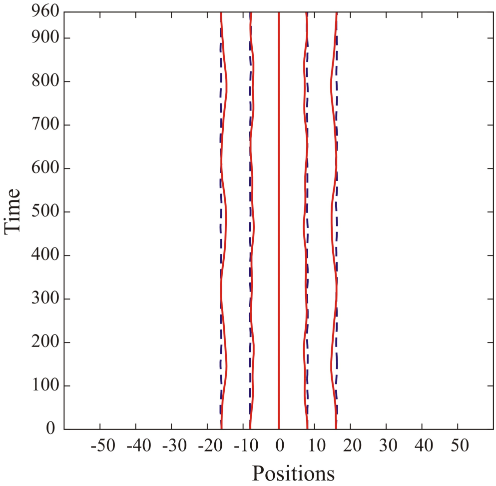

Figure: 4. $V_{H_{\rm per}} = A\cos\frac{\pi}{4}x,\ A = -0.0075,\ \frac{|H_{V_{\rm per}}|}{|H_0|} \approx 3.53\%.$

Example 5

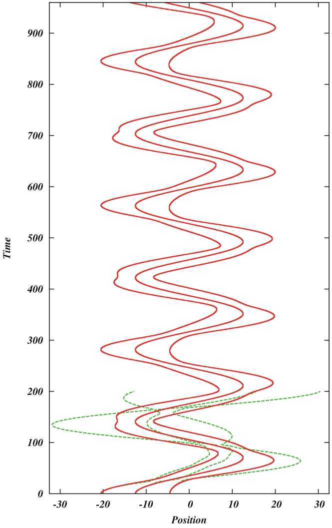

Figure: 5. $V(x) = 0.001x^2$, $\frac{|H_{V_h}|}{|H_0|} = 1198\%$

Example 6

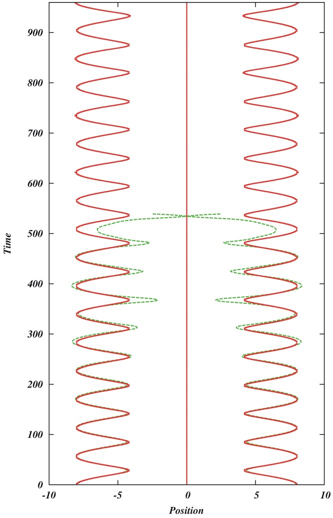

Figure: 6. $V_{H_{\mathrm{h}}} = 0.001x^2$, $\frac{|H_{V_{\mathrm{h}}}|}{|H_0|} = 26.1\%$

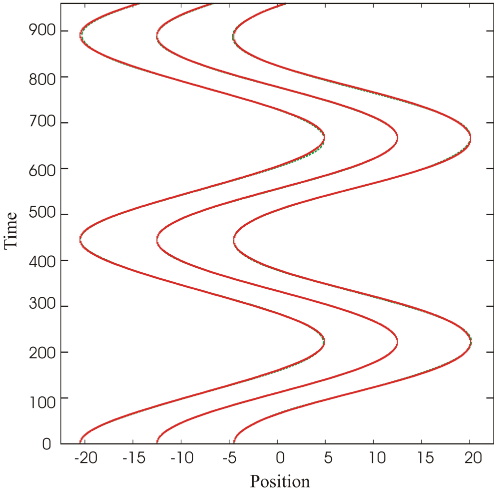

Figure: 7. $V_{H_\mathrm{h}} = 0.0001x^2$, $\frac{|H_{V_\mathrm{h}}|}{|H_0|} = 11.98\%$.

Example 8

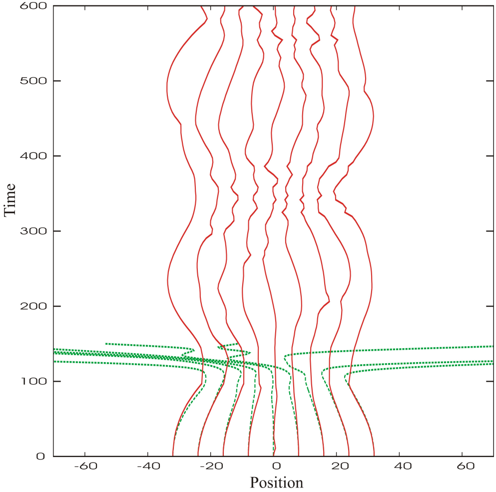

Figure: 8. $V_{H_\mathrm{h}} = 0.000036x^2$, $\frac{|H_{V_\mathrm{h}}|}{|H_0|} = 12.87\%$.

Example 9

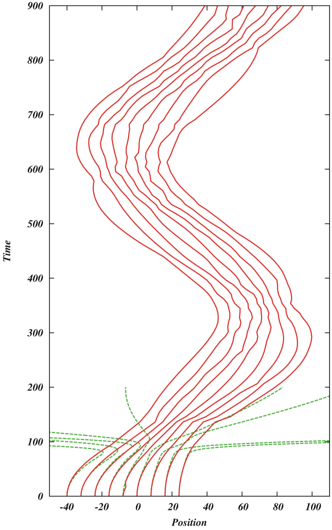

Figure: 9. $V_{H_\mathrm{h}} = 0.00005(x - 32)^2$, $\frac{|H_{V_\mathrm{h}}|}{|H_0|} = 60.82\%$.

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Thanks

Thank you for your kind attention !