

# Multisoliton Interactions for the Manakov System under Composite External Potentials

Michail D. Todorov

Department of Applied Mathematics and Computer Science  
Technical University of Sofia, Bulgaria

*Work done in collaboration with V. Gerdjikov and A. Kyuldjiev from  
INRNE-BAS, Sofia*

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# NLS & Soliton Trains

The idea of the adiabatic approximation to the soliton interactions (Karpman&Solov'ev (1981)) led to effective modeling of the  $N$ -soliton trains of the perturbed scalar NLS eq.:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2u(x, t) = iR[u]. \quad (1)$$

By  $N$ -soliton train we mean a solution of the NLSE (1) with initial condition

$$u(x, t = 0) = \sum_{k=1}^N u_k(x, t = 0), \quad (2)$$

$$u_k(x, t) = 2\nu_k e^{i\phi_k} \operatorname{sech} z_k, \quad z_k = 2\nu_k(x - \xi_k(t)), \quad \xi_k(t) = 2\mu_k t + \xi_{k,0},$$

$$\phi_k = \frac{\mu_k}{\nu_k} z_k + \delta_k(t), \quad \delta_k(t) = 2(\mu_k^2 + \nu_k^2)t + \delta_{k,0}.$$

Here  $\mu_k$  are the amplitudes,  $\nu_k$  – the velocities,  $\delta_k$  – the phase shifts,  $\xi_k$  – the centers of solitons.

# NLS & Adiabatic Approximation

The adiabatic approximation holds if the soliton parameters satisfy the restrictions

$$|\nu_k - \nu_0| \ll \nu_0, \quad |\mu_k - \mu_0| \ll \mu_0, \quad |\nu_k - \nu_0| |\xi_{k+1,0} - \xi_{k,0}| \gg 1, \quad (3)$$

where  $\nu_0$  and  $\mu_0$  are the average amplitude and velocity respectively. In fact we have two different scales:

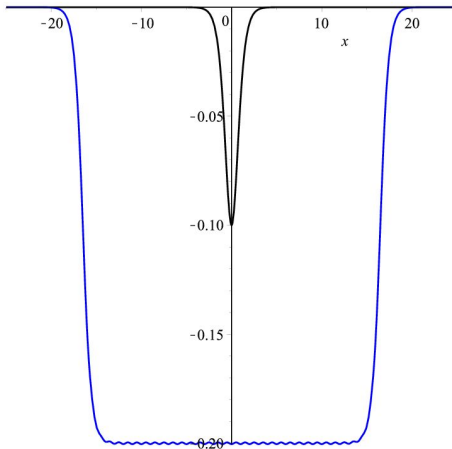
$$|\nu_k - \nu_0| \simeq \varepsilon_0^{1/2}, \quad |\mu_k - \mu_0| \simeq \varepsilon_0^{1/2}, \quad |\xi_{k+1,0} - \xi_{k,0}| \simeq \varepsilon_0^{-1}.$$

In this approximation the dynamics of the  $N$ -soliton train is described by a dynamical system for the  $4N$  soliton parameters. In our previous works we investigate it in presence of periodic and polynomial potentials. Now, we are interested in what follow perturbation(s) by external sech-potentials:

$$iR[u] \equiv V(x)u(x, t), \quad V(x) = \sum_s c_s \operatorname{sech}^2(2\nu_0 x - y_s). \quad (4)$$

The latter allows us to realize the idea about localized potential wells(depressions) and humps.

# Potentials



**Figure:** 1. Single sech-potential vs. composite potential well  
 $V(x) = \sum_{s=0}^{32} c_s \operatorname{sech}^2(x - x_s)$ ,  $c_s = -10^{-1}$ ,  $x_s = -16 + sh$ ,  $h = 1$ ,  
 $s = 0, \dots, 32$ .

# Perturbed Vector NLS

Here we generalize to the perturbed vector NLS

$$i\vec{u}_t + \frac{1}{2}\vec{u}_{xx} + (\vec{u}^\dagger, \vec{u})\vec{u}(x, t) = iR[\vec{u}]. \quad (5)$$

The corresponding vector  $N$ -soliton train is determined by the initial condition

$$\vec{u}(x, t = 0) = \sum_{k=1}^N \vec{u}_k(x, t = 0), \quad \vec{u}_k(x, t) = 2\nu_k e^{i\phi_k} \operatorname{sech} z_k \vec{n}_k, \quad (6)$$

and the amplitudes, the velocities, the phase shifts, and the centers of solitons are as in Eq.(2). The phenomenology, however, is enriched by introducing a constant polarization vectors  $\vec{n}_k$  that are normalized by the conditions

$$(\vec{n}_k^\dagger, \vec{n}_k) = 1, \quad \sum_{s=1}^n \arg \vec{n}_{k;s} = 0.$$

# Variational approach and evolution

We use the variational approach (Anderson and Lisak (1986)) and derive the GCTC model. Like the (unperturbed) CTC, GCTC is a finite dimensional completely integrable model allowing Lax representation.

The Lagrangian of the vector NLS perturbed by external potential is:

$$\begin{aligned}\mathcal{L}[\vec{u}] &= \int_{-\infty}^{\infty} dt \frac{i}{2} \left[ (\vec{u}^\dagger, \vec{u}_t) - (\vec{u}_t^\dagger, \vec{u}) \right] - H, \\ H[\vec{u}] &= \int_{-\infty}^{\infty} dx \left[ -\frac{1}{2}(\vec{u}_x^\dagger, \vec{u}_x) + \frac{1}{2}(\vec{u}^\dagger, \vec{u})^2 - (\vec{u}^\dagger, \vec{u})V(x) \right].\end{aligned}\tag{7}$$

Then the Lagrange equations of motion:

$$\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \vec{u}_t^\dagger} - \frac{\delta \mathcal{L}}{\delta \vec{u}^\dagger} = 0,\tag{8}$$

coincide with the vector NLS with external potential  $V(x)$ .

# Variational approach and evolution

Next we insert  $\vec{u}(x, t) = \sum_{k=1}^N \vec{u}_k(x, t)$  (see eq. (6)) and integrate over  $x$  neglecting all terms of order  $\epsilon$  and higher.

Thus after long calculations we obtain:

$$\mathcal{L} = \sum_{k=1}^N \mathcal{L}_k + \sum_{k=1}^N \sum_{n=k\pm 1} \tilde{\mathcal{L}}_{k,n}, \quad \mathcal{L}_{k,n} = 16\nu_0^3 e^{-\Delta_{k,n}} (R_{k,n} + R_{k,n}^*),$$

$$R_{k,n} = e^{i(\tilde{\delta}_n - \tilde{\delta}_k)} (\vec{n}_k^\dagger \vec{n}_n), \quad \tilde{\delta}_k = \delta_k - 2\mu_0 \xi_k,$$

$$\Delta_{k,n} = 2s_{k,n}\nu_0(\xi_k - \xi_n) \gg 1, \quad s_{k,k+1} = -1, \quad s_{k,k-1} = 1. \quad (9)$$

where

$$\begin{aligned} \mathcal{L}_k = & -2i\nu_k \left( (\vec{n}_{k,t}^\dagger, \vec{n}_k) - (\vec{n}_k^\dagger, \vec{n}_{k,t}) \right) + 8\mu_k \nu_k \frac{d\xi_k}{dt} \\ & - 4\nu_k \frac{d\delta_k}{dt} + \dots \end{aligned} \quad (10)$$



The equations of motion are given by:

$$\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta p_{k,t}} - \frac{\delta \mathcal{L}}{\delta p_k} = 0, \quad (11)$$

where  $p_k = \{\delta_k, \xi_k, \mu_k, \nu_k, \vec{n}_k^\dagger\}$ .

Let  $\lambda_k = \mu_k + i\nu_k$ ,  $X_k = 2\mu_k \Xi_k + D_k$  and

$$Q_k = -2\nu_0 \xi_k + k \ln 4\nu_0^2 - i(\delta_k + \delta_0 + k\pi - 2\mu_0 \xi_k). \quad (12)$$

Then:

$$\begin{aligned}\frac{d\lambda_k}{dt} &= -4\nu_0 \left( e^{Q_{k+1}-Q_k} (\vec{n}_{k+1}^\dagger, \vec{n}_k) - e^{Q_k-Q_{k-1}} (\vec{n}_k^\dagger, \vec{n}_{k-1}) \right) + M_k + iN_k, \\ \frac{dQ_k}{dt} &= -4\nu_0 \lambda_k + 2i(\mu_0 + i\nu_0) \Xi_k - iX_k,\end{aligned}\tag{13}$$

where

$$N_k[u] = \frac{1}{2} \Re \int_{-\infty}^{\infty} R[u_k] \operatorname{sech} z_k e^{-i\phi_k} dz_k,$$

$$M_k[u] = \frac{1}{2} \Im \int_{-\infty}^{\infty} R[u_k] \tanh z_k \operatorname{sech} z_k e^{-i\phi_k} dz_k,$$

$$\Xi_k[u] = \frac{1}{4} \Re \int_{-\infty}^{\infty} R[u_k] z_k \operatorname{sech} z_k e^{-i\phi_k} dz_k,$$

$$D_k[u] = \frac{1}{2\nu_k} \Im \int_{-\infty}^{\infty} R[u_k] (1 - z_k \tanh z_k) \operatorname{sech} z_k e^{-i\phi_k} dz_k,$$

So, we have a generalization of CTC:

$$\begin{aligned}\frac{d\lambda_k}{dt} &= -4\nu_0 \left( e^{Q_{k+1}-Q_k} (\vec{n}_{k+1}^\dagger, \vec{n}_k) - e^{Q_k-Q_{k-1}} (\vec{n}_k^\dagger, \vec{n}_{k-1}) \right) + M_k + iN_k, \\ \frac{dQ_k}{dt} &= -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \quad \frac{d\vec{n}_k}{dt} = \mathcal{O}(\epsilon),\end{aligned}\tag{14}$$

The explicit form of  $M_k$ ,  $N_k$ ,  $\Xi_k$  and  $D_k$  for the potential chosen is given by

$$\begin{aligned}M_k &= \sum_s 2c_s \nu_k P(\Delta_{k,s}), & N_k &= 0, \\ \Xi_k &= 0, & D_k &= \sum_s c_s R(\Delta_{k,s}).\end{aligned}\tag{15}$$

where  $\Delta_{k,s} = 2\nu_0\xi_k - y_s$  and the functions  $P(\Delta)$  and  $R(\Delta)$  are known explicitly.

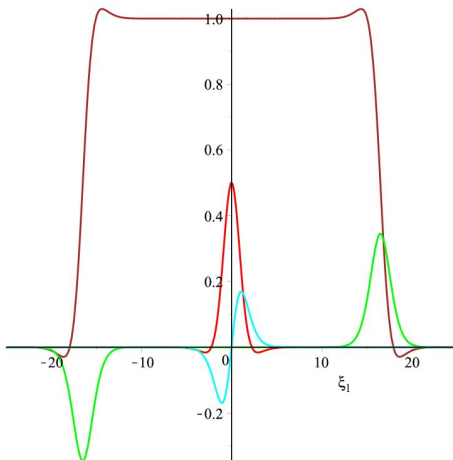


Figure: 2.  $P$  and  $R$  functions: for a single sech-potential and for the composite potential.

For the potential well

$$V(x, y_s) = c_s [\tanh(2\nu_0 x - y_i) - \tanh(2\nu_0 x - y_f)]$$

functions

$$P(\Delta) = \frac{\sinh \Delta - \Delta \cosh \Delta}{\sinh^3 \Delta}$$

and

$$R(\Delta) = \frac{\exp(-3\Delta) + (4\Delta^2 - 1) \cosh \Delta + (3 - 8\Delta) \sinh \Delta}{8 \sinh^3 \Delta}.$$

# Lax representation of CTC

The CTC is completely integrable model; it allows Lax representation  $L_t = [A.L]$ , where:

$$\begin{aligned} L &= \sum_{s=1}^N (b_s E_{ss} + a_s (E_{s,s+1} + E_{s+1,s})), \\ A &= \sum_{s=1}^N (a_s (E_{s,s+1} - E_{s+1,s})), \end{aligned} \tag{16}$$

where  $a_s = \exp((Q_{s+1} - Q_s)/2)$ ,  $b_s = \frac{1}{2} (\mu_{s,t} + i\nu_{s,t})$  and the matrices  $E_{ks}$  are determined by  $(E_{ks})_{pj} = \delta_{kp} \delta_{sj}$ . The eigenvalues of  $L$  are integrals of motion and determine the asymptotic velocities.

The GCTC is also a completely integrable model because it allows Lax representation  $\tilde{L}_t = [\tilde{A}, \tilde{L}]$ , where:

$$\begin{aligned}\tilde{L} &= \sum_{s=1}^N (b_s E_{ss} + \tilde{a}_s (E_{s,s+1} + E_{s+1,s})), \\ \tilde{A} &= \sum_{s=1}^N (\tilde{a}_s (E_{s,s+1} - E_{s+1,s})),\end{aligned}\tag{17}$$

where  $\tilde{a}_s = m_{0k} e^{i\phi_{0k}} a_s$ ,  $b_s = \mu_{s,t} + i\nu_{s,t}$ . Like for the scalar case, the eigenvalues of  $\tilde{L}$  are integrals of motion. If we denote by  $\zeta_s = \kappa_s + i\eta_s$  (resp.  $\tilde{\zeta}_s = \tilde{\kappa}_s + i\tilde{\eta}_s$ ) the set of eigenvalues of  $L$  (resp.  $\tilde{L}$ ) then their real parts  $\kappa_s$  (resp.  $\tilde{\kappa}_s$ ) determine the asymptotic velocities for the soliton train described by CTC (resp. GCTC).



While for the RTC the set of eigenvalues  $\zeta_s$  of the Lax matrix are all real, for the CTC they generically take complex values, e.g.,  $\zeta_s = \kappa_s + i\eta_s$ .

Hence, the only possible asymptotic behavior in the RTC is an asymptotically separating, free motion of the solitons. In opposite, for the CTC the real parts  $\kappa_s \equiv \Re\zeta_s$  of eigenvalues of the Lax matrix  $\zeta_s$  determines the asymptotic velocity of the sth soliton.

# Effects of the polarization vectors on the soliton interaction

Thus, starting from the set of initial soliton parameters we can calculate  $L|_{t=0}$  (resp.  $\tilde{L}|_{t=0}$ ), evaluate the real parts of their eigenvalues and thus determine the asymptotic regime of the soliton train.

- Regime (i)**  $\kappa_k \neq \kappa_j$  ( $\tilde{\kappa}_k \neq \tilde{\kappa}_j$ ) for  $k \neq j$  – asymptotically separating, free solitons;
- Regime (ii)**  $\kappa_1 = \kappa_2 = \dots = \kappa_N = 0$   
( $\tilde{\kappa}_1 = \tilde{\kappa}_2 = \dots = \tilde{\kappa}_N = 0$ ) – a “bound state;”
- Regime (iii)** group of particles move with the same mean asymptotic velocity and the rest of the particles will have free asymptotic motion.

Varying only the polarization vectors one can change the asymptotic regime of the soliton train.

# Problem Formulation: Conservation Laws

Define “mass”,  $M$ , (pseudo)momentum,  $P$ , and energy,  $E$ :

$$M \stackrel{\text{def}}{=} \frac{1}{2\beta} \int_{-L_1}^{L_2} (|\psi|^2 + |\phi|^2) dx, \quad P \stackrel{\text{def}}{=} - \int_{-L_1}^{L_2} \mathcal{I}(\psi \bar{\psi}_x + \phi \bar{\phi}_x) dx,$$
$$E \stackrel{\text{def}}{=} \int_{-L_1}^{L_2} \mathcal{H} dx, \quad \text{where} \quad (18)$$

$$\mathcal{H} \stackrel{\text{def}}{=} \beta (|\psi_x|^2 + |\phi_x|^2) - \frac{1}{2} \alpha_1 (|\psi|^4 + |\phi|^4) \\ - (\alpha_1 + 2\alpha_2) (|\phi|^2 |\psi|^2) - 2\Gamma [\Re(\bar{\psi}\bar{\phi})]$$

is the Hamiltonian density of the system. Here  $-L_1$  and  $L_2$  are the left end and the right end of the interval under consideration.

The following conservation/balance laws hold, namely

$$\frac{dM}{dt} = 0, \quad \frac{dP}{dt} = \mathcal{H}|_{x=L_2} - \mathcal{H}|_{x=-L_1}, \quad \frac{dE}{dt} = 0, \quad (19)$$

To solve the main problem numerically, we use an implicit conservative scheme in complex arithmetic.

$$\begin{aligned} i \frac{\psi_i^{n+1} - \psi_i^n}{\tau} &= \frac{\beta}{2h^2} (\psi_{i-1}^{n+1} - 2\psi_i^{n+1} + \psi_{i+1}^{n+1} + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n) \\ &+ \frac{\psi_i^{n+1} + \psi_i^n}{4} \left[ \alpha_1 (|\psi_i^{n+1}|^2 + |\psi_i^n|^2) + (\alpha_1 + 2\alpha_2) (|\phi_i^{n+1}|^2 + |\phi_i^n|^2) \right] \\ &- \frac{1}{2} \Gamma (\phi_i^{n+1} + \phi_i^n), \end{aligned}$$

$$\begin{aligned} i \frac{\phi_i^{n+1} - \phi_i^n}{\tau} &= \frac{\beta}{2h^2} (\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1} + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n) \\ &+ \frac{\phi_i^{n+1} + \phi_i^n}{4} \left[ \alpha_1 (|\phi_i^{n+1}|^2 + |\phi_i^n|^2) + (\alpha_1 + 2\alpha_2) (|\psi_i^{n+1}|^2 + |\psi_i^n|^2) \right] \\ &- \frac{1}{2} \Gamma (\psi_i^{n+1} + \psi_i^n). \end{aligned}$$

$$\begin{aligned}
 i \frac{\psi_i^{n+1,k+1} - \psi_i^n}{\tau} &= \frac{\beta}{2h^2} \left( \psi_{i-1}^{n+1,k+1} - 2\psi_i^{n+1,k+1} + \psi_{i+1}^{n+1,k+1} \right. \\
 &\quad \left. + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n \right) \\
 &+ \frac{\psi_i^{n+1,k} + \psi_i^n}{4} \left[ \alpha_1 (|\psi_i^{n+1,k+1}| |\psi_i^{n+1,k}| + |\psi_i^n|^2) \right. \\
 &\quad \left. + (\alpha_1 + 2\alpha_2) (|\phi_i^{n+1,k+1}| |\phi_i^{n+1,k}| + |\phi_i^n|^2) \right] \\
 i \frac{\phi_i^{n+1,k+1} - \phi_i^n}{\tau} &= \frac{\beta}{2h^2} \left( \phi_{i-1}^{n+1,k+1} - 2\phi_i^{n+1,k+1} + \phi_{i+1}^{n+1,k+1} \right. \\
 &\quad \left. + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n \right) \\
 &+ \frac{\phi_i^{n+1,k} + \phi_i^n}{4} \left[ \alpha_1 (|\phi_i^{n+1,k+1}| |\phi_i^{n+1,k}| + |\phi_i^n|^2) \right. \\
 &\quad \left. + (\alpha_1 + 2\alpha_2) (|\psi_i^{n+1,k+1}| |\psi_i^{n+1,k}| + |\psi_i^n|^2) \right].
 \end{aligned}$$

# Numerical Method: Conservation Properties

It is not only convergent (consistent and stable), but also conserves mass and energy, i.e., there exist discrete analogs for (19), which arise from the scheme.

$$M^n = \sum_{i=2}^{N-1} (|\psi_i^n|^2 + |\phi_i^n|^2) = \text{const},$$

$$E^n = \sum_{i=2}^{N-1} \frac{-\beta}{2h^2} (|\psi_{i+1}^n - \psi_i^n|^2 + |\phi_{i+1}^n - \phi_i^n|^2) + \frac{\alpha_1}{4} (|\psi_i^n|^4 + |\phi_i^n|^4) \\ + \frac{1}{2}(\alpha_1 + 2\alpha_2) (|\psi_i^n|^2 |\phi_i^n|^2) - \Gamma \Re[\bar{\phi}_i^n \psi_i^n] = \text{const}, \\ \text{for all } n \geq 0.$$

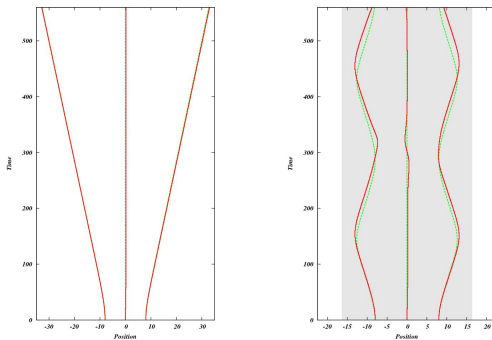
These values are kept constant during the time stepping. The above scheme is of Crank-Nicolson type for the linear terms and we employ internal iterations to achieve implicit approximation of the nonlinear terms, i.e., we use its linearized implementation.

# Effects of the external potentials. Numeric checks vs Variational approach

The predictions and validity of the CTC and GCTC are compared and verified with the numerical solutions of the corresponding CNSE using fully implicit difference scheme of Crank-Nicolson type, which conserves the energy, the mass, and the pseudomomentum. The scheme is implemented in a complex arithmetics. Such comparison is conducted for all dynamical regimes considered.

- First we study the soliton interaction of the pure Manakov model (without perturbations,  $V(x) \equiv 0$ ) and with vanishing cross-modulation  $\alpha_2 = 0$ ;
- 3- and 9-soliton configurations and transitions between different asymptotic regimes under the effect of well-like external potential as well as under the effect of self-dispersion (linear coupling).

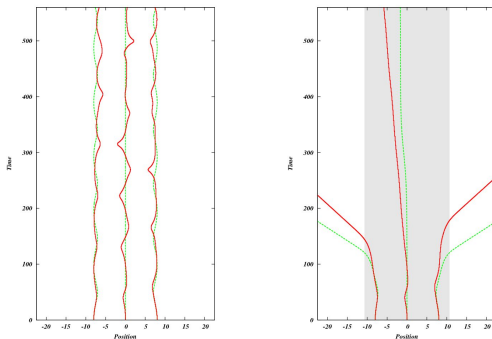
# Three-soliton configuration in free asymptotic regime corresponding to real parts of eigenvalues of the Lax pair $\text{Re}\zeta_1 = -0.0116$ , $\text{Re}\zeta_2 = 0$ , $\text{Re}\zeta_3 = 0.0116$



**Figure:** 3. Free potential behavior (left); External potential well  $V(x) = \sum_{s=0}^{32} c_s \text{sech}^2(x - x_s)$ ,  $c_s = -10^{-1}$ ,  $x_s = -16 + sh$ ,  $h = 1$ ,  $s = 0, \dots, 32$ (right).

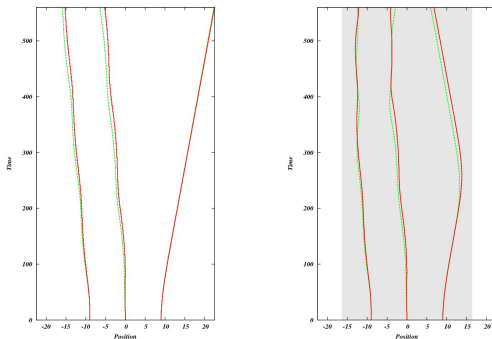


# Three-soliton configuration in bound state regime corresponding to real parts of eigenvalues of the Lax pair $\text{Re}\zeta_1 = \text{Re}\zeta_2 = \text{Re}\zeta_3 = 0$



**Figure:** 4. Free potential behavior (left); External potential hump  $V(x) = \sum_{s=0}^{12} c_s \text{sech}^2(x - x_s)$ ,  $c_s = 10^{-2}$ ,  $x_s = -10 + sh$ ,  $h = 5/3$ ,  $s = 0, \dots, 12$  (right).

# Three-soliton configuration in mixed asymptotic regime corresponding to real parts of eigenvalues of the Lax pair $\text{Re}\zeta_1 = \text{Re}\zeta_2 = -0.00321$ , $\text{Re}\zeta_3 = 0.00642$



**Figure:** 5. Free potential behavior (left); External potential well  $V(x) = \sum_{s=0}^{32} c_s \text{sech}^2(x - x_s)$ ,  $c_s = -10^{-2}$ ,  $x_s = -16 + sh$ ,  $h = 1$ ,  $s = 0, \dots, 32$ .

$$\begin{aligned} \text{Nine-soliton asymptotic regime: } \xi_k &= -45 + 9k, \\ \nu_k &= 0.4625 + 0.0075(k - 1), \quad \mu_k = 0, \quad \delta_k = k\pi, \\ k &= 1, \dots, 9, \quad \theta_{k+1} = \theta_k - \frac{\pi}{10}, \quad k = 1, \dots, 8, \quad \theta_1 = \frac{9\pi}{10} \end{aligned}$$

Calculated eigenvalues of the potentialfree Lax pair are

$$\zeta_1 = -0.005720 + 0.239562i$$

$$\zeta_2 = 0.005720 + 0.239562i$$

$$\zeta_3 = -0.001564 + 0.245551i$$

$$\zeta_4 = 0.001564 + 0.245551i$$

$$\zeta_5 = -0.005720 + 0.260438i$$

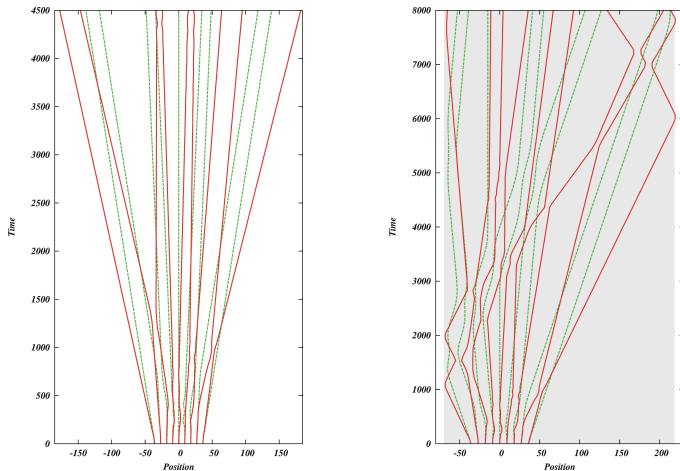
$$\zeta_6 = -2.939394 \times 10^{-19} + 0.250000i$$

$$\zeta_7 = -0.001564 + 0.254449i$$

$$\zeta_8 = 0.001564 + 0.254449i$$

$$\zeta_9 = 0.005720 + 0.260438i$$

# Nine-soliton configuration



**Figure:** 6. Free potential behavior (left); External potential well  $V(x) = c_s[\tanh(2\nu_0x - y_i) - \tanh(2\nu_0x - y_f)]$ ,  $c_s = -0.004$ ,  $y_i = -69.5$ ,  $y_f = 218.5$  (right).

$$\begin{aligned} \text{Nine-soliton asymptotic regime: } \xi_k &= -40 + 8k, \\ \nu_k &= 0.4625 + 0.0075(k - 1), \quad \mu_k = 0, \quad \delta_k = k\pi, \\ k &= 1, \dots, 9, \quad \theta_{k+1} = \theta_k - \frac{\pi}{10}, \quad k = 1, \dots, 8, \quad \theta_1 = \frac{9\pi}{10} \end{aligned}$$

Calculated eigenvalues of the potentialfree Lax pair are

$$\zeta_1 = -0.011877 + 0.241165i$$

$$\zeta_2 = 0.011877 + 0.241165i$$

$$\zeta_3 = -0.011877 + 0.258835i$$

$$\zeta_4 = -0.006926 + 0.248312i$$

$$\zeta_5 = -0.006926 + 0.251688i$$

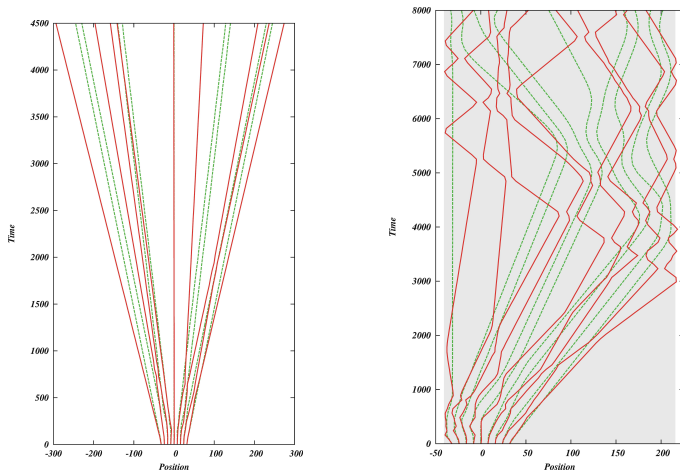
$$\zeta_6 = -8.679330 \times 10^{-20} + 0.250000i$$

$$\zeta_7 = 0.006926 + 0.248312i$$

$$\zeta_8 = 0.006926 + 0.251688i$$

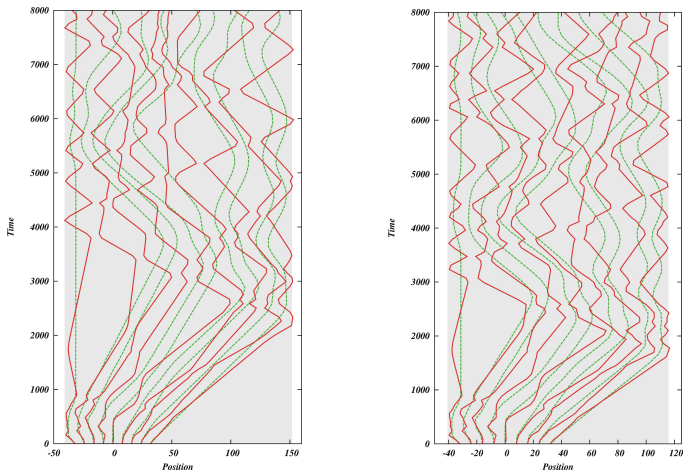
$$\zeta_9 = 0.011877 + 0.258835i$$

# Nine-soliton configuration



**Figure:** 7. Free potential behavior (left); External potential well  $V(x) = c_s[\tanh(2\nu_0x - y_i) - \tanh(2\nu_0x - y_f)]$ ,  $c_s = -0.004$ ,  $y_i = -40.5$ ,  $y_f = 215.5$  (right).

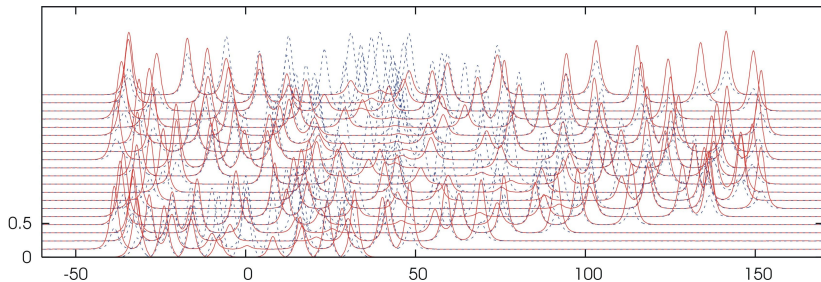
# Nine-soliton configuration



**Figure:** 8. External potential well

$V(x) = c_s[\tanh(2\nu_0 x - y_i) - \tanh(2\nu_0 x - y_f)]$ ,  $c_s = -0.004$ :  $y_i = -40.5$ ,  
 $y_f = 151.5$  (left);  $y_i = -40.5$ ,  $y_f = 115.5$  (right).

# Nine-soliton configuration



**Figure:** 9. External potential well

$V(x) = c_s[\tanh(2\nu_0 x - y_i) - \tanh(2\nu_0 x - y_f)]$ ,  $c_s = -0.004$ :  $y_i = -40.5$ ,  
 $y_f = 151.5$ .



# Three-soliton adiabatic interaction

$$\mu_i = 0, \quad i = 1, 2, 3, \quad \nu_2 = 0.5, \quad \nu_{1,3} = \nu_2 \pm \Delta\nu, \quad \Delta\nu = 0.7, \quad \delta_1 = 0, \\ \delta_{2,3} = \pm \frac{\pi}{2}, \quad \theta_1 = \frac{3\pi}{10}, \quad \theta_i = \theta_{i-1} - \frac{\pi}{10}, \quad i = 2, 3, \quad V(x) \equiv 0.$$

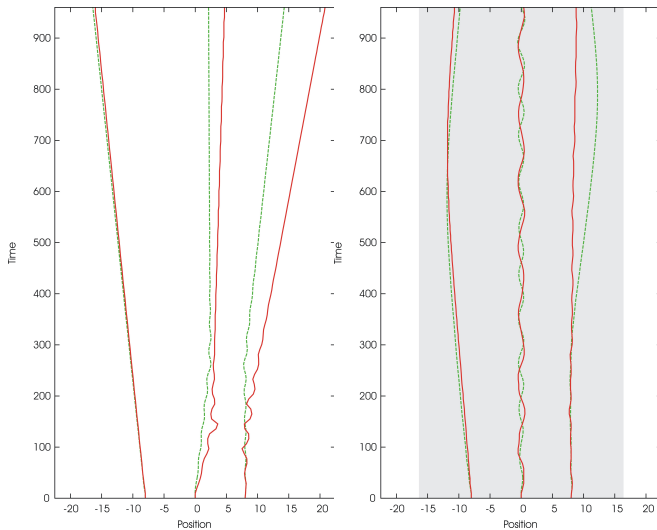
Lax eigenvalues

$$\zeta_1 = 0.00229 + 0.25258i, \quad \zeta_2 = -0.0001 + 0.28255i, \\ \zeta_3 = -0.00219 + 0.21487i - \text{free asymptotic regime.}$$

Composite potential well of kind “well-in-well”

$$V(x) = \sum_{s=0}^{32} c_s \operatorname{sech}^2(x - x_s), \quad x_s = -16 + sh, \quad h = 1, \quad s = 0, \dots, 32 \\ c_s = \begin{cases} -0.01 & \text{for } 0 \leq s \leq 12 \quad \text{and} \quad 20 \leq s \leq 33, \\ -0.02 & \text{for } 13 < s < 19. \end{cases}$$

# Three-soliton adiabatic interaction



**Figure:** 10. Potential free interaction (left); Composite potential well of kind “well-in-well”.

The perturbed CTC for this type of perturbation becomes

$$\frac{d\xi_k}{dt} = 2\mu_k, \quad (21)$$

$$\frac{d\delta_k}{dt} = 2\mu_k^2 + 2\nu_k^2 + c_1 D[u_k], \quad (22)$$

$$\frac{d\nu_k}{dt} = 8\nu_0^3 \sum_n e^{-\Delta_{kn}} i(R_{kn} - R_{kn}^*) + c_1 N[u_k], \quad (23)$$

$$\frac{d\mu_k}{dt} = -8\nu_0^3 \sum_n e^{-\Delta_{kn}} (R_{kn} + R_{kn}^*) + c_1 M[u_k], \quad (24)$$

$$\begin{aligned} \frac{d\vec{n}_k}{dt} = & 4\nu_0^2 i \sum_{n=k\pm 1} e^{-\Delta_{kn}} \left[ e^{i(\tilde{\delta}_n - \tilde{\delta}_k)} \vec{n}_n - R_{kn} \vec{n}_k + e^{i(\tilde{\delta}_n - \tilde{\delta}_k)} \vec{n}_n + R_{kn}^* \vec{n}_k \right] \\ & - ic_1 \nu_0 \vec{n}_k. \end{aligned} \quad (25)$$

For the case when  $iR[u] = \sigma_1 \bar{u}$  we get:

$$\begin{aligned}
 M[u_k] &= -\nu_k \int_{-\infty}^{\infty} \frac{dz_k \sinh(z_k)}{\cosh^3(z_k)} \left( c_1 \vec{n}_k^\dagger \sigma_1 \vec{n}_k \right) = 0, \\
 N[u_k] &= \nu_k \int_{-\infty}^{\infty} \frac{dz_k}{\cosh^2(z_k)} \left( c_1 \vec{n}_k^\dagger \sigma_1 \vec{n}_k \right) = 0, \\
 \Xi[u_k] &= -\frac{1}{2\nu_k} \int_{-\infty}^{\infty} \frac{dz_k z_k}{\cosh^2(z_k)} \left( c_1 \vec{n}_k^\dagger \sigma_1 \vec{n}_k \right) = 0, \\
 D[u_k] &= -\int_{-\infty}^{\infty} \frac{dz_k (1 - z_k \tanh(z_k))}{\cosh^2(z_k)} \left( c_1 \vec{n}_k^\dagger \sigma_1 \vec{n}_k \right) \\
 &= -\frac{1}{4} \left( c_1 \vec{n}_k^\dagger \sigma_1 \vec{n}_k \right).
 \end{aligned} \tag{26}$$

# Three solitons. $iR[u] = c_1\sigma_1\vec{u}$

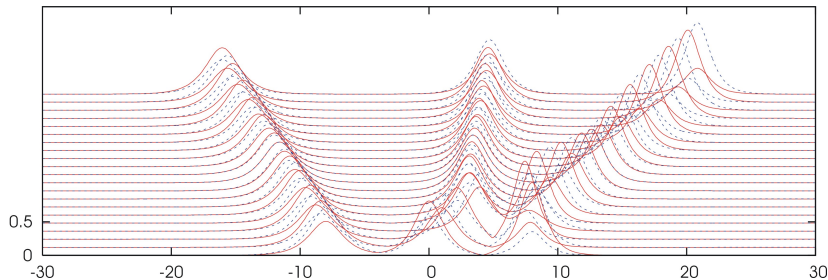


Figure: 11.  $c_1 = 0.1$ , Pauli matrix  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

# Three solitons. $iR[u] = c_1\sigma_1\vec{u}$

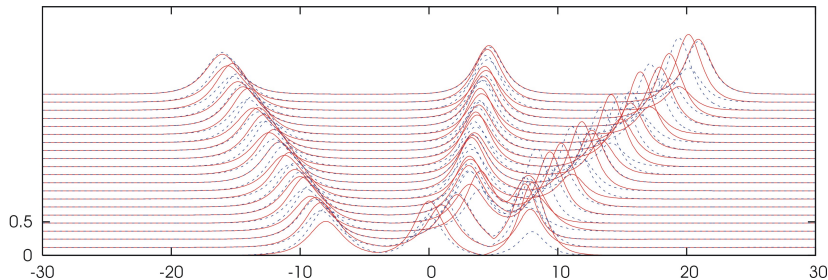


Figure: 12.  $c_1 = 0.5$ , Pauli matrix  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

# Three solitons. $iR[u] = c_1\sigma_1\vec{u}$

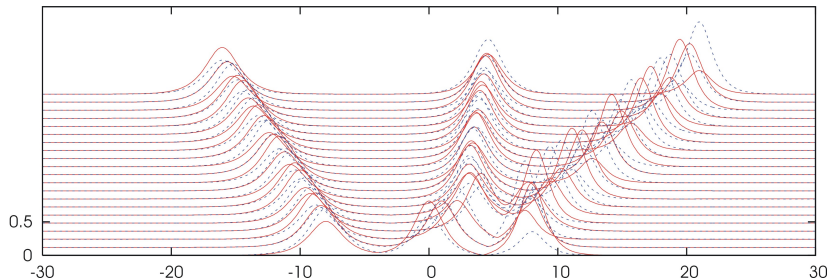
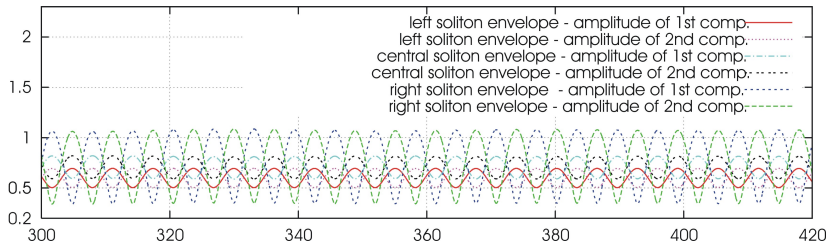


Figure: 13.  $c_1 = 1$ , Pauli matrix  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

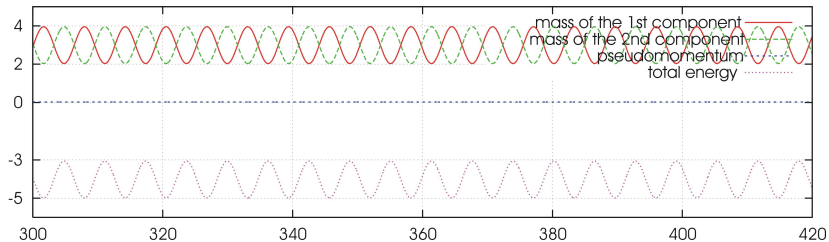
# Three solitons. $iR[u] = c_1\sigma_1\vec{u}$



**Figure:** 14. Adiabatic approximation. Component amplitudes of Breathing Solitons for  $c_1 = 0.5$ .

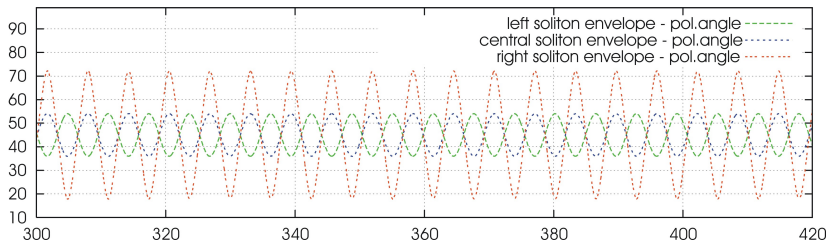


# Three solitons. $iR[u] = c_1\sigma_1\vec{u}$



**Figure:** 15. Adiabatic approximation. Component masses, pseudomomentum, and total energy for  $c_1 = 0.5$ .

# Three solitons. $iR[u] = c_1\sigma_1\vec{u}$



**Figure:** 16. Adiabatic approximation. Polarization angles of envelopes for  $c_1 = 0.5$ .

# Effects of external potentials

We analyze the effects of three types of external potentials: anharmonic, periodic and wide-well:

$$V_{\text{anh}}(x) = \sum_{s=0}^4 V_s x^s, \quad V_{\text{per}}(x) = A \cos(\Omega x + \Omega_0), \quad (27)$$

$$V_{\text{ww}}(x) = \sum_s c_s (\tanh(2\nu_0 x + y_f) - \tanh(2\nu_0 x - y_i)).$$

Test a criteria under which given the effect of the potential on the soliton train can be viewed as an adiabatic perturbation, namely  $|H_V| \ll |H_0|$

$$H_V = \int_{-\infty}^{\infty} dx V(x) (\vec{u}^\dagger, \vec{u})(x, t), \quad H_0 = \int_{-\infty}^{\infty} dx \left( (\vec{u}_x^\dagger, \vec{u}_x) - \frac{1}{2} (\vec{u}^\dagger, \vec{u})^2 \right) \quad (28)$$

# Criterion for perturbation

Let us consider the ratio  $|H_V| \ll |H_0|$ . Both integrals can be evaluated through the parameters of the soliton train. While the leading term of  $|H_0|$  depends only on  $\nu_k$  and  $\mu_k$ ,  $|H_V|$  depends substantially also on the positions  $\xi_k$  of the solitons as well as on their number. If the ratio  $|H_V|/|H_0| \simeq \varepsilon$  then the perturbed CTC matches very well the soliton trajectories of the perturbed Manakov model.

# Criterion for perturbation. Example 1

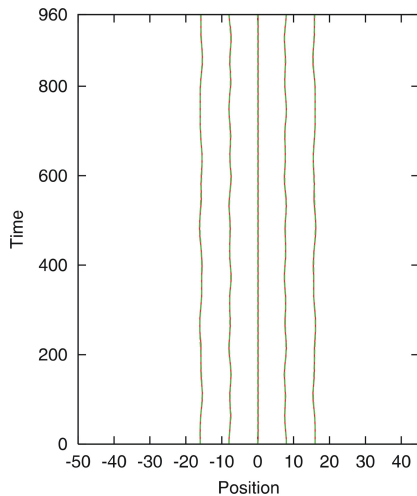


Figure: 17.  $V_{H_h} = 0.000036x^2$ ,  $\frac{|H_{V_h}|}{|H_0|} \approx 2.78\%$

# Criterion for perturbation. Example 2

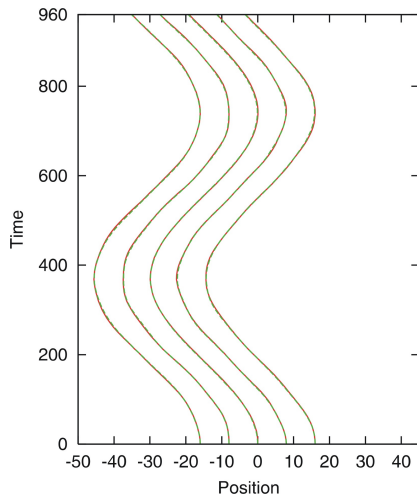


Figure: 18.  $V_{H_h} = 0.000036(x + 15)^2$ ,  $\frac{|H_{V_h}|}{|H_0|} \approx 7.64\%$

# Criterion for perturbation. Example 3

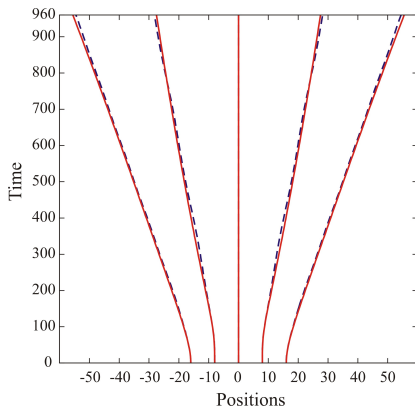


Figure: 19.  $V_{\text{per}} = A \cos \frac{\pi}{4} x$ ,  $A = -0.0001$ ,  $\frac{|H_{V_{\text{per}}}|}{|H_0|} \approx 0.047\%$

# Criterion for perturbation. Example 4

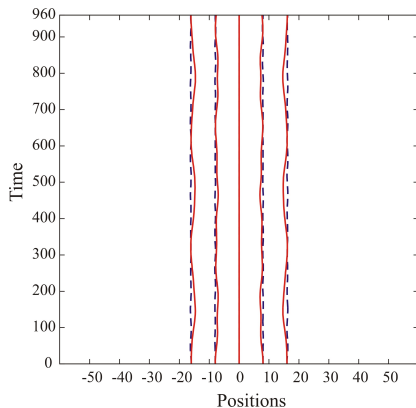


Figure: 20.  $V_{\text{per}} = A \cos \frac{\pi}{4} x$ ,  $A = -0.0075$ ,  $\frac{|H_{V_{\text{per}}}|}{|H_0|} \approx 3.53\%$



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Thank you for your attention!