A Numerical Approach to Price Path Dependent Asian Options

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Outline

Option. Call and Put options

Option

An option is a contract between the writer and the holder of the option about trading the stock at a prespecified fixed price K (exercise price) within a specified period (from the date of signing the contract to the maturity date T).

Depending on what an option concern: Call and Put options

The call option gives the holder the right (but not the obligation) to buy the stock for the price K by the date (or at the date) of the maturity.
The put option gives the holder the right (but not the obligation) to sell the stock for the price K by the date (or at the date) of the maturity.

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European and American style of option; Asian option

Depending on when an option may be exercised

European option exercise is only at the date of the maturity. American style of option can be exercised at any time up to and including the date of the maturity. The payoff depends on the underlying asset price in the moment of its exercise.

Asian option

An Asian option can be of European or American style. An Asian option is an option whose payoff depends on the average of an underlying asset price over some time period, for example $A = A(t) = \frac{1}{t}\int_0^t S(\theta)d\theta$, where $S(\theta)$ is the price of the underlying stock.

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Mathematical model

Asian call option of European style

P. Wilmott et al.,Mathematical Models and Computation, (1993):

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma_1^2 \bar{S}^{\gamma} \frac{\partial^2 V}{\partial \bar{S}^2} + r\bar{S}\frac{\partial V}{\partial \bar{S}} - \bar{S}\frac{\partial V}{\partial \bar{x}} - rV,$$
$$(\bar{S}, \bar{x}, \tau) \in (0, \infty) \times (0, \infty) \times (0, T],$$

V is the Asian option prise; $\bar{S}$ is the underlying stock price;
$\tau = T - t$, is the time to maturity T (t is the time);
σ_1 is the volatility; r is the interest rate;
$\bar{x} = \bar{x}(t) = \int_0^t \bar{S}(\theta)d\theta$, γ is the order of degeneracy, $0 < \gamma \leq 2$;
$(\bar{S}, \bar{x}, \tau) \in (0, S_{max}) \times (0, x_{max}) \times (0, T]$.

Mathematical model

Initial and boundary conditions

$$V(\bar{S},\bar{x},0) = \max\{X(\bar{x}) - K,\ 0\} \equiv V_0(\bar{S},\bar{x}),$$
$$V(0,\bar{x},\tau) = e^{-r\tau}\max\{X(\bar{x}) - K,\ 0\} \equiv V_1(\bar{x},\tau),$$
$$V(S_{\max},\bar{x},\tau) = \max\left\{e^{-r\tau}\left(X(\bar{x}) - K\right) + \frac{S_{\max}}{rT}\left(1 - e^{-r\tau}\right),\ 0\right\}$$
$$\equiv V_2(\bar{x},\tau),$$
$$V(\bar{S},0,\tau) = \frac{\bar{S}}{rT}\left(1 - e^{-r\tau}\right) \equiv V_3(\bar{S},\tau),$$
$$X(\bar{x}) = (x_{\max} - \bar{x})/T.$$

Previous Work

FDM and FEM, constructed for ultra-parabolic equations *without degeneration*:

Vabishchevich, P. N.: The numerical simulation of unsteady convective-diffusion processes in a countercurrent. Zh. Vychisl. Mat. Mat. Fiz. 35 (1), 46–52 (1995)

Akrivis, G., Grouzlix, M., Thomee, V.: Numerical methods for ultra-parabolic equations. CALCOLO 31, 179–190 (1996)

Ashyralyev, A., Yilmaz, S.: Modified Crank-Nicholson difference schemes for ultra-parabolic equations. Comp. and Math. Appls. 64, 2756–2764 (2012)

Previous Work

A number of techniques to price Asian options have been proposed:

- Monte-Carlo method (Y.-K.Kwok, R.Seydel);
- analytical methods (I.Sengypta, M.Fu, D.Madan, T.Wang);
- modified binomial tree approach (P.Wilmott, J.Dewyne, S. Howison);
- finite difference schemes (Z.Cen, A.Le, A.Xu, J.Hugger, T.Chernogorova, L.Vulkov);
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Equation in dimensionless variables and splitting method

Equation in dimensionless variables

$$S = \frac{\bar{S}}{x_{\max}}, \quad x = \frac{\bar{x}}{x_{\max}}, \quad \sigma = \sigma_1 x_{\max}^{\frac{\gamma-2}{2}} :$$

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^\gamma \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - S\frac{\partial V}{\partial x} - rV, \;\; x \in (0,1), \;\; S \in (0, S_0).$$

Splitting

– the first one with respect to (S, τ);
– the second one with respect to (x, τ).

$$0 = \tau_1 < \tau_2 < \cdots < \tau_n < \tau_{n+1} < \ldots \tau_{P+1} = T, \;\; \triangle\tau_n = \tau_{n+1} - \tau_n.$$

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Parabolic subproblem

Formulation

x - fixed, $V(S, x, \tau_n)$ - given,

$$? \; u(S, x, \tau), \;\; (S, x, \tau) \in (0, S_0) \times (0, 1) \times (\tau_n, \tau_{n+1/2}],$$

$$\frac{1}{2}\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 S^\gamma \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru,$$

$$u(S, x, \tau_n) = V(S, x, \tau_n),$$

$$u(0, x, \tau_{n+1/2}) = V_1(x, \tau_{n+1/2}),$$

$$u(S_0, x, \tau_{n+1/2}) = V_2(x, \tau_{n+1/2}).$$

Hyperbolic subproblem

Formulation

S - fixed, $u(S, x, \tau_{n+1/2})$ - given

$$? \; V(S, x, \tau), \; (S, x, \tau) \in (0, S_0) \times (0, 1) \times (\tau_{n+1/2}, \tau_{n+1}],$$

$$\frac{1}{2}\frac{\partial V}{\partial \tau} + S\frac{\partial V}{\partial x} = 0,$$

$$V(S, x, \tau_{n+1/2}) = u(S, x, \tau_{n+1/2}),$$

$$V(S, 0, \tau_{n+1}) = V_3(S, \tau_{n+1}).$$

First difference approximation of the Parabolic subproblem

Non-uniform meshes in $[0, 1]$ and $[0, S_0]$:

$$0 = x_1 < x_2 < \ldots < x_j < x_{j+1} < \ldots < x_{M+1} = 1,$$

$$h_j^x = x_{j+1} - x_j;$$

$$I_i = \left[S_i, S_{i+1}\right], \; i = 1, 2, \ldots, N,$$

$$0 = S_1 < S_2 < \ldots < S_{N+1} = S_0.$$

The secondary mesh:

$$S_{i+1/2} = 0.5(S_i + S_{i+1}), i = 1, 2, \ldots, N;$$

$$h_i = S_{i+1} - S_i, \quad \hbar_i = S_{i+1/2} - S_{i-1/2}.$$

First difference approximation of the Parabolic subproblem

Divergent form of the equation

$$a(S) = \frac{1}{2}\sigma^2 S^{\gamma-1}, \quad b(S) = rS - \gamma a(S),$$

$$c(S) = 2r - \frac{1}{2}\gamma(\gamma-1)S^{\gamma-2}\sigma^2,$$

$$\frac{1}{2}\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 S^{\gamma}\frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru \rightarrow$$

$$\frac{1}{2}\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial S}\left(aS\frac{\partial u}{\partial S} + bu\right) - cu.$$

First difference approximation of the Parabolic subproblem

Finite volume method; x, τ - fixed

$$\frac{1}{2}\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial S}\left(aS\frac{\partial u}{\partial S} + bu\right) - cu,$$

$$\left[S_{i-1/2},\ S_{i+1/2}\right],\ i = 2, 3, \ldots, N,$$

$$\frac{1}{2}\left.\frac{\partial u}{\partial \tau}\right|_{S_i} \hbar_i \approx \rho(u)|_{S_{i+1/2}} - \rho(u)|_{S_{i-1/2}} - c_i u_i \hbar_i,$$

$$\rho(u) = aS\frac{\partial u}{\partial S} + bu, \quad c_i = c(S_i, x, \tau).$$

Approximation of $\rho(u)$ at $S_{i+1/2}$, $2 \leq i \leq N$; x, τ - fixed

Fitted finite volume method

Allen & Southwell (1955); Miller & Wang (1994);
Wang (1997; 2004); Angermann & Wang (2003)

$$\rho(u) = aS\frac{\partial u}{\partial S} + bu, \quad S \in I_i = [S_i, S_{i+1}],$$

$$\left(a_{i+1/2}Sw' + b_{i+1/2}w\right)' = 0,$$

$$w(S_i) = u_i, \quad w(S_{i+1}) = u_{i+1},$$

$$a_{i+1/2}Sw' + b_{i+1/2}w = C_1, \quad w = C_2 S^{-\alpha_i} + \frac{C_1}{b_{i+1/2}},$$

$$\rho_i(u) = C_1 = b_{i+1/2}\frac{S_{i+1}^{\alpha_i}u_{i+1} - S_i^{\alpha_i}u_i}{S_{i+1}^{\alpha_i} - S_i^{\alpha_i}}, \ \alpha_i = \frac{b_{i+1/2}}{a_{i+1/2}}.$$

Approximation of the flux $\rho(u)$ at $S_{3/2}$, x, τ - fixed

Fitted finite volume method

$$\left(a_{3/2} S w' + b_{3/2} w\right)' = C_1, \quad S \in I_1,$$

$$w(0) = u_1, \quad w(S_2) = u_2.$$

$$w = u_1 + \frac{u_2 - u_1}{S_2} S,$$

$$\rho_1(u) = \frac{1}{2}\left[\left(a_{3/2} + b_{3/2}\right) u_2 - \left(a_{3/2} - b_{3/2}\right) u_1\right].$$

The fully implicit difference scheme

For differential equation:

$\bar{u}_{i,j}$ – the approximate solution on the level $n + 1/2$;
$u_{i,j}$ – the approximate solution on the level n.

$$\frac{\bar{u}_{2,j} - u_{2,j}}{\Delta\tau_n}\hbar_2 = b_{5/2}\frac{S_3^{\alpha_2}\bar{u}_{3,j} - S_2^{\alpha_2}\bar{u}_{2,j}}{S_3^{\alpha_2} - S_2^{\alpha_2}}$$

$$-\frac{1}{2}\cdot\left[\left(a_{3/2} + b_{3/2}\right)\bar{u}_{2,j} - \left(a_{3/2} - b_{3/2}\right)\bar{u}_{1,j}\right] - \hbar_2 c_2 \bar{u}_{2,j},$$

$$\frac{\bar{u}_{i,j} - u_{i,j}}{\Delta\tau_n}\hbar_i = b_{i+\frac{1}{2}}\frac{S_{i+1}^{\alpha_i}\bar{u}_{i+1} - S_i^{\alpha_i}\bar{u}_i}{S_{i+1}^{\alpha_i} - S_i^{\alpha_i}} - b_{i-\frac{1}{2}}\frac{S_i^{\alpha_{i-1}}\bar{u}_i - S_{i-1}^{\alpha_{i-1}}\bar{u}_{i-1}}{S_i^{\alpha_{i-1}} - S_{i-1}^{\alpha_{i-1}}}$$

$$-\hbar_i c_i \bar{u}_{i,j},\ i = 3, 4, \ldots, N,\ j = 2, 3, \ldots, M,$$

+ approximation of additional conditions.

Theoretical results

Truncation error of the scheme:

$$O(\triangle\tau + h), \quad h = \max_{1\le j\le M} h_j, \quad \triangle\tau = \max_{1\le n\le P} \triangle\tau_n.$$

Lemma 1.

Suppose that $u_{i,j} \ge 0,\ i = 1, 2, \dots, N+1,\ j = 1, 2, \dots, M+1$. Then for sufficiently small $\triangle\tau$ we have $\overline{u}_{i,j} \ge 0,\ i = 1, 2, \dots, N+1,\ j = 1, 2, \dots, M+1$.

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Second difference approximation for Parabolic subproblem (the classical monotone scheme of A. A. Samarskii)

Divergent form of equation:

$$\frac{1}{2}\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 S^\gamma \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru,$$

$$\frac{1}{2}\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial S}\left(k(S)\frac{\partial u}{\partial S}\right) + p(S)\frac{\partial u}{\partial S} - ru,$$

$$k(S) = \frac{1}{2}\sigma^2 S^\gamma, \quad p(S) = rS - \frac{1}{2}\gamma S^{\gamma-1}\sigma^2.$$

$$\bar{\omega}_h = \{S_i = (i-1)h, \quad i = 1, 2, \ldots, N+1, \quad h = S_0/N\}.$$

The classical monotone scheme of A. A. Samarskii)

The fully implicit monotone difference scheme with truncation error $O(\triangle\tau + h^2)$:

$$\frac{\bar{u}_{i,j} - u_{i,j}}{\triangle\tau_n} = \bar{\rho}_i \frac{1}{h}\left[a_{i+1}\frac{\bar{u}_{i+1,j} - \bar{u}_{i,j}}{h} - a_i\frac{\bar{u}_{i,j} - \bar{u}_{i-1,j}}{h}\right]$$

$$+ b_i^+ a_{i+1}\frac{\bar{u}_{i+1,j} - \bar{u}_{i,j}}{h} + b_i^- a_i\frac{\bar{u}_{i,j} - \bar{u}_{i-1,j}}{h} - r\bar{u}_{i,j},$$

$$i = 2, 3, \dots, N, \quad j = 1, 2, \dots, M,$$

$$\bar{\rho}_i = \frac{1}{1 + \frac{1}{2}h\frac{|p(S_i)|}{k(S_i)}}, \quad a_i = k(S_i - h/2), \quad b_i^+ = \frac{p^+(S_i)}{k(S_i)},$$

$$b_i^- = \frac{p^-(S_i)}{k(S_i)}, \quad p^- = \frac{p - |p|}{2}, \quad p^+ = \frac{p + |p|}{2}.$$

Difference approximation for Hyperbolic subproblem

An implicit difference scheme:

$$BC: \quad \hat{V}_{i,1} = V_3(S_i, x_1), \quad i = 2, 3, \dots, N.$$

$$IC: \quad V(S_i, x_j, \tau_{n+1/2}) = u(S_i, x_j, \tau_{n+1/2}).$$

For the equation (an implicit backward scheme):

$$\frac{\hat{V}_{i,j} - \bar{u}_{i,j}}{\Delta\tau_n} + S_i \frac{\hat{V}_{i,j} - \hat{V}_{i,j-1}}{h^x_{j-1}} = 0, i = 2, \dots, N, \; j = 2, \dots, M+1.$$

The truncation error: $O(\triangle\tau + h)$.
The scheme is unconditionally stable.
Theorem. For sufficiently small $\triangle\tau$, the numerical solutions, obtained by the two methods, are non-negative.

Numerical experiments

An analytical solution and the fixed values of the parameters

$V_a(S, x, \tau) = (2 - x)\,(S/S_0)^2\, e^{-r\tau}$;

$S_0 = 2$, $x \in [0, 1]$, $T = 1$, $K = 1$, $r = 0.05$, $\sigma = 0.4$ (J. Hugger, ANZIAM J. 45 (E), pp. C215–C231, 2004)

Numerical experiments were performed for the different values of γ, $\gamma \in (0, 2]$.

For every one of the experiments the time-step decreases until establishment of the first four significant digits of the relative C-norm of the error at the last time level $\tau = T$.

The rate of convergence (RC) is calculated using the double mesh principle.

First discretization, $\gamma = 1.5$

Space steps	Relative C-norm of the error	RC	L_2-norm of the error	RC
0.1	1.440 E-4	-	2.481 E-4	-
0.05	3.836 E-5	1.91	6.409 E-5	1.95
0.025	9.986 E-6	1.94	1.627 E-5	1.98
0.0125	2.563 E-6	1.96	4.089 E-6	1.99
0.00625	6.489 E-7	1.98	1.021 E-6	2.00

First discretization, $\gamma = 1$

Space steps	Relative C-norm of the error	RC	L_2-norm of the error	RC
0.1	1.406 E-3	-	2.114 E-3	-
0.05	5.388 E-4	1.38	7.995 E-4	1.40
0.025	1.655 E-4	1.70	2.431 E-4	1.72
0.0125	4.434 E-5	1.90	6.476 E-5	1.91
0.00625	1.152 E-5	1.94	1.678 E-5	1.95

First discretization, $\gamma = 0.8$

Space steps	Relative C-norm of the error	RC	L_2-norm of the error	RC
0.1	1.675 E-3	-	2.501 E-3	-
0.05	7.956 E-4	1.08	1.164 E-3	1.10
0.025	3.452 E-4	1.20	4.910 E-4	1.24
0.0125	1.248 E-4	1.47	1.705 E-4	1.53
0.00625	3.698 E-5	1.75	4.865 E-5	1.81

$$\frac{S_{i+1}^{\alpha_i} u_{i+1} - S_i^{\alpha_i} u_i}{S_{i+1}^{\alpha_i} - S_i^{\alpha_i}}$$

Second discretization, $\gamma = 1.5$

Space steps	Relative C-norm of the error	RC	L_2-norm of the error	RC
0.1	4.338 E-4	-	6.969 E-4	-
0.05	1.263 E-4	1.78	1.994 E-4	1.80
0.025	3.462 E-5	1.87	5.374 E-5	1.89
0.0125	9.144 E-6	1.92	1.391 E-5	1.95
0.00625	2.369 E-6	1.95	3.567 E-6	1.96

Second discretization, $\gamma = 1$

Space steps	Relative C-norm of the error	RC	L_2-norm of the error	RC
0.1	1.472 E-3	-	2.198 E-3	-
0.05	6.354 E-4	1.21	9.383 E-4	1.23
0.025	2.468 E-4	1.36	3.609 E-4	1.38
0.0125	8.485 E-5	1.55	1.234 E-4	1.55
0.00625	2.467 E-5	1.78	3.619 E-5	1.77

Second discretization, $\gamma = 0.8$

Space steps	Relative C-norm of the error	RC	L_2-norm of the error	RC
0.1	1.496 E-3	-	2.230 E-3	-
0.05	7.190 E-4	1.06	1.048 E-3	1.09
0.025	3.231 E-4	1.16	4.644 E-4	1.18
0.0125	1.338 E-4	1.27	1.884 E-4	1.30
0.00625	4.967 E-5	1.43	6.839 E-5	1.46

Second discretization, $\gamma = 0.1$

Space steps	Relative C-norm of the error	RC	L_2-norm of the error	RC
0.1	9.692 E-4	-	1.442 E-3	-
0.05	4.962 E-4	0.96	7.312 E-4	0.98
0.025	2.508 E-4	0.99	3.665 E-4	1.00
0.0125	1.256 E-4	1.00	1.822 E-4	1.01
0.00625	6.240 E-5	1.01	8.970 E-5	1.02

Summary

- The first scheme works properly for $0.8 \leq \gamma \leq 2$.
- In the interval $0.8 \leq \gamma \leq 2$, in general, the first scheme is more accurate and has bigger rate of convergence than the second discretization.
- For the values $0 < \gamma < 0.8$ the first discretization is not applicable.
- The second scheme can be used for all values $0 < \gamma \leq 2$.
- For the two discretizations the rate of convergence decreases, when γ decreases.

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